The Dilogarithm

Cf: • D. Zagier, "The Dilogarithm Function"

(Frontiers in Number Theory, Physics, Geometry II

Springer 2007, pp 3-65)

• S. Bloch, "Higher regulators, ..."

AMS 2000

Euler's dilogarithm: (As opposed to a more general one to come later)

$$Li_2(z) = \sum \frac{z^n}{n^2}$$, converges $|z|<1$

[Etymology: recall $-\log(1-z) = \sum \frac{z^n}{n}$ for $|z|<1$]

Thus we also have:

$$Li_k(z) = \sum \frac{z^n}{n^k}$$

for other k.

$$Li_2(z) = -\int_0^z \log(1-u)\,\frac{du}{u}$$ is an alternate representation

$\Rightarrow$ defined on $\mathbb{C} \setminus [1, \infty)$; the branch cut makes it single valued.

Special Values

There are (only) 8 values z where z & $Li_2(z)$ can be written in closed form.

$$Li_2(0) = 0$$

$$Li_2(1) = \pi^2/6$$

$$Li_2(-1) = -\frac{\pi^2}{12}$$

$$Li_2\left(\tfrac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2)$$

$$Li_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

$$Li_2\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

$$Li_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{-\pi^2}{15} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

$$Li_2\left(\frac{-1-\sqrt{5}}{2}\right) = \frac{-\pi^2}{10} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

Identities

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)$$

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \log z \cdot \log(1-z)$$

$$\boxed{\begin{array}{c} \mathrm{Li}_2(z),\ \mathrm{Li}_2\left(\frac{1}{1-z}\right),\ \mathrm{Li}\left(\frac{z-1}{z}\right),\ -\mathrm{Li}_2\left(\frac{1}{z}\right) \\ -\mathrm{Li}(1-z),\ -\mathrm{Li}_2\left(\frac{z}{z-1}\right) \\ \text{all differ by elementary functions.} \end{array}}$$

[These expressions arise when trying to compute Moduli of 4 pts in $\mathbb{P}^1$]

$$\mathrm{Li}_2(z^2) = 2\left(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)\right)$$

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z)$$

5-term relation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right) =$$

$$\frac{\pi^2}{6} - \log x \log(1-x) - \log y \log(1-y) +$$

$$\log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$$

Known to many in the 19^{th} century.

6 term relation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2(z) = \frac{1}{2}\left[\mathrm{Li}_2\left(-\frac{xy}{z}\right) + \mathrm{Li}_2\left(-\frac{yz}{x}\right) + \mathrm{Li}_2\left(-\frac{zx}{y}\right)\right]$$

Bloch-Wigner dilogarithm

Goal: Improve on the fact that we have the branch cut now.

Set:

$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \arg(1-z)\log|z|)$ removes the branch cut.

$D(z)$ is real analytic on $\mathbb{C}\setminus\{0,1\}$

$r \log r$ at $0,1$

The previous identities/relations give new idstrels on D:

$$D(z) = D\left(\frac{1}{1-z}\right) = D\left(\frac{z-1}{z}\right) = -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{z}{z-1}\right)$$

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0 \qquad (*)$$

Now, consider:

$$\tilde{D}(z_0, z_1, z_2, z_3) = D\left(\frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}\right)$$

$$\sum_{i=0}^{4} (-1)^i \tilde{D}(z_0, \ldots, \hat{z}_i, z_4) = 0$$

giving an interpretation of the funny relation $(*)$.

Regulators for class number formula

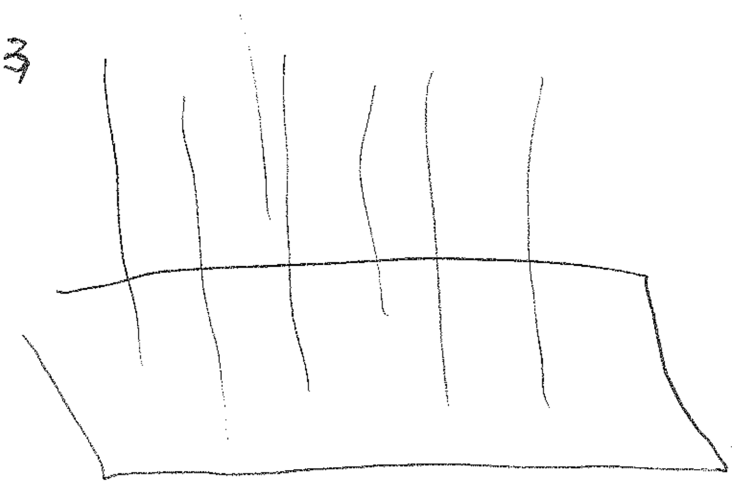
alg. K-theory

number fields...

