K3-Surfaces

- What is a K3 surface?

(1) topologist. compact simply connected 4-manifold

$H^2(X,\mathbb{Z}) \cong \Lambda^{3,19}$ even unimodular lattice of signature (3,19)

one diffeomorphism class. "The" K3-manifold.

(2) alg.geo.: compact complex surface with no holomorphic 1-forms and nowhere vanishing holomorphic 2-form Ω

Kodaira thm: any the manifold is the K3 manifold and the $\mathbb{C}$-manifold is Kähler(ian) (by Siu-Todorov)

The Kähler metrics are always hyperkähler. holonomy is $SU(2)$

(3) diff.geo.:

$$\left\{ \begin{matrix} \text{hyperkähler metrics on } X \\ \text{of volume 1} \end{matrix} \right\} \Big/ \mathrm{Diff}(X)$$

Fact (Donaldson / Borcea / other guy)

$$\mathrm{Diff}(X)/\mathrm{Diff}_0(X) \underset{\substack{\text{index} \\ 2}}{\subseteq} O(\Lambda^{3,19}, \mathbb{Z})$$

picking orientation of positive 3-plane.

Period Map (alg.geo.) X complex K3 surface (with choice from $H^2(X,\mathbb{Z}) \cong \Lambda^{3,19}$)

$$X \longmapsto \left[\int_{\gamma_1} \Omega, \ldots, \int_{\gamma_{22}} \Omega \right] \in \mathbb{P}^{21}_{\mathbb{C}}$$

image lies in $\{ \alpha \mid (\alpha,\alpha)=0, (\alpha,\bar{\alpha})>0 \}$, $\cong O(3,19)/O(2)\times O(1,19)$ ← symmetric space

20-dim complex space.

$$\{\text{moduli of K3}\} \longrightarrow O(\Lambda^{3,19}, \mathbb{Z}) \backslash O(3,19) / O(2)\times O(1,19)$$

"roughly" an isomorphism (local iso. / global surj.)

$\boxed{\text{Warm-up}}$

T^2 2-torus — topologist ✓

complex geometer : elliptic curve ✓

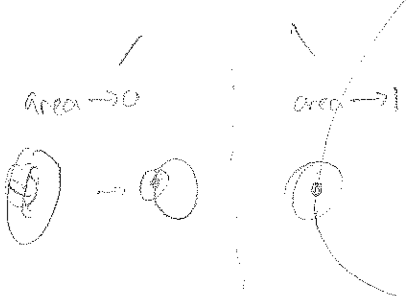
diff. geo. : flat metric ✓

$\mathcal{H} / SL(2,\mathbb{Z}) \cong$ j-line

topological property of this family : realizes Dehn twist

⇝ a smooth 4-manifold.

(param. space of curves with a simple node)



(Gromov-Hausdorff convergence)

Baily-Borel compactification

Borel-Serre compactification.

ALE space: of the form $\mathbb{C}^2/G$ $\quad G \subset_{\text{finite}} SU(2)$

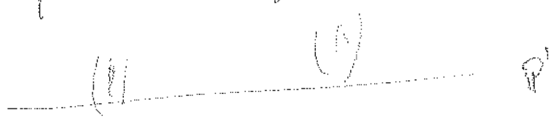
Torelli type theorem for these

ALF: no Torelli theorem yet.

ALG, ALH: understood by work of Chen-Chen.

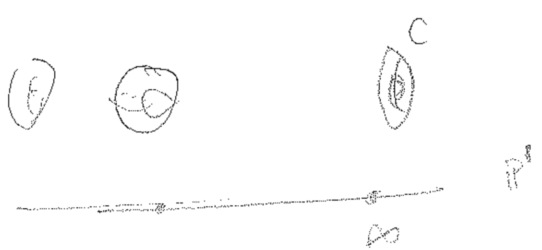
ALG, ALH: Start with rational elliptic surface

$$y^2 = x^3 + f(s)x + g(s)$$



$$\deg f(s) = 4k, \quad \deg g(s) = 6k$$

$k=2$: get K3 surface
$k=1$: get rational elliptic surface. S



$S - C$ = "noncompact K3 surface"

Tian-Yau $\Rightarrow$ $\exists$ complete kähler Ricci-flat metric on $S - C$

the metric is either ALG, ALH or doesn't collapse fast enough.

$\boxed{\text{M-Theory}}$ (10+1)-spacetime Physical theory

g_{ij} metric

C_{ijk} (3-form) E_8 -gauge field

$X \times \mathbb{R}^{3,6}$

$\uparrow$

$R_x(g_{ij})=0$

harmonic $C_{ijk}\,dx^i dx^j dx^k$

- On a noncompact manifold get E_8 gauge field on boundary