1. Hamiltonian formalism for "particles"

(1) canonical variables q_i, p_i

(2) Hamiltonian $\mathcal{H}(q_i, p_i)$

- Poisson bracket $\{\cdot,\cdot\}$

e.g. $f(q,p)$, $g(q,p)$

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - f \leftrightarrow g$$

$$\{q_i, p_j\} = \delta_{ij}$$

Hamilton equation

$$\begin{cases} \dot{q}_i = \{q_i, H\} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \{p_i, H\} = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Lagrangian:

$$L(q_i, \dot{q}_i) = \sum_i p_i \dot{q}_i - \mathcal{H}$$

Action:

$$I = \int dt\, L(q_i, \dot{q}_i)$$

Hamilton eq $\iff \delta I = 0$ $\qquad I[q(t)]$

- Conserved quantities $Q(p,q)$

$$0 = \dot{Q}(p,q) = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p}$$

$$= \{Q, \mathcal{H}\}$$

$\mathcal{H}$ generates time translation

$$\| \quad -\{\mathcal{H}, Q\}$$

$$\delta \mathcal{H} = \{\mathcal{H}, Q\}$$

$$\delta I = \int dt \left(\delta(p\dot{q}) - \cancel{\delta \mathcal{H}}\right) \to 0$$

$$= \int dt \, \underbrace{(\delta p \cdot \dot{q} - \dot{p}\,\delta q)}_{= \frac{dQ}{dt}} + \int dt \frac{d}{dt}(p\,\delta q) = Q - p\,\delta q$$

e.g. $\mathcal{H} = \frac{p^2}{2m}$ $\quad$ momentum is conserved

$$Q(p,q) = p.$$

$$\delta q = \{q, \varepsilon Q\} = \varepsilon.$$

2. Hamiltonian formalism of "fields"

(1) canonical variables $\phi(x), \pi(x)$

(2) Hamiltonian $H = \int d^d x\, \mathcal{H}(\phi(x), \pi(x))$

(3) Poisson bracket $f[\phi, \pi]$, $g[\phi, \pi]$.

$$\{f, g\} = \int d^d x \left[\frac{\delta f}{\delta \phi(x)} \frac{\delta g}{\delta \pi(x)} - f \leftrightarrow g \right]$$

(4) Global symmetry $Q[\phi, \pi]$

$$\frac{dQ}{dt} = 0 \iff \begin{array}{l} \delta\phi = \{\phi, Q\} \\ \delta\pi = \{\pi, Q\} \end{array} \quad \text{leaving } H \text{ invariant}$$

(5) Gauge symmetry

$$\Phi(\phi, \pi) = 0 \Big|_{\text{on shell}} \implies \begin{array}{l} \delta\phi = \{\phi, \Phi\} \\ \delta\pi = \{\pi, \Phi\} \end{array}$$

---

3) Hamiltonian formulation for GR

$g = \det g_{\mu\nu}$
$= (\det g_{ij}) |N|^2$

$$I[g] = \int d^{d+1} x \sqrt{g} \cdot (R - \Lambda)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(N^2 + N^i N_i) dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$\int dx \sqrt{g}\,(R-\Lambda) =$$

$N =$ "lapse"

$$= \int dt \int N \cdot d^3x \sqrt{|g_{ij}|}\left[K_{ij}K^{ij} - K^2 + R(g_{ij})\right]$$

where indices are raised by g_{ij} and

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + N_{i/j} + N_{j/i}\right)$$ is the

extrinsic curvature. (Note: / denotes covariant derivative.)

Canonical variable: g_{ij} Π_{ij}

$$\Pi_{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \sqrt{g}\,(K_{ij} - g_{ij}K)$$

$$H = \int \pi^{ij}\dot{g}_{ij} - L$$

$$= \int d^dx\,(N\mathcal{H} + N^i\mathcal{H}_i)$$

where $\mathcal{H}(\Pi, g) = \frac{1}{\sqrt{g}}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right) - \sqrt{g}\,R$

$$\mathcal{H}_i(\pi, g) = -2\pi^{\delta}{}_{i/j}$$

$$I[N, N^i, g_{ij}, \Pi^{ij}] \quad ; \quad \delta I = 0$$

$$\begin{cases} \mathcal{H} = 0 & \text{on shell} \\ \mathcal{H}_i = 0 & \text{on shell} \end{cases}$$

(N, N^i are Lagrange multipliers)

$$\delta g = \{ g_{ij}, \xi^i \mathcal{H}_i \} = g_{ij,k}\xi^k + g_{ik}\xi^k_{,j} + g_{kj}\xi^k_{..}$$

$$\tilde{x} = x^i + \xi^i$$

"Lie Brackets"

$$\left.\begin{array}{c} \{\mathcal{H}, \mathcal{H}_i\} \\ \{\mathcal{H}_i, \mathcal{H}_i\} \\ \{\mathcal{H}, \mathcal{H}\} \end{array}\right\} \to \text{closed algebra}$$

* $\tilde{x}^i = x^i + \xi^i$

Gauge symmetry transmuted to global symmetry

---

4 Finding the global symmetry change

$$\delta H_0 = \int d^d x \, [A_{ij}\, \delta\pi^{ij} + B_{ij}\, \delta g_{ij}]$$

$$+ \int_{\text{boundary}} dS_\ell \, [G^{ijk\ell}(N\, \delta g_{ij,k} - N_{,k}\, \delta g_{ij}) + 2N_i\, \delta\pi^{i\ell} + (2N^i \pi^{k\ell} - N^{\ell}\pi^{jk}) \cdot \delta g_{jk}]$$

$\delta B[N, N^i]$

Redefine $\mathcal{H} = \mathcal{H}_0 - B[N, N^i]$

5. AdS_3, asymptotic symmetry, & central extension

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2$$

has isometry given by $O(2,2)$ 6 Killing vectors

$$ds^2\Big|_{r\to\infty} = r^2(-dt^2 + d\phi^2) + \frac{1}{r^2}dr^2$$

$$\xi^t = T(\alpha,\phi) + \frac{1}{r^2}\bar{T}(\alpha,\phi) + O(1/r^4)$$

$$\xi^r = r R(t,\phi) + O(1/r$$

$$\xi^{(\varphi)} = \Phi(t,\phi) + \frac{1}{r^2}\bar{\Phi}(t,\phi) + O(1/r^4)$$

$$T_t = \Phi_\phi$$

$$\Phi_t = T_{,\varphi}.$$

$$\delta B(N,N^i) \to B = 0\Big|_{AdS_3}$$

$$\{B[\xi], B[\eta]\} = B[[\xi,\eta]] + K[\xi,\eta]$$

$$\underset{\displaystyle \delta_\eta B[\xi]}{\shortparallel}$$