Higher Chow Groups, van Geemen lines & Mirror Symmetry for Open Strings

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Calabi-Yaus X, Y 3-folds

On X: n_d = # of rational curves of degree d on X.

We form generating series:

$$\sum n_d \frac{d^3 q^d}{1-q^d} \qquad (q \text{ a formal parameter})$$

On Y: (Y near some boundary pt in moduli space)

$z = 0$ coord. near boundary pts

We write ϖ (\varpi) for periods

$$\varpi_j = \int_{\gamma_j} \Omega(z)$$

We introduce t s.t. $\log t = \frac{\omega_1}{\omega_0}$

$$q = e^{2\pi i t}$$

$\int_\gamma \nabla_z \nabla_z \Omega(z) \wedge \Omega(z)$ has same expansion as

the the prev. generating series (using $q = e^{2\pi i t}$)

—

Quintic 3-fold

We get:

$$5 + 2875 \frac{q}{1-q} + 609250 \frac{2^3 q^2}{-(q^2-1)} + 317206375 \frac{3^3 q^3}{-(q^3-1)} + \cdots$$

Computing the numbers:

Find diff. eq. related to period.

$$\mathcal{D}_{PF} \int_\gamma \Omega(z) = \int_\gamma \mathcal{D}_{PF} \Omega(z) = \int_\gamma \text{exact} = 0$$

Picard-Fuchs

Look at $x_1^5 + x_2^5 + \cdots + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0$

$$z = (5\psi)^{-5}$$

Then $\mathcal{D}_{PF} = \Theta^4 - 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)$

$$\Theta = z\frac{d}{dz}$$

annihilates the periods.

This was classical mirror symmetry.

Open mirror symmetry in '96 compares IIA/IIB string theory on X

w/ IIB/IIA string theory on Y.

<u>Problem</u>

Find holomorphic discs ending on Z. $(Z \subseteq X)$

Z special Lagrangian

submanifold.

$$\mathrm{Im}(\Omega\, x)\big|_Z = 0.$$

Can make a generating function in this case too.

On Y:

$D^b(\mathrm{Coh}\, Y) \longleftrightarrow$ matrix factorizations $\longrightarrow$ alg. cycles homologous to zero.

Walcher - DRM: Took Z real quintic $\subseteq$ complex quintic X (connected)

Then $Z \cong \mathbb{RP}^3$

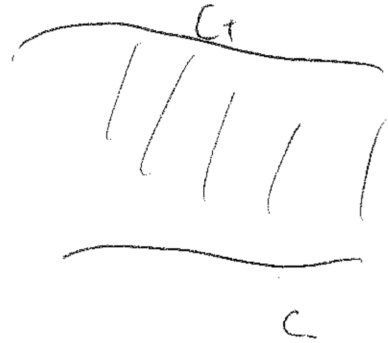
$$\downarrow$$

$\mathcal{D}_{PF}(\mathcal{Z}(z)) = f(z)$ non zero

Found C_+, C_- conic on Y s.t. $C_+ - C_- = \partial\Gamma$

$$"\int_{C^-}^{C^+} \Omega(z)" = \int_{\Gamma} \Omega(z)$$

$(\partial\Gamma = C^+ - C^{-})$



$$\Rightarrow \theta_{PF} \int_{\Gamma} \Omega(z) = \theta_{PF} \int_{\Gamma+\gamma} \Omega(z)$$

[θ_{PF} annihilates the period]

3-chain Γ, $\partial\Gamma$ = comb. of alg. cycles.

We get $\int_{\Gamma} \Omega(z) \in \mathcal{J}(z)$

Intermediate Jacobian.

Goal

Find Z (finding sp. Langr. sub is hard)

Want to use properties of Y story to find a reasonable guess for Z — should be some 3-manifold.

$$\left[\text{Mirror quintic is } V\left(\sum x_i^5 - 5\psi x_1 \cdots x_5\right) \Big/ \mathbb{Z}_5^3\right]$$

$[\omega = e^{2\pi i/3}]$

Van Geemen:

$$\begin{cases} x_1 + \omega x_2 + \omega^2 x_3 = 0 \\ a(x_1 + x_2 + x_3) = 3x_4 \\ b(x_1 + x_2 + x_3) = 3x_5 \end{cases}$$

This is a line on the mirror quintic

$\Longleftrightarrow$

$ab\psi = 6$

$a^5 + b^5 = 27$

$C_\omega - C_{\omega^2}$ = alg. cycle homologous to zero

$\int_\Gamma \Omega(z)$ hard to compute.

[Closed string detail:
Periods $\varpi_j = \int_{\gamma_j} \Omega$ can be studied by taking $\lim z \to 0$]

Note: $\lim_{z\to 0} \varpi_i = \dfrac{200\,\zeta(3)}{(2\pi i)^3}$ $\quad 200 = \chi(Y)$

Conjecture

$$\frac{195\sqrt{-3}}{2\pi^3}\, L\left(2, \left(\frac{-3}{\cdot}\right)\right)$$ is the function

Numbers live in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ ~ Other examples involve other number fields.

$$\begin{tikzcd} \{ \mathcal{J}(Y(z)) \} \arrow[d] & \subseteq & \overline{\{ \quad \}} \arrow[d] \\ \{ z \neq 0 \} & \subseteq & \{ z \} \end{tikzcd}$$

$$\mathrm{Ch}(Y(z))_0 \longrightarrow \mathcal{J}(Y(z))$$

Green-Griffiths-Kerr use higher Chow groups
(Things live in $Y \times \square^k$)

Blow-up singular locus of mirror quintic; 5 points total.

Lift vG lines; follow the double complex
to the blow-up

$$L^{33}_{125} \big|_{x_3=0} = 5 \left[\frac{-(1-u)^5(1-v)}{w^2[\ldots]} \right]$$

Related to regulators in K-theory.

$$AJ(L) = D\left(\underset{\sim}{1-u}\right)$$

$$\sum AJ(L) = 2.195\, i\, D(e^{2\pi i/3})$$