↑ Need to talk about this next

Hyperbolic Geometry

$\mathbb{H}^3$



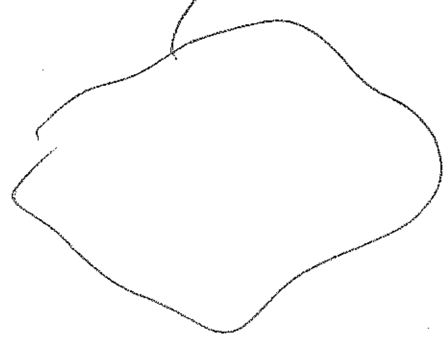
$\mathbb{R}^2$

$\downarrow$

$\mathbb{H}^3/\Gamma$

(where Γ acts freely & cocompactly)

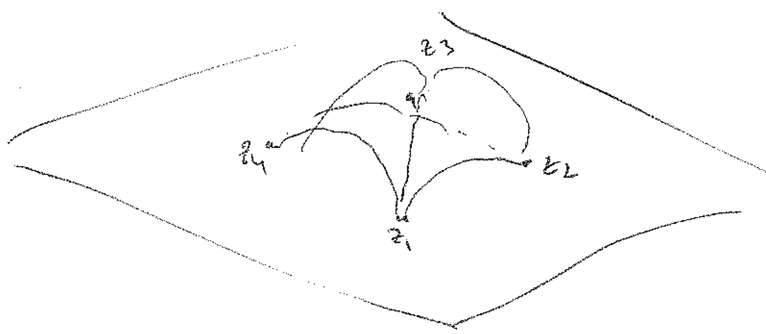
Hyperbolic 3-manifold



Volume is well-defined after we normalize to constant curvature -1.

These manifolds can also be built using ideal tetrahedra (maybe after removing finite # of geodesic circles).

Triangles in the Triangulation look like:



(Analogous to  in $\mathbb{H}^2$)

$$\mathrm{Vol}(T(z_1, z_2, z_3, z_4)) = \mathcal{D}(z_1, \ldots, z_4)$$
$$= \mathcal{D}(\text{cross ratio of } z_1 \ldots z_4)$$

These build up the manifold & we can compute the volumes as above, so we can compute the volume of the whole thing given a triangulation.

Not any collection of ideal tetrahedra give a 3-manifold —

Let G be abelian group.

$$\Lambda^2 G = \langle g \wedge h \rangle \big/ \langle g \wedge g,\ gh \wedge k - g \wedge k - h \wedge k \rangle$$

Apply this to $G = \mathbb{C}^*$:

$$\Lambda^2 \mathbb{C}^* = \langle x \wedge y \rangle \big/ \langle x \wedge x,\ xy \wedge z - x \wedge z - y \wedge z \rangle$$

If $z_1, \dots, z_\nu$ are vertices in an ideal tetrahedralization of a hyperbolic 3-manifold, then

$$\sum_{\nu=1}^{n} z_\nu \wedge (1 - z_\nu) = 0 \quad \text{for} \quad z_j \in \mathbb{C} \setminus \{0,1\}$$

Bloch Group

[Orig. used for Alg. K-Theory]

Formal sums $\left\{ [z_1] + \cdots + [z_n] \;\middle|\; \begin{array}{l} z_i \in \mathbb{C} \setminus \{0,1\} \\ \text{s.t. } \sum z_i \wedge (1-z_i) = 0 \end{array} \right\}$

$$\Big/ \left([x] + [1/x],\ [x] + [1-x],\ [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \right)$$

This is called the Bloch group for $\mathbb{C}$ — we call it $B_{\mathbb{C}}$.

D is well-defined on $B_{\mathbb{C}}$:

$$\left(\sum \mathrm{vol}(\Delta) = \sum D(z) \right)$$

$$D : B_{\mathbb{C}} \longrightarrow \mathbb{R}$$

$$[z_1] + \cdots + [z_n] \longrightarrow D(z_1) + \cdots + D(z_n)$$

Theorem

$$D(B_{\mathbb{Q}}) = D(B_{\overline{\mathbb{Q}}})$$

Corollary

$D(B_{\mathbb{Q}})$ is countable.

Corollary 2

$\{\mathrm{vol}(\mathbb{H}^2/\Gamma)\}$ is countable.

$$\alpha = \frac{1-\sqrt{-7}}