K3 Surfaces · Lecture II (cf "Some remarks on moduli of K3 surfaces" - K81) ①

Goal: Constructing manifolds with holonomy G_2

Preview:

$X = K3$ surface $\qquad X \subseteq \underset{\substack{\uparrow \\ \text{Algebraic 3-fold}}}{Z} \quad$ s.t. $\quad \omega_Z^{-1} = \mathcal{O}_Z(X)$

If ω_Z^{-1} is ample, Z is called Fano; we need "weak Fano"

Let $\tilde{Z}$ = blowup of base locus of $|X|$

$$\begin{array}{c} \tilde{Z} \\ \downarrow \pi \\ \mathbb{P}^1 \end{array} \qquad \pi^{-1}(\infty) = X$$

$(\tilde{Z} \setminus X)$ is a (non-projective) variety w/ trivial canonical bundle.

$\Rightarrow$ (a Thm of Yau) $\exists$ a Ricci flat metric in $\tilde{Z} \setminus X$, asymptotically conical

Twisted Connect Sum (Kovalev)

$$(\tilde{Z} \setminus X) \times S^1 \bigcup_{X \times T^2} (\tilde{W} \setminus X) \times S^1$$

$S^1 \subset T^2 \supset S^1$

different S^1's

same metric, diff complex structures

X a K3 surface

Kähler, $\dim_{\mathbb{C}} X = 2$, $\exists$ nowhere vanishing holomorphic 2-form Ω

no global, nonzero holomorphic 1-forms

$\Rightarrow \pi_1(X)$ is trivial; all such X are diffeomorphic

Metric: For each Kähler class $[\omega] \in H^2_{DR}(X, \mathbb{R})$,

$\exists!$ Ricci-metric in that class.

$\mathrm{Span}_{\mathbb{R}}\{Re(\Omega), Im(\Omega), \omega\} \subseteq H^2_{DR}(X, \mathbb{R})$ is a max'l pos. def subspace.

Any $\varphi \in SO(3)$ can be applied to the subspace without changing the metric.

Alg. Geo Approach

Basis $\gamma_1, \ldots, \gamma_{22}$ of $H_2(X, \mathbb{Z})$

$$\cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8(-1) \oplus E_8(-1)$$

Then we can study ~~the~~ period map:

$$\left[\int_{\gamma_1} \Omega, \ldots, \int_{\gamma_{22}} \Omega \right] \in \mathbb{P}^{21}_{\mathbb{C}} \left(= \mathbb{P} H^2(X, \mathbb{C}) \right)$$

Image satisfies:

$\langle v, v \rangle = 0 \qquad \langle v, \bar{v} \rangle > 0$

$$H^2(X, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

$H^{2,0} = \mathbb{C} \cdot \Omega \quad ; \quad H^{0,2} = \mathbb{C} \cdot \bar{\Omega} \qquad H^{1,1}_{\mathbb{C}} = (H^{2,0} \oplus H^{0,2})^{\perp}$

With Given 2 complex K3's surfaces X, Y (Kählerian), each with a chosen basis of $H_2(-, \mathbb{Z})$ & an isomorphism $\varphi: H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ that preserves Hodge structure & intersection pairing (After $\otimes \mathbb{R}$)

(after $\otimes \mathbb{C}$)

Global Torelli

(Background)

- Let $V^+(X)$ be component of $\{H \in H^2(X,\mathbb{R}) \mid \langle H, H \rangle > 0\}$

  Containing Kähler classes.

  (The image of $V^+(X)$ should be contained in $V^+(Y)$)

- $\Delta^+(X) = \{ \delta \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R}) \mid \delta$ is the class of an effective curve in $X \}$

  $C \subseteq X$, effective irred $\Rightarrow C^2 = 2g-2$; $g \equiv g(C)$

$\therefore$ Given $\delta \in H^2(X,\mathbb{Z}) \in H^{1,1}(X,\mathbb{R})$, δ is represented by an algebraic class;

$\pm\delta$ is the class of an effective divisor (Riemann-Roch).

$\Rightarrow \Delta(X) = \Delta^+(X) \cap \{\delta \mid \delta^2 = -2\}$

Fact

If $\delta_1, \ldots, \delta_n, \ldots$ are the irr classes of curves

then $\forall \delta \in \Delta(x)$, $\delta = \sum n_i \delta_i$

$n_i \in \mathbb{Z}_{\geq 0}$

$\varphi : H^2(X, \mathbb{Z}) \longrightarrow H^2(Y, \mathbb{Z})$ must satisfy:

$$\varphi(V^+(X)) = V^+(Y)$$

$$\varphi(\Delta^+(X)) = \Delta^+(Y)$$

$\Rightarrow \exists\ \Phi : X \to Y$ iso s.t. $\Phi^* = \varphi$.

$\boxed{\underline{\text{Global Torelli à la Burns-Rapaport}}}$ holds.

