No. UCSB
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Dave Morrison - Degeneration of K3 Surfaces, Gravitational Instantons and M-Theory

K3 Surfaces

What is a K3 surface?

1) Topologist: Compact simply connected 4-manifold

$$H^2(X, \mathbb{Z}) \cong \Lambda^{3,19}$$

Even unimodular lattice of signature (3, 19)
One diffeomorphism class of these, "K3 Manifold"

2) Algebraic Geometer: Compact complex surfaces with no holomorphic 1-form and nowhere vanishing holomorphic 2-form.
$\Rightarrow$ manifold is the K3 manifold (Kodaira)
$\Rightarrow$ Complex manifold is Kahlerian (Siu)
$\Rightarrow$ Kahler metric is hyperKahler $\Leftrightarrow$ holonomy $SU(2)$

3) Differential Geometer: $\{$hyperKahler metric on X of Vol 1$\} / \mathrm{Diff}(X)$

Fact (Donaldson, Burcea,):

$$\mathrm{Diff}(X) / \mathrm{Diff}_0(X) \underset{\text{index } 2}{\subseteq} O(\Lambda^{3,19}; \mathbb{Z})$$

pick an orientation of positive subspace.

Period Map (Alg. geo)

$$X,\ H^2(X, \mathbb{Z}) = \text{complex K3 surface} \longmapsto \left[\int_{\gamma_1} \Omega, \dots \int_{\gamma_{22}} \Omega\right] \in \mathbb{P}^{21}$$

$\downarrow \wr$
$\Lambda^{3,19}$

Image lies in $\{\alpha \mid (\alpha, \alpha) = 0, (\alpha, \bar{\alpha}) > 0\}$

$$\begin{Bmatrix}\text{moduli of} \\ K3\end{Bmatrix} \longrightarrow O^+(\Lambda^{3,19}, \mathbb{Z}) \backslash O(3,19) / O(2) \times O(1,19)$$

Roughly an "isomorphism"

Period map (diff geo)

$$\{\text{hK metric of vol } 1\} / \mathrm{Diff} \longrightarrow Gr^+(3,19) / O^+(\Lambda^{3,19}, \mathbb{Z})$$

<u>Theorem</u> (Anderson)

1) There is an L^2-metric on LHS which makes this an isometric embedding
2) If we expand LHS to include orbifold metric, we get a global isometry

Back to Alg Geo.

Theorem (Kulikov) Let $\overline{X} \xrightarrow{\pi} \Delta = \{z \mid |z| < 1\}$, $X_0 = \pi^{-1}(0)$

$\{\text{Moduli of } K3\}$

$$\begin{array}{ccc} X & \subseteq & \overline{X} \\ \downarrow & & \downarrow \\ M & \subseteq & \overline{M} \end{array}$$

1-parameter semi-stable family of K3 surfaces are of type

quartic in $\mathbb{P}^3$ I) $\pi^{-1}(0) = K3$

quadric ∪ quadric II) $\pi^{-1}(0) = RES \cup \ldots$ elliptic ruled $\cup$ RES (elliptic curves)

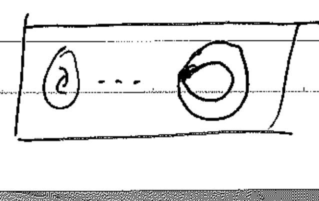
plane ∪ plane ∪ plane (III) $\pi^{-1}(0) = \bigcup S_j$ each S_j, each $S_j \cap S_k \cong$ topology is S^2

<u>Warm up</u> T^2 topologist – Two torus

Complex geometry – elliptic curve ✓

Com. Alg. Geo. version diff. geo – flat metric ✓

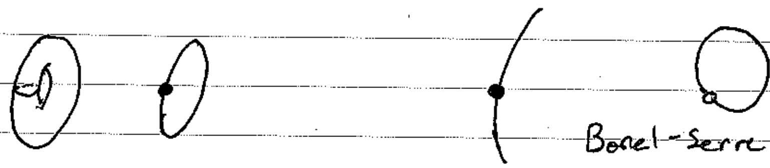
$\mathfrak{H} / SL(2, \mathbb{Z}) \subseteq j$-line



