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The Symmetries of Type IIB

String Theory

We will talk about:

Type IIB string theory

§0. Gauge theories

Given a physical theory with symmetry, where the symmetry can vary from point to point

Described by a Gauge group G; Maps (X, G)
a space time

G should not be directly observable, but we look for gauge-invariant quantities

$$\begin{array}{ll} B & G\text{-bundle} \\ \big\downarrow & A = \text{connection on bundle} \\ X & F = \text{curvature of } A \text{; this is a gauge-invariant.} \end{array}$$

§. 1. Spinors & Spin^c-structures

Again, X is a spacetime (pseudo-Riemannian manifold, tangent space has a nondeg. inner product)

Not nec. pos. def.

We can form the frame bundle:

$$\begin{array}{l} \mathcal{F} \\ \downarrow \\ X \end{array}$$

$\mathcal{F}$ = Choices of orthonormal frame in each tangent space

The orthogonal group acts transitively on the frame bundle, so we have a principle SO-bundle.

Example

$X = \mathbb{R}^{p,q}$ (flat) $\left((p,q) = (n,0) \text{ or } (p,q) = (n-1,1) \right)$

There's $SO(p,q)$-'s worth of symmetries, but also translations:

$$SO(p,q) \ltimes \mathbb{R}^{p,q} = \begin{cases} \text{Poincaré} & \text{if } (n-1,1) \\ \text{Euclidean} & \text{if } (n,0) \end{cases}$$

Representation of Poincaré $\longleftrightarrow$ particles/fields

These are typically labelled by reps of subgroups of $SO(p,q)$

We can use this framework to formulate theories w/ photons, bosonic fields, but have some trouble w/ fermionic fields (e.g. electrons)

Fun fact from Lie theory

$$\pi_1(SO(p,q)) = \mathbb{Z}/2 \quad \text{if } p+q > 2$$

$\Rightarrow$ There's a double cover

$$\begin{array}{c} Spin(p,q) \\ \downarrow \\ SO(p,q) \end{array}$$

$$SU(2) \longrightarrow \mathbb{C}^2,\ Sym^k\,\mathbb{C}^2 \qquad (Sym^k\,\mathbb{C}^2 \text{ has dim } k+1)$$

$$\begin{array}{c} SU(2) \\ \downarrow \\ SO(3) \\ \downarrow \\ \text{"vector rep"} \end{array}$$

$q=1$ - $SO(3)$

"vector rep"

3-dim'l

- $-Id$ acts non-trivially on $\mathbb{C}^2$, $Sym^3\,\mathbb{C}^2$, $Sym^5\,\mathbb{C}^2$
- Trivially on $Sym^2\,\mathbb{C}^2$, $Sym^4\,\mathbb{C}^2$, ...

This type of analysis leads to representations that could correspond to the fermionic ones.

Ex 2

$$\begin{array}{c} Spin(2) \\ \downarrow \\ SO(2) \end{array} \qquad \pi_1(SO(2)) = \mathbb{Z}$$

§ 2. Supersymmetric Theories

Odd symmetry : exchanges bosons & femiors

Need a lift of the frame bundle to a spin bundle.

Math description:

Clifford algebra of $SO(p,q)$

$SO(p,q) : V \circlearrowleft$

Take free tensor algebra, quotient.

$$Cl(Q) = \bigoplus_{n=0}^{\infty} \otimes^n V \Big/ x \otimes x - Q(x)\mathbb{1}$$

eg. If $Q(x) \equiv 0$, then $Cl(Q) = \Lambda^\bullet V$, since we're only quotienting by $x \otimes x$; the general case is a "deform" of the usual alternating algebra.

Clifford modules:

$$Cl(Q) \times M \longrightarrow M$$

$$Spin(p,q) \subseteq C\ell(Q)^\times$$

Complex clifford modules:

$$C\ell(Q) \times M_{\mathbb{C}} \longrightarrow M_{\mathbb{C}}$$

$$Spin(p,q) \times U(1)$$

$$(-1,-1)$$

Interested in case where this is a gauge symmetry

$$\Longrightarrow \quad Spin^c(p,q) := Spin(p,q) \times U(1) \big/ (-1,-1)$$

Having a gauge symm. usually means we have a bundle

$$\begin{array}{c} G \\ \downarrow \\ X \end{array}$$ being acted on.

