

# The Symmetries of Type IIB

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## String Theory

We will talk about:

Type IIB string theory

### §0. Gauge theories

Given a physical theory with symmetry, where the symmetry can vary from point to point

described by a gauge group  $G$ ;  $\text{Maps}(X, G)$   
a spacetime

$G$  should not be directly observable, but we look

for gauge-invariant quantities

$B$   
 $\downarrow$   
 $X$

$G$ -bundle

$A$  = connection on bundle

~~$A$~~

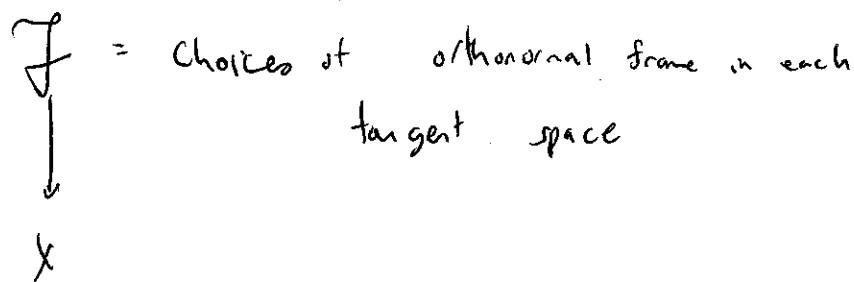
$F$  = curvature of  $A$ ; this is a gauge-invariant.

# § 1. Spinors & Spin<sup>c</sup>-structures

Again,  $X$  is a spacetime (pseudo-Riemannian manifold, tangent space has a nondeg. inner product)

(  
Not nec. pos. def.)

We can form the frame bundle:



The orthogonal group acts transitively on the frame bundle, so we have a principle  $SO$ -bundle.

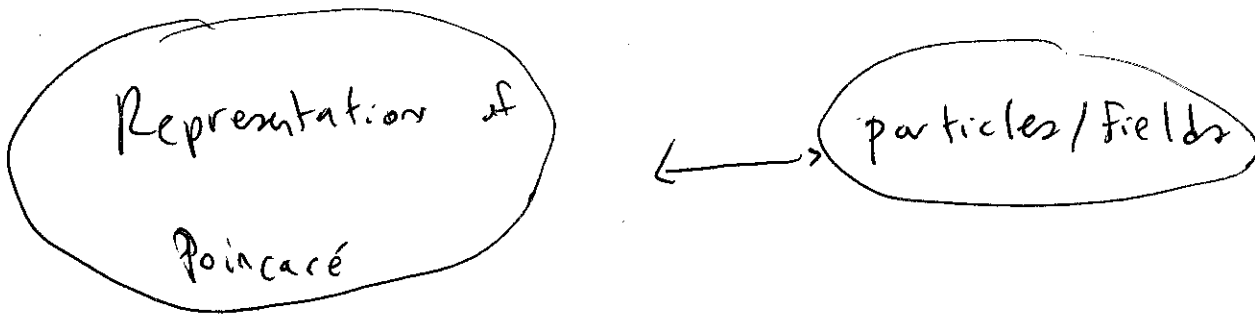
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Example

$$X = \mathbb{R}^{pq} \quad (\text{flat}) \quad ((p,q) = (n,0) \text{ or } (p,q) = (n-1,1))$$

There's  $SO(p,q)$ 's worth of symmetries, but also translations:

$$SO(p,q) \times \mathbb{R}^{pq} = \begin{cases} \text{Poincaré} & \text{if } (n-1,1) \\ \text{Euclidean} & \text{if } (n,0) \end{cases}$$



These are typically labelled by reps of subgroups of  $SO(p,q)$

We can use this framework to formulate theories w/ photons, bosonic fields, but have some trouble w/ fermionic fields (eg. electrons)

# Fun fact from Lie theory

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$$\pi_1(SO(p,q)) = \mathbb{Z}/2 \quad \text{if } p+q \geq 2$$

$\Rightarrow$  There's a double cover  $Spin(p,q)$   
 $\downarrow$   
 $SO(p,q)$

$$SU(2) \sim \mathbb{C}^2, \text{Sym}^k \mathbb{C}^2 \quad (\text{Sym}^k \mathbb{C}^2 \text{ has dim } k+1)$$

$\downarrow$   
 $n=1$  -  $SO(3)$   
 $\downarrow$   
"vector rep"  
3-dim'l

$-Id$  acts non-trivially on  $\mathbb{C}^2$   
 $\text{Sym}^3 \mathbb{C}^2$   
 $\text{Sym}^5 \mathbb{C}^2$

- Trivially on  $\text{Sym}^2 \mathbb{C}^2, \text{Sym}^4 \mathbb{C}^2, \dots$

This type of analysis leads to representations that could correspond to the fermionic ones.

Ex 2

$\text{Spin}(2)$

$\downarrow$

$\text{SO}(2)$

$$\pi_1(\text{SO}(2)) = \mathbb{Z}$$

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## § 2. Super symmetric Theories

Odd symmetry: exchanges bosons & fermions

Need a lift of the frame bundle to a spin bundle.

Math description:

Clifford algebra of  $\text{SO}(p, q)$

$$\text{SO}(p, q): V \otimes V$$

Take free tensor algebra, quotient. ~~not~~

$$\text{Cl}(Q) = \bigoplus_{n=0}^{\infty} \otimes^n V \Big/ x \otimes x - Q(x) \mathbb{1}$$

eg. If  $Q(x) \equiv 0$ , then  $Cl(Q) = \wedge^* V$ , since we're

only quotienting by  $x \otimes x$ ; the general case is a "deform" of the usual alternating algebra.

Clifford modules:

$$Cl(Q) \times M \longrightarrow M$$

$$Spin(p, q) \subseteq Cl(Q)^*$$

~~Complex Clifford modules:~~

Complex Clifford modules:

$$Cl(Q) \times M_{\mathbb{C}} \longrightarrow M_{\mathbb{C}}$$

$$Spin(p, q) \times U(1)$$

$$(-1, -1)$$

$$\Rightarrow Spin^{\mathbb{C}}(p, q) := \frac{Spin(p, q) \times U(1)}{(-1, -1)}$$

Interested in case where this is a gauge symmetry

Having a gauge symm. usually means we have a bundle

$G$   
↓  
 $X$  being acted on.

If all fermions have  $\frac{1}{2}\mathbb{Z}$  charge, bosons have  $\mathbb{Z}$  charge

should have  $Spin^c$  (prop 1).

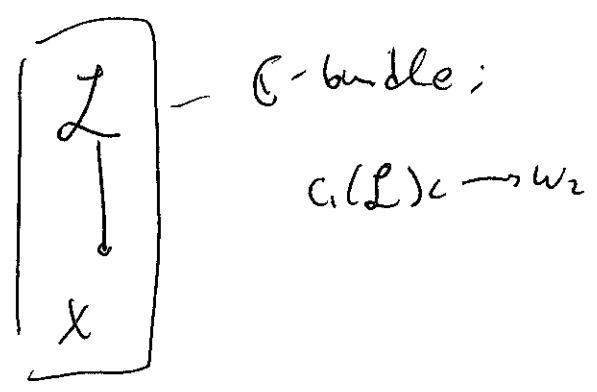
$$\int_C \frac{det}{2\pi} = \frac{1}{2} \int_C w_2 \text{ mod } \mathbb{Z}$$

↳ 2<sup>nd</sup> Stiefel-Whitney class

Lesson

Give a  $U(1)$ -gauge theory with this spin / charge, can put it

on  $Spin^c$ -backgrounds.



# Type IIB supergravity (10 dimensions)

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- fields (bosonic & fermionic)
- 2-d Supersymmetry transformation  
↑  
2 real dimensional space of

- $SO(2)$  rotates the supersymmetries

The formulation, due to Green & Schwarz, is made easier using the fact that ~~we~~ we have  $SL(2, \mathbb{R})$  global symmetry.

