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The BTZ black hole

$$S = \frac{1}{2\pi} \int d^3x \sqrt{-g}\,(R - 2\Lambda) + \text{boundary terms}$$

$$\Lambda = -\frac{1}{\ell^2}\,, \qquad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

$$8G = 1$$

$$\hbar = 1$$

BTZ

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (d\phi + N^\phi dt)^2$$

$$N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{2r^2}$$

$$N^\phi = \frac{J}{2r^2} \qquad\qquad M > 0, \ |J| < M\ell$$

$$N = 0 \Rightarrow \quad r_\pm^2 = \frac{1}{2} M\ell^2 \left(1 - \sqrt{1 - \left(\frac{J}{M\ell}\right)^2} \right)$$

$$M = \frac{r_+^2 + r_-^2}{\ell^2} \qquad J = \frac{2 r_+ r_-}{\ell}$$

Beckenstein-Hawking:

$$S_{BH} = \frac{A_H}{4G} = \frac{1}{4G} \int_{\substack{t=cst \\ r=r_+}} \sqrt{g_{\phi\phi}}\, d\phi = \frac{2\pi r_+}{4G}$$

$$Q[\partial_t] \overset{?}{=} M \, ; \qquad Q[\partial_\phi] \overset{?}{=} J$$

Yes:

$$\delta Q[\xi] = \int_{\partial(t=\text{const})} d^{n-2} s_\ell \Big[G^{ijk\ell} \left(\xi_c^0\, \delta g_{ij;k} - \xi_{c,k}\, \delta g_{ij} \right)$$

$$+ 2 \xi_{c\,k}\, \delta \pi^{k\ell} + \left(2 \xi_c^k \pi^{j\ell} - \xi_c^\ell \pi^{jk} \right) \delta g_{ij} \Big]$$

$$\xi_c^0 = N \xi^t, \quad \xi_c^i = \xi^i + N^i \xi^t$$

$$G^{ijk\ell} = \frac{1}{2}\sqrt{g} \left(g^{ik} g^{j\ell} + g^{i\ell} g^{jk} - 2 g^{ij} g^{k\ell} \right)$$

$$\Pi^{ij} = \frac{\delta S}{\delta g_{ij}} = \sqrt{g}\left(K^{ij} - K g^{ij}\right)$$

$$\delta M, \ \delta J \Rightarrow \delta g_{ij}$$

$$\xi^t = l(T^+ + T^-) + \frac{l^3}{2r^3}(T^{+''} + T^{-''}) + O(1/r^4)$$

$$\xi^r = -r(T^{+'} + T^{-'}) + O(1/r)$$

$$\xi^\phi = T^+ - T^- - \frac{l^2}{2r^2}(T^{+''} - T^{-''}) + O(1/r^4)$$

$$T^\pm\left(\frac{t}{l} \pm \phi\right)$$

$$T^\pm \sim \frac{1}{2} e^{in\left(\frac{t}{l} \pm \phi\right)}$$

$$\left[\xi^\pm_m, \xi^\pm_n\right]_{Lie} = i(n-m)\,\xi_{m+n} \quad \text{(Witt algebra)}$$

$$\left\{Q[\xi^\pm_m], Q[\xi^\pm_n]\right\}_{D.B.} = i(n-m)\,Q[\xi^\pm_{m+n}] + 2\pi i\, l n(n^2+1)\,\delta_{m+n}$$

$$\{\,,\,\}_{D.B.} = -i[\,,\,], \quad L_n = Q[\xi^\pm_n]$$

$$[L^\pm_m, L^\pm_n] = (m-n)L^\pm_{m+n} + \frac{c^\pm}{12} m(m^2-1)\,\delta_{m+n,0}$$

$$\boxed{c^{\pm} = \frac{3\ell}{2G}}$$

$$S_{Cardy} = 2\pi\sqrt{\frac{c^+ L_0^+}{6}} + 2\pi\sqrt{\frac{c^- L_0^-}{6}}$$

Note: $\xi_0^{\pm} = \frac{\ell}{2}\left(\partial_t \pm \partial_\phi\right)$

$M = Q[\partial_t]\,,\ J = Q[\partial_\phi] \Rightarrow$ $\quad \ell M = L_0^+ + L_0^-$

$\quad J = L_0^+ - L_0^-$

Thus,

$$S_{Cardy} = \pi\sqrt{\frac{\ell(\ell M + J)}{2G}} + \pi\sqrt{\frac{\ell(\ell M - J)}{2G}}$$

$$= S_{BH}$$

4D: $\quad 10 - 4 - 4 = 2$ $\quad$ (2 polarizations of grav. wave)

$g_{\mu\nu}$ $\quad \mathcal{L}_\xi = 0$, $\mathcal{H}_\xi = 0$

3D: $\quad 6 - 3 - 3 = 0$

$R_{\mu\nu\lambda\rho} \propto g_{\lambda\rho} R_{\mu\nu}$ + perms.