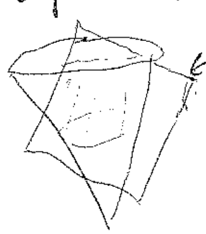
$$\mathcal{D} \subseteq Q \subseteq \mathbb{P}^{21}$$

$$\underline{\mathcal{D} / O^+(\Lambda^{3,19})}$$

$K(X)$ = Kähler cone

$= \{ H \in V^+(X) \mid \langle H, \delta \rangle > 0\ \forall \delta \in \Delta(X) \}$

~~There are no points in~~



$\langle H, \delta \rangle = 0$

$$\{ V^+ \} \neq \{ (X, K) \}$$

$\big\downarrow$ b/c $H \notin$ wall $\delta^\perp$

$\mathcal{D}$

We want to stay away from these walls.

Approaching walls

$$\lim_{i\to\infty} K_i = K \Rightarrow \langle K, \delta\rangle = 0 \Rightarrow \int_C \omega = 0 \quad ;$$

To resolve this, we blow down:

$$\underset{C_1,\ldots,C_k}{X} \longrightarrow \underset{p_1,\ldots,p_k}{\bar{X}} \leftarrow \text{controlled singularities}$$

Kobayashi-Todorov:

There exists a Ricci-flat orbifold metric on $\bar{X}$

$\Delta(X)$ determines a reflection group; $W(X)$ $\left(\subseteq O^+(\Lambda^{3,19})\right)$ acts transitively on the regions of $K(X) \setminus \bigcup_{\delta\in\Delta(X)} \delta^\perp$

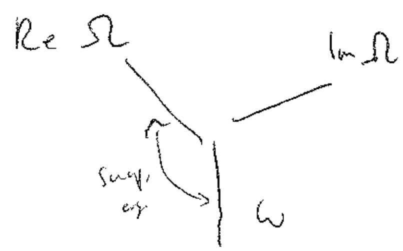
$$X \subset Z$$
$$\mathcal{L}|_X \quad \mathcal{L}$$
$$\uparrow$$
$$H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R}) \in K(X)$$

Computing periods

$$\left.\left(\frac{d}{dt}\int_\gamma \Omega(t)\right)\right|_{t=0} \in H^{2,0}\oplus H^{1,1}$$

$$\left.\frac{d^2}{dt^2}\left(\int_\gamma \Omega(t)\right)\right|_{t=0} \in H^{2,0}\oplus H^{1,1}$$

Other tool: Changing $\mathbb{C}$-structure



$$\{U^+(X)\} \big/ O^+(\Lambda^{3,19}) \hookleftarrow \{(X,K)\}$$

$$\{U^+(X)\} \to \mathcal{D}$$

$\downarrow$

3-dim'l subspace

$H^{1,1}(X,\mathbb{R}) = \mathbb{R}[\omega] \oplus [\omega]^{\perp}$

$H^{2,0} \oplus H^{0,2} \cap H^2(X,\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}\,\Omega, \ \mathrm{Im}\,\Omega\}$

$\therefore$ 3 dim'l space is $(H^{2,0} \oplus H^{0,2} \oplus \mathbb{R}[\omega]) \cap H^2(X,\mathbb{R})$

$$\begin{tikzcd} \{U^+(X)\} \arrow[d] \arrow[r] & Gr^+(3, \mathbb{R}^{3,19}) \times \mathbb{R} \\ \emptyset & \end{tikzcd}$$

vol wrt ω

is an isomorphism.

$O^+(\Lambda^{3,19})$ acts naturally on $\circlearrowleft$

$$\bigcup_{\delta \in \Delta(X)} \delta^{\perp} \longrightarrow \bigcup_{\delta \mid \delta^2 = -2} \delta^{\perp}$$

contains the fixed points of elements of $O^+(\Lambda^{3,19})$

Ricci flat metric on a K3 $\rightsquigarrow$ $\{$harmonic, self-dual 2-forms$\}$ $\in Gr^+(3,\mathbb{R}^{3,19})$

$\cap$

$H^2(X,\mathbb{R})$

Ricci flat orbifold metric on X:

$$\longrightarrow \varepsilon \bigcup \delta^{\perp}$$

If $\exists\, \delta$ s.t. $\langle K, \delta \rangle = 0$, then:
$\langle \delta, \delta \rangle = -2$

$$\{ \delta \mid \langle \delta, \delta \rangle = -2,\ \langle K, \delta \rangle = 0 \}$$ is finite

Connected components of such a collection of δ's

satisfy $\langle \delta_j, \delta_j \rangle = -2$; $\langle \delta_j, \delta_k \rangle \geq 0$, $j \neq k$

This problem is equivalent to classifying the

Simply-laced Dynkin diagrams.

This leads to:

A_n:
D_n:
E_6:
E_7:
E_8: