Twisted connect sum constructions of G2 manifolds

Sources

Corti, Hardström, et al.

1206.2277 (alg geom)

1207.4470 (G_2 manifolds)

Beauville

math.AG/0211313

Algebraic Geometry prelim

S - K3 $\qquad C \subseteq S$ alg. curve, nonsingular

$C^2 = 2g-2$ so if $g \geq 2$, $C^2 > 0$

$\varphi_{|C|} : \bar{S} \longrightarrow \underset{\underset{\mathbb{P}^g}{\cap}}{\bar{S}}$, K3 with rational double points at worst

We have $H \cap \varphi(S) \equiv C' \equiv C$ and $K_{C'} = (K_S + e') |_{C'} = N_{C'/S}$
$= \pi^* \mathcal{O}_{\mathbb{P}^g}(1)$

$|C| \big|_{C'} = |K_{C'}| \implies \varphi_{|K_{C'}|} : C' \hookrightarrow H \cong \mathbb{P}^{g-1}$

Question: Is every algebraic curve the hyperplane section of some K3 surface?

$$\dim \mathcal{M}_{K3}^{\langle 2g-2 \rangle} = 19 \Rightarrow \cancel{\dim}(\mathbb{P}^g)^* = g$$

$$\Rightarrow \dim\{C \mid C \subset S\} \leq 19 + g$$

$$\dim \mathcal{M}_g = 3g-3 \quad \& \quad 19+g \geq 3g-3 \iff g \leq 11$$

Therefore no hope that every curve of genus $11 <$ has no chance.

Q: Is

$$\{(S,C)\}/_{iso} \longrightarrow \{C\}/_{iso}$$

surjective in other cases?

If Y is a Fano 3-fold, i.e., $-K_Y$ is ample and S is a smooth divisor in $|-K_Y|$, then

$$K_S = (K_Y + S)|_S = (K_Y + -K_Y)|_S = 0$$

(In fact, S is K3)

$$Y \supset S \supset C$$

$C = Y \cap H \cap H' \Rightarrow S' = Y \cap \{c_1 H_1 + c_2 H_2\}$ is a K3 cont. C

$\Rightarrow$ we have a 1 parameter family in the fiber of C

$$\mathcal{M}_{(Y,S)} \longrightarrow \mathcal{M}^{N}_{K3} = \{S \sim K3 \text{ w/ } N \subseteq \mathrm{Pic}(S)\}/\sim$$

$$\mathrm{Pic}(Y) \xrightarrow{\alpha} \mathrm{Pic}(S) \subseteq \Lambda^{3,19} \qquad N = \mathrm{im}(\alpha) \qquad \text{eg } N = \langle 2, -2\rangle$$

Beauville's theorem $\Rightarrow$ this map is generically surjective for any Fano 3-fold

Generally, $\dim \mathcal{M}^{N}_{K3} = 20 - \mathrm{rk}(N)$

CHNP: generalize to Y being semi-Fano.

(End AG prelim)

$\boxed{\text{G2 manifolds}}$

Given 11-D spacetime ~~...~~ that to write as

((compact 7-Riemann manifold) $\times$ Minkowski 4-space

g_{ij}

Spinor
ζ
covariantly constant wrt
g_{ij}

Spinors

If we can then the holonomy is contained in G_2.

We can also do this with a 10-dim'l Spacetime; the analogous result says that the holonomy is contained in SU([illegible]). which eventually leads to the Calabi-Yau condition.

Holonomy:

Take loops in space

parallel transport along loop & get a map $T_p \to T_p$ preserving inner product $\Rightarrow \in SO(T_p)$

$SO(T_p)$ has a double cover called spin group, which has

Spinor representation

$$\begin{tikzcd} SO(T_p) & Spin(T_p) \arrow[l] \arrow[d] \\ & \text{Spinor rep} \end{tikzcd}$$

[illegible]

Question: Given $SO(7) \longleftarrow \mathrm{Spin}(7) \circlearrowleft$, what is the stabilizer of a given spinor? It turns out that the answer is G_2, regardless of the given spinor.