$R = 2\Lambda$

$$R_{\mu\nu\lambda\rho} = \Lambda\left(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}\right)$$

BTZ is locally AdS_3

$$x^a = (v, u, x, y)$$

AdS_3:

$$-u^2 - v^2 + x^2 + y^2 = 0$$

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2$$

$SO(2,2)$

$$J_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a}$$

$$\xi = \frac{1}{2}\omega^{ab} J_{ab}, \quad \omega^{ab} = -\omega^{ba}$$

$$I_1 = \omega_{ab}\omega^{ab}, \quad I_2 = \frac{1}{2}\varepsilon^{abcd}\omega_{ab}\omega_{cd}$$

$$P \sim e^{t\xi} P \qquad t = 0, \pm 2\pi, \pm 4\pi, \dots$$

If $\xi \cdot \xi < 0$ in some region $\Rightarrow$ CTC

$$\xi \cdot \xi = 0, \quad r = 0.$$

$$\xi \cdot \xi > 0$$

Make the BTZ

$$\xi = \frac{r_+}{\ell} J_{12} - \frac{r_-}{\ell} J_{03}$$

$$I_1 = \frac{-2(r_+^2 + r_-^2)}{\ell} = -2M$$

$$I_2 = \frac{-4 r_+ r_-}{\ell} = -2J/\ell$$

$$\xi \cdot \xi > 0 : \quad -\frac{r_-^2 \ell^2}{r_+^2 - r_-^2} < u^2 - x^2 < \infty$$

I. $r_+^2 < \xi \cdot \xi < \infty$ $\qquad r_+ < r < \infty$

II. $r_-^2 < \xi \cdot \xi < r_+^2$ $\qquad r_- < r < r_+$

III. $0 < \xi \cdot \xi < r_-^2$ $\qquad 0 < r < r_-$

bounded by null surfaces

$$u^2 - x^2 = 0, \quad v^2 - y^2 - \ell^2 - (u^2 - x^2) = 0.$$

In all three regions, find (t, r, ϕ) s.t.

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (d\phi + N^\phi dt)^2.$$

where

$$N^2 = -\frac{r_+^2 + r_-^2}{\ell^2} + \frac{r^2}{\ell^2} + \frac{r_+^2 r_-^2}{\ell^2 r^2}$$

$$N^\phi = -\frac{r_+ r_-}{\ell r^2}$$

using (in I):

$$u = \sqrt{A} \cosh \tilde{\phi}$$
$$x = \sqrt{A} \sinh \tilde{\phi}$$
$$y = \sqrt{B} \cosh \tilde{t}$$
$$v = \sqrt{B} \sinh \tilde{t}$$

$$A = \ell^2 \frac{r^2 - r_-^2}{r_+^2 - r_-^2}$$

$$B = \ell^2 \frac{r^2 - r_+^2}{r_+^2 - r_-^2}$$

$$\tilde{t} = \frac{1}{\ell}\left(\frac{t r_+}{\ell} - r_- \phi\right)$$

$$\tilde{\phi} = \frac{1}{\ell}\left(-\frac{t r_-}{\ell} + r_+ \phi\right)$$

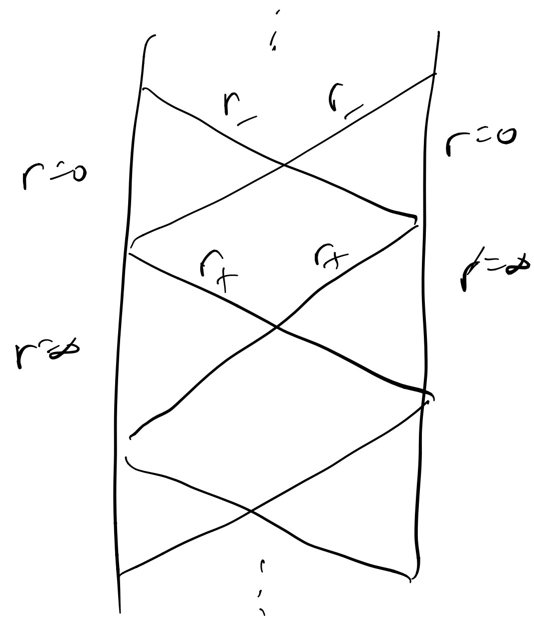
Ranges of coordinates:

$$0 < r < \infty$$
$$-\infty < t < \infty$$
$$-\infty < \phi < \infty$$

to make the BH: $\phi \sim \phi + 2\pi$

Penrose diagram

$r_+ > r_-$



$J=0,\ r_-=0$