Characteristic Classes, Gamma functions, sigma models

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$$E \xrightarrow{\pi} X$$

E = complex vector bundle $\pi^{-1}(U) \cong U \times \mathbb{C}^N$

X = compact smooth manifold transition functions in $GL(N, \mathbb{C})$

A connection on E is a way of differentiating sections

$$\nabla : E \longrightarrow E \times \Omega^1$$

$s \in \Gamma(U, E)$

$fs \in \Gamma(U, E)$ f function on U

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

$$F^{\nabla} \in \Gamma(X, \Omega^2_X \otimes \mathrm{End}(E))$$

$\mathrm{End}(E) \leftarrow gl(N, \mathbb{C})$

$$F^{\nabla}_{(s)}(X,Y) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

Topological Invariants of the bundle

Want: invariant functions on $gl(N, \mathbb{C})$ under conjugation

$$f(g^{-1} Z g) = f(Z)$$

$$Z \in gl(N, \mathbb{C}), \quad g \in GL(N, \mathbb{C})$$

$f(F^{\nabla})$ = an even diff. form on X

Lemma (Chern)

$$d f(F^{\nabla}) = 0$$

$$[f(F^{\nabla})] \in H^{even}(X, \mathbb{C})$$

$$c(E) = c(F^\nabla) = \det\left(I + \tfrac{i}{2\pi} F^\nabla\right)$$

$$(\text{total chern class}) = 1 + c_1(F^\nabla) + c_2(F^\nabla) + \cdots$$

$$c_j(E) = H^{2j}(X, \mathbb{Z})/_{\text{tors.}}$$

$$= [c_j(F^\nabla)]$$

If X is a Kähler manifold E is a hermitian bundle

$\Rightarrow \exists$ connection ∇ s.t.

$$c_j(F^\nabla) \in H^{j,j}(X).$$

If X is a kähler manifold, X has a "holo ~~tangent~~ bundle"

$$T^{1,0}X, \quad c_j(T^{1,0}_X) =: c_j(X)$$

<u>Hirzebruch - Riemann - Roch</u>

$$E, \quad \dim H^0(X,E) - \dim H^1(X,E) + \dim H^2(X,E) - \cdots$$

$$= \chi(X,E)$$

$$= \int_X ch(E) \wedge td_X \text{ , where}$$

$$ch(E) = [ch(F^\nabla)] = tr\left(\exp\left(\tfrac{i}{2\pi} F^\nabla\right)\right)$$

<u>Splitting Principle (Grothendieck)</u>

We may pretend $E = L_1 \oplus \cdots \oplus L_N$, $\operatorname{rank}(L_j) = 1$
$(N = \operatorname{rank}(E))$

$$c(E) = \prod (1 + c_1(L_j)) = \sum c_j(E)$$

$$ch(E) = \sum e^{c_1(L_j)}$$

<u>Multiplicative Characteristic Classes</u>:

Ingredient: power series in z with constant term 1
$p(z)$

Method: $\prod (p(c_1(L_j)))$

$$p(z) = 1 + z, \quad c(E) = \prod (1 + c_1(L_j))$$

$$td_X \longleftrightarrow \frac{z}{1 - e^{-z}}$$

only even powers

$$\hat{A}_X \longleftrightarrow \frac{z/2}{\sinh(z/2)}, \quad \boxed{td_X = e^{c_1(X)/2} \hat{A}_X}$$

$$\left[\prod \frac{c_1(L_j)/2}{\sinh(c_1(L_j)/2)} \right]$$

<u>Mukai</u>: realized $\mathcal{E}, \mathcal{F}$ coherent sheaves on Alg. Variety

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \ldots \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \ldots$$

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_i (-1)^i \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$$

$ch: K(X) \longrightarrow H^{even}(X, \mathbb{Z})$ is an isomorphism after $\otimes \mathbb{Q}$.

$ch(\,)$ on $K(X)$ define pairing by

$$\langle \mathcal{E} | \mathcal{F} \rangle = \chi(\mathcal{E}, \mathcal{F}) = \int_X ch(\mathcal{E}^\vee) \wedge ch(\mathcal{F}) \wedge td_X$$

$$\widetilde{ch}(E) = ch(E) \wedge \underbrace{\sqrt{td_X} \wedge e^{i\Lambda}}_{\Gamma_X}$$

$$V^\vee = \frac{\tau(V)}{\sqrt{ch(\omega_X)}}$$, where $\tau(V)$ multiplies by $(\sqrt{-1})^k$ on $H^k(X)$.

$$\langle v | w \rangle = \int_X v^\vee \wedge w$$

$$\frac{\pi w}{\sin(\pi w)} = \Gamma(1+w)\Gamma(1-w)$$

$\sqrt{td_X}$ or $\sqrt{\hat{A}_X}$ $\rightsquigarrow$ use $\Gamma(1+z)$ as a mult. char. class. Γ_X

$$\left[\Gamma\left(1 + \frac{i}{2\pi} z\right) \right]$$

$$\Lambda(z) = \operatorname{Im} \log \Gamma\left(1 + \frac{z}{2\pi i}\right)$$

$$= \frac{\gamma}{2\pi} z + \sum (-1)^k \frac{\zeta(2k+1)}{2k+1} \left(\frac{z}{2\pi}\right)^{2k+1}$$

$$\hat{\Lambda}(z) = \Lambda(z) - \frac{\gamma}{2\pi} z$$

Renormalization of $\overset{N=(2,2)}{\wedge}$ SSY NLSM in 1+1 dim

$$\Sigma \longrightarrow X$$

Action depends on a Kähler metric on X

Renormalization measures response to scale change.

"β-function" of theory, depends on coupling constant α'

β vanishes to zeroth order in α'

$$\frac{1}{\alpha'}\beta_{ij} = Ric_{ij} + \alpha'(\cdots)$$

$$= Ric_{ij} + \phi\alpha' + \phi\alpha'^2 + \frac{\zeta(3)}{48} T_{ij}\alpha'^3 + O\alpha'^4 + \cdots$$

Conjectured Answer:

$$e^{i\lambda(z)} = *\zeta(3)c_3 + *(\zeta(5)c_5 - \zeta(3)c_3c_2) + \cdots$$