$\dim G_2 = 14$

$V \cong T_p$

G_2 acts on V; irred 7-dim'l rep ($\underline{7}$)

$G_2 \curvearrowright \Lambda^2 V = \underline{7} \oplus \mathrm{adj}(G_2)$

$G_2 \curvearrowright (S^2 V)_0 =$ irred. 27-dim'l rep : $\underline{27}$

$G_2 \curvearrowright (\Lambda^3 V) = \underline{27} \oplus \underline{7} \oplus \underline{1}$

$\Rightarrow G_2$ stabilizes a 3-form on 7 manifold

In fact, $G_2 \cong \mathrm{Stab}$(This 3-form in $\Lambda^3 \mathbb{R}^7$)

$$\{3\text{-forms w/ Stab} \cong G_2\} \underset{\text{open}}{\subset} \Lambda^3 \mathbb{R}^7$$

$$\underbrace{}_{\text{"Positive 3-forms"}}$$

$SO(p,q)_{\mathbb{C}} \cong SO(p+q, \mathbb{C})$

Examples

1) $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ $\quad$ hol $= \{1\} \subseteq G_2$

2) $T^3 \times S$, $S = K3$ $\quad$ hol $= SU(2) \subseteq G_2$

$$\Phi = d\theta_1 \wedge d\theta_2 \wedge d\theta_3 + \sum_{j=1}^{3} d\theta_j \wedge d\omega_j$$

where $\omega_1, \omega_2, \omega_3$ are orthonormal basis of positive harmonic forms

3) $S^1 \times Z$, where Z is a CY 3-fold

$$\Phi = d\theta \wedge \omega + Re(\Omega)$$, where ω is Kahler form of Z

Ω = holo 3-form.

holonomy $= SU(3) \leq G_2$

1′) T^7/G (where G is a finite group

whose fixed points in real codimension 4

$\mathrm{Stab}(pt) \subseteq SU(2) \subseteq SO(4)$)

<u>Joyce orbifolds</u> (G fixes a pos 3-form)

$T^7/G \leftarrow W$ (like a res. of singularities)

This gave the first class of G_2 manifolds

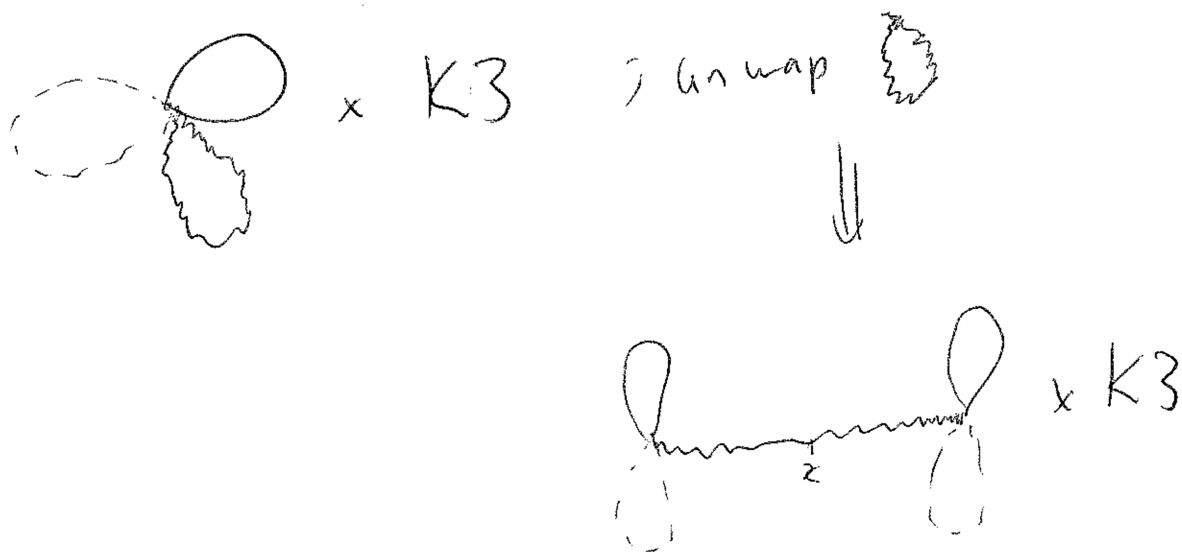
Holonomy $= G_2$!

2′) $(S^1 \times Z)/\mathbb{Z}_2$: nonplex $\mathbb{Z}_2$ acts by complex on Z and $\theta \to -\theta$ on S^1 (if Z is defined over $\mathbb{R}$)

<u>This is smooth iff $Z(\mathbb{R}) = \emptyset$; if singular, can resolve</u> (barely G_2; holonomy is $SU(3) \rtimes \mathbb{Z}_2 \subseteq G_2$)

