20 January 2017
Z. Wang

3D Quantum Gravity

1. Correspondence of 3D $\leftrightarrow$ Chern-Simons gauge theory
   Witten '88
2. BTZ hole '93
3. AdS_3/CFT_2 Brown-Henneaux '86
   Chao-Ming Jian
4. 2/24 A. Kitaev SYK model
5. 2/9 M. Hastings Quant. code

Einstein-Hilbert action

X^3 space-time, g = gravitational field (metric)

$$I(g) = \int_{X^3} d^3x \sqrt{g}\,(R - 2\Lambda)$$

R : scalar curvature, Λ : cosmological constant

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

3D anti-de Sitter space

$$\left\{ -\sum_{i=1}^{2} x_i^2 + \sum_{j=3}^{4} x_j^2 = -\ell^2 \right\} \subset \mathbb{R}^4$$

$$\ell > 0$$

$$ds^2 = -\sum_{i=1}^{2} dx_i^2 + \sum_{j=3}^{4} dx_j^2$$

$$R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu} \; ; \quad R = -\frac{6}{\ell^2} \qquad \Lambda = -\frac{1}{\ell^2} < 0$$

solves Einstein's equation

Topologically, anti-de Sitter is $S^1 \times \mathbb{R}^2$.

Choose $(x_3, x_4) \in \mathbb{R}^2$

$$\sum_{i=1}^{2} x_i^2 = \ell^2 + \sum_{3}^{4} x_j^2$$

= Solid torus minus the boundary

Also true geometrically: Use $\mathbb{H}^2$ not $\mathbb{R}^2$.

$$I(X^3, g) = \int \cdots$$

Let X^3 be a closed oriented 3-manifold.

Lorentzian

ω = spin connection

Then $TX^3 \cong X^3 \times \mathbb{R}^3$

$$e : TX^3 \to X^3 \times \mathbb{R}^3 \quad \text{framing}$$

~~vier~~bein

drei

$$(\omega, e) \to \begin{pmatrix} \omega & e \\ -e & 0 \end{pmatrix} = SO(2,2) \text{ gauge field}$$

$$A_\pm = \omega \pm \frac{e}{\ell}, \qquad \Lambda = -\frac{1}{\ell^2}, \quad \ell > 0$$

$$= \frac{k_L}{4\pi} \int_{X^3} \underbrace{\mathrm{tr}\left(A_+ dA_+ + \frac{2}{3} A_+^3\right)}_{CS(A_+)}$$

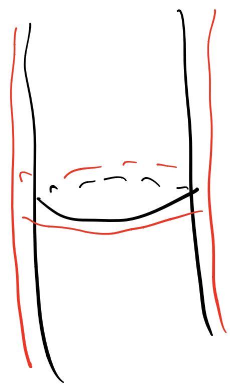
$$- \frac{k_R}{4\pi} \int_{X^3} \underbrace{\mathrm{tr}\left(A_- dA_- + \frac{2}{3} A_-^3\right)}_{CS(A_-)}$$

Classical 3D gravity with $\Lambda < 0 \leftrightarrow$ Doubled CS theory

$k_L = k_R = \frac{3G}{2\ell}$ "level" of CS theory

Quantum 3D pure gravity with $\Lambda < 0$

① $$Z(X^3) = \int_g \mathcal{D}g \, e^{iI(g)}$$

② 

$S^1 \times \mathbb{R}^2$

conformal boundary

AdS_3/CFT_2. Find out CFT_2 ?

Consequences

① Volume conjecture (suggested)

② AdS/CFT

Classical physics

Quantum mechanics

Quantum field theory

Quantum gravity

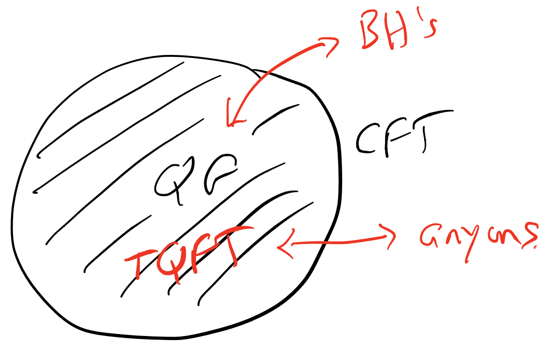
computing

Turing machine

Quantum computer

?

③ Irrational TQFT



Volume conjecture

Two cases

1. Kashaev conjecture for hyperbolic knots
2. closed hyperbolic 3-manifold

Perelman-Thurston

X^3 closed 3-manifold

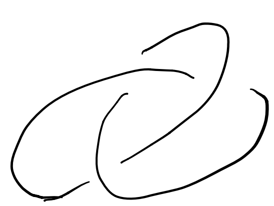
$|\pi_1| = \infty$, irreducible, atoroidal $\Rightarrow$ hyperbolic

$X \neq X_1 \#_{S^2} X_2$ with $X_i \neq S^3$

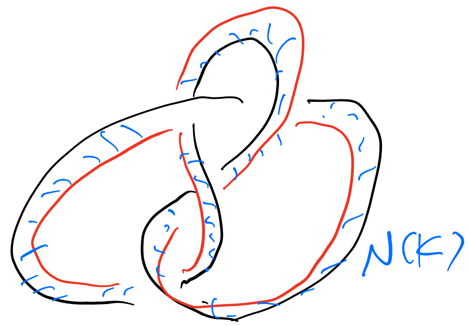
no embedded incompressible $T^2 \subset X$

i.e. $\nexists$ $\pi_1 T^2 \to \pi_1 X$ injective

Construct them by Dehn surgery on knots.



not hyperbolic (trefoil)



hyperbolic (figure 8)

$$\partial(S^3 \setminus \mathring{N}(K)) \cong T^2 \qquad \varphi: T^2 \to T^2$$

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2\mathbb{Z}$$

$$(S^3 \setminus \mathring{N}(K)) \cup_\varphi (S^1 \times D^2).$$

"p/q surgery"

$$Z(X^3) = \int_g \mathcal{D}g \, e^{\frac{-i}{\hbar} J(g)} \underset{\hbar \to 0}{\sim} e^{V(X^3)}$$

Wick rotation : Lorentzian → Riemannian

$SO(2,2) \longrightarrow SO(3,1) \cong SL(2,\mathbb{C})$

"unitary ~~irrational~~ TQFTs"

$SU(2)_k$ unitary TQFTs $\qquad k = r-2$

$$|Z(X^3)| \leq D^{g-1} \qquad D \sim r^{3/2}$$

↑ Heegaard genus g

$$\frac{\log|Z|}{r} \to V$$ (So cannot drop "structural")

Can you drop "unitary"?

Non-unitary TQFTs $\rightsquigarrow$ volume conjecture might hold.

T. Yang, QT Chen

$$SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

$$2\pi \frac{\log|Z(X^3)|}{r} \xrightarrow[r\to\infty]{} \mathrm{vol}(X^3)$$