- bosonic fields:  $z \in \mathbb{H}$  <sup>upper half plane</sup>  
pair of 2-form fields  
|  
have field  
Strength: pair of 2-forms  
 $H_1, H_2$   
  
4-form field w/  
self-dual field strength  
  
fluctuating metric



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<u>Fermions</u>	<u>Spin</u>
2 gravitinos	3/2
2 fermions	1/2

Now recall  $\mathcal{Y} = SL_2(\mathbb{R}) / SO(2)$

$$\left( \begin{array}{l} \text{For } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \text{ action on } z, z \mapsto \frac{az+b}{cz+d} \\ \\ \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \end{array} \right)$$

The covariant formalism uses  $SO(2)$  gauge group

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \omega_j \in \mathbb{C}$$

$$\ln(\omega_1 \bar{\omega}_2) = 1$$

$$SL_2(\mathbb{R}) \simeq SO(2)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} e^{i\theta} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} \ln(\omega_1) & \operatorname{Re}(\omega_1) \\ \ln(\omega_2) & \operatorname{Re}(\omega_2) \end{pmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

⇒ SO(2) is rotating the super geometries.

⇒ SO(2) - charges

⇒ Charge - Spin relation

$$\mathcal{A} = \frac{1}{2} (\bar{\omega}_1 d\omega_2 - \omega_2 d\omega_1) = d\theta - \frac{1}{2} (\ln(z))^2 d\operatorname{Re}(z)$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \begin{matrix} \text{Gauge} \\ \text{equiv} \end{matrix} \begin{bmatrix} \frac{1}{\sqrt{\ln z}} z e^{i\theta} \\ \frac{1}{\sqrt{\ln z}} e^{i\theta} \end{bmatrix}$$

Gauge - fix:  $\omega_2 \in \mathbb{R}^+$  is the Gauge fixing

~~the gauge fixing~~ ~~is~~

# Observation (Gaberdiel - Green)

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This is not compatible w/  $SL_2(\mathbb{R})$ .

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{Im z}} & z \\ \frac{1}{\sqrt{Im z}} & \frac{1}{\sqrt{Im z}} \end{bmatrix} = \begin{bmatrix} \frac{az + b}{\sqrt{Im z}} \\ \frac{cz + d}{\sqrt{Im z}} \end{bmatrix}$$

↙ not in  $\mathbb{Z}$

To get rid of phase, multiply by  ~~$\frac{1}{\sqrt{Im z}}$~~  some  $e^{i\theta}$ .

$M_p(\mathbb{Z}, \mathbb{R})$  - Met-electric

Boronic symmetry group ~ name for  $SL_2(\mathbb{R}) \times SO(2)$

Full symmetry:  $\tilde{C}_1 =$  Special Shimura

Super gravity

$$SL(2, \mathbb{R}) \xrightarrow{\text{String Theory}} SL(2, \mathbb{Z})$$

$$\sim \text{Look at } SL_2(\mathbb{Z}) \times SO(2)$$

$$\uparrow$$

$$M_p(\mathbb{Z}, \mathbb{Z})$$

$$\pi_1(SL(2, \mathbb{Z})) = \mathbb{Z}$$

The double cover corresponds to  $M_p(\mathbb{Z}, \mathbb{Z}) \rightarrow SL_2(\mathbb{Z})$ .

$g \in M_p(\mathbb{Z}, \mathbb{Z}) \sim g = (A, \varphi(z))$  for

$$A = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \text{ then } \varphi(z) = cz + d$$

(Motivated by study of modular forms of half integral weight,  
e.g. Dedekind  $\eta$ )

### Implication

1) IB on  $X$  possible if  $X$  has  $\text{Spin}^c$ -bundle

e.g.  $X = \mathbb{Z}^n$  <sup>complex</sup> noncompact

$$e_1(T_{\mathbb{Z}}) = K_{\mathbb{Z}}$$

$$L = \wedge^{\dim \mathbb{Z}} T_{\mathbb{Z}}$$

~~Example~~  $f \in L^{0-4}$   
 $g \in L^{0-6}$

$$z = \int \frac{dx}{\sqrt{x^2 + dx + c}}$$