Baily-Borel-stable

Flat metric

area $\to 0$ area $\to 1$



Borel-Serre

Diff. geo versus

$$Gr^+(3,19)\big/O^+(\Lambda^{3,19};\mathbb{Z})$$

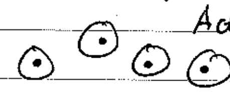
Symmetric space theory determines: $\boxed{\text{pick } r}$

$$Gr^+(3,19) \cong \mathbb{R}^+ \times Gr^+(3-r,19-r) \times SL(r)/SO(r) \times \Lambda^2\mathbb{R}^r \times \mathbb{R}^{3-r,19-r}\otimes\mathbb{R}^r$$

Parabolic subgroup of $O^+(3,19)$

Anderson

If $\{g_{ij}\}$ is a divergent (in L^2-norm) sequence of vol. 1 h.K metric then there is a subsequence which undergoes Cheeger-Gromov collapse in the complement of a finite set $\{p_i\}$

$X \dashrightarrow$  Acted upon by tori

B

$X = T^4/\mathbb{Z}_2$ "Kummer surface"

16 singular points: Asymptotically locally euclidean (ALE) spaces of type A_1

each contributes $3/2$ to $\chi(\tilde{X})$

$$16\cdot\tfrac{3}{2} = 24 = 1+22+1$$

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$(X \setminus \text{sing pts})$ There are "gravitational instanton" which fill in space

$\big\downarrow T^k$

$T^{4-k}/\mathbb{Z}_2$

$k=1$ ALF (Free)

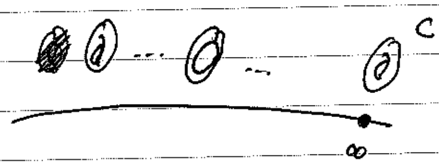
$k=2$ ALG

$k=3$ ALH

$\mathbb{C}^2/G$ $\quad G \underset{\text{finite}}{\subseteq} SU(2)$ Torelli-type theorem

ALG, ALH: Start with a Rational Elliptic Surface

Understand via Weierstrauss model on $\mathbb{P}^1$



$y^2 = x^3 + f(s)x + g(s)$

$\deg f = 4k$

$\deg g = 6k$

$k=2$ K3 surface

$k=1$ Rational elliptic surface

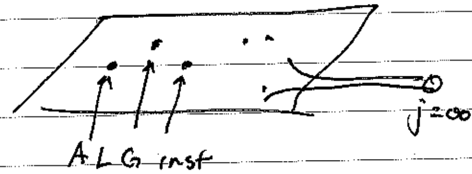
$S-C$ is a non-compact K3

Tian-Yau $\Rightarrow$ $\exists$ complete Ricci flat Kahler metric on $S-C$

Metric is ALG, ALH, ω doesn't collapse fibers

- If C is non-singular ALH
- If C is singular, $\lim_{s\to\infty} j(s) < \infty$ then ALG
- If C is singular, $\lim_{s\to\infty} g(s) = \infty$ then not ALE, ALF, ALG, ALH

<u>ALG</u>: Limit is a complex surface base of a T^2-fibration



(~~Gross~~ Gross-Wilson)

$Gr^+(2,18)$

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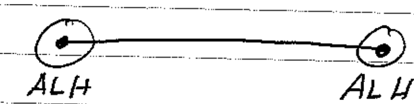
ALH :



$Gr^+(0,16)$

ALF: Beliet $Gr^+(1,17)$

M-Theory

Metric + 3-form $(g_{ij}, C_{ijk}$ (3-form)$)$

$\cancel{}$ E_8-gauge field

$X \times \mathbb{R}^{1,6}$

$\uparrow$

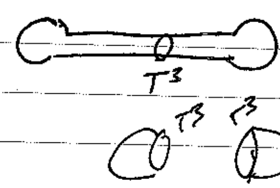
On a non-compact manifold get E_8 gauge field in boundary

$Ric(g_{ij}) = 0$

$(C_{ijk}\, dx^i dx^j dx^k)$ ~~harmonic~~

↳ harmonic

$g_{ij} \xrightarrow{\text{goes to}}$ boundary via ALH $(K=3)$



delta at $\mathbb{R}^8 \otimes \mathbb{R}^3$