If all fermions have $\frac{1}{2}\mathbb{Z}$ charge, bosons have $\mathbb{Z}$ charge

should have $Spin^c(p,q)$.

$$\int_C \frac{dA}{2\pi} = \frac{1}{2}\int_C w_2 \mod \mathbb{Z}$$

w_2: 2nd Stiefel-Whitney class

Lesson

Give a $U(1)$-gauge theory with this spin/charge, can put it

on $Spin^c$-backgrounds. $\boxed{\begin{array}{c} \mathcal{L} \\ \downarrow \\ X \end{array}}$ – $\mathbb{C}$-bundle;

$c_1(\mathcal{L})c \longrightarrow w_2$

Type IIB supergravity (10 dimensions)

- fields (bosonic & fermionic)

- 2d Supersymmetry transformation

2 real dimensional space of

- SO(2) rotates the supersymmetries

The formulation, due to Green & Schwarz, is made easier using the fact that ~~at~~ we have $SL(2,\mathbb{R})$ global symmetry.

- bosonic fields: $\tau \in \mathfrak{h}$ upper half plane

pair of 2-form fields

have field strength: pair of 3-forms H_1, H_2

4-form field w/ self-dual field strength

fluctuating metric

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Fermios	Spin
2 gravitions	3/2
2 fermions	1/2

Now recall $\mathcal{H} = SL_2(\mathbb{R}) / SO(2)$

(For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, action on τ, $\tau \mapsto \frac{a\tau + b}{c\tau + d}$

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$)

The covariant formalism makes $SO(2)$ gauge group

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad \omega_j \in \mathbb{C}$$

$$\operatorname{Im}(\omega_1 \overline{\omega_2}) = 1$$

$SL_2(\mathbb{R}) \times SO(2)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} e^{i\theta} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathrm{Im}(\omega_1) & \mathrm{Re}(\omega_1) \\ \mathrm{Im}(\omega_2) & \mathrm{Re}(\omega_2) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\Rightarrow$ SO(2) is rotating the supersymmetries.

$\Rightarrow$ SO(2) - charges

$\Rightarrow$ Charge - spin relation

$$\mathcal{A} = \frac{1}{2}(\bar{\omega}_1 \, d\omega_2 - \bar{\omega}_2 \, d\omega_1) = d\theta - \frac{1}{2}(\mathrm{Im}(z)^{-1} d\,\mathrm{Re}(z))$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \underset{\text{equiv}}{\overset{\text{Gauge}}{\sim}} \begin{bmatrix} \frac{1}{\sqrt{\mathrm{Im}\, z}} \, z \, e^{i\theta} \\ \frac{1}{\sqrt{\mathrm{Im}\, z}} \, e^{i\theta} \end{bmatrix}$$

Gauge - fix: $\omega_2 \in \mathbb{R}^+$ is the Gauge fixing

Observation (Gaberdiel - Green)

This is not compatible w/ $SL_2(\mathbb{R})$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\operatorname{Im}\tau}} & \tau \\ & \frac{1}{\sqrt{\operatorname{Im}\tau}} \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{c\bar{\tau}+d}{\sqrt{\operatorname{Im}\tau}}}_{\text{not in } \mathcal{Y}} \end{bmatrix}$$

To get rid of phase, multiply by some $e^{i\theta}$.

$Mp(2,\mathbb{R})$ — Metaplectic

$\downarrow$

Bosonic symmetry group ~ name for $SL_2(\mathbb{R}) \times SO(2)$

full symmetry: G_1 = Special Shimura

Super gravity

$$SL(2,\mathbb{R}) \xrightarrow[\text{Theory}]{\text{String}} SL(2,\mathbb{Z})$$

$\rightsquigarrow$ Look at $SL_2(\mathbb{Z}) \times SO(2)$

$\uparrow$

$Mp(2,\mathbb{Z})$

$\pi_1(SL(2,\mathbb{Z})) = \mathbb{Z}$

The double cover corresponds to $Mp(2,\mathbb{Z}) \to SL_2(\mathbb{Z})$.

$g \in Mp(2,\mathbb{Z}) \rightsquigarrow g = (A, \varphi(\tau))$ for

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\varphi(\tau)^2 = c\tau + d$

(Motivated by study of modular forms of half integral weight, e.g. Dedekind η)

Implication

1) IIB on X possible if X has Spin^c-bundle

e.g. $X = Z \times$ noncompact (Z complex)

$c_1(T_Z) = K_Z \qquad \mathcal{L} = \Lambda^{\dim Z} T_Z$

$f \in \mathcal{L}^{\otimes -4}$
$g \in \mathcal{L}^{\otimes -6}$

$\tau = \int \frac{dx}{\sqrt{x^3 + fx + g}}$