<u>$\mathbb{Z}_2$ locus.</u>

2) G_2 cylinders



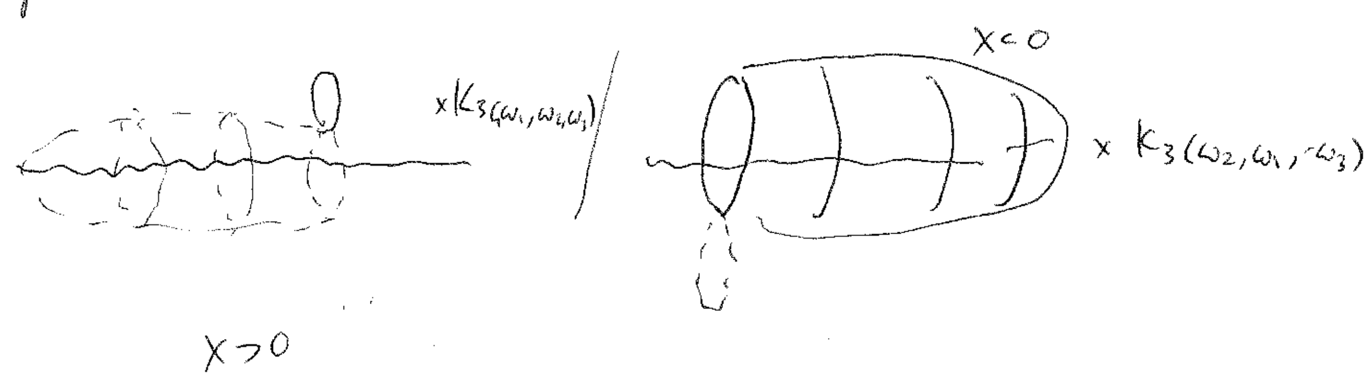
$$\Phi = d\theta_1 \wedge d\theta_2 \wedge dx + d\theta_1 \wedge \omega_1 + d\theta_2 \wedge \omega_2 + dx \wedge \omega_3$$

Donaldson matching:

$x \longmapsto -x \quad \theta_1 \longleftrightarrow \theta_2 \,, \quad \{\omega_1, \omega_2, \omega_3\} \longrightarrow \{\omega_2, \omega_1, -\omega_3\}$

Preserves Φ

Asymptotically cylindrical G_2-manifold



Construction:

$(Y_+, S_+) \times S^1 \longrightarrow Cyl_+ \times S^1$ (9)

$(Y_-, S_-) \times S^1 \longrightarrow Cyl_- \times S^1$

$Bl_+ Y = Z_+ \qquad Z_+ \setminus S_+$ = noncompact CY 3-fold

Variant of Yau's theorem $\Rightarrow$ there's a Ricci flat metric, asymptotically canonical

• A weak Fano 3-fold is non-singular Y such that

$-K_Y \cdot C \geq 0 \;\; \forall C, \qquad (-K_Y)^3 > 0$ ($\Leftarrow$ "big")
"nef"

Choose S, S' generating an anticanonical pencil;

$Z = Bl_{S \cap S'} Y$, Z fibered over $\mathbb{P}^1$, fibers are K3 surfaces.

$\omega_1(S^+) = $ Kähler

$\omega_1 + i\omega_3 = $ holo 2-form

$\omega_2(S^-) = $ Kähler

$\omega_2 + i\omega_3 = $ holo 2-form

$\mathrm{Span}(\omega_1, \overbrace{\omega_2, \omega_3}) \oplus (H^{2,0} + H^{0,2})_{\mathbb{R}}$

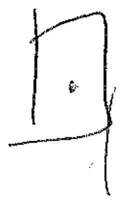
(Semi-Fano
$\Updownarrow$
anticanon. eff. + Y contains no div. e

$\mathcal{M}_{K3}^{N}$ $\quad N \subseteq H^{1,1}(S)$ Kähler class

$\mathcal{M}_{K3}^{N_+}$ $\quad N_+ \subseteq H^{1,1}(S_+)$ $\quad (1, \mathrm{rk}\, N_+ - 1)$

$\mathcal{M}_{K3}^{N_-}$ $\quad N_- \subseteq H^{1,1}(S_+)$ $\quad (1, \mathrm{rk}\, N_- - 1)$

$N_+^{\perp}$ $-$ $SO(2, 20 - \mathrm{rk}\, N_+) / SO(2) \times SO(20 - \mathrm{rk}\, N_+)$



(Assuming we have $N_+ \oplus N_- \subseteq \Lambda^{3,19}$)

Ample class in N_-

Example

Y_+: Start with a quartic 3-fold in $\mathbb{P}^4$

which contains a $\mathbb{P}^2$ $\Rightarrow \exists F_1, F_2$

$\Rightarrow Q_4 = x_1 F_1 + x_2 F_2 \Rightarrow \deg F_1 = \deg F_2 = 3$

Singular at $F_1 = F_2 = Q_4$. (36 pts)

Small blowup gives Y_+ of Picard # 2,

lattice $N_+ = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}$

For Y_-, Pic # 1 e.g. int. cubic & quadric in $\mathbb{P}^5$.

$\Rightarrow N_- = (6)$

$\Rightarrow N_+ \oplus N_- \hookrightarrow \Lambda_{K3} = \Lambda^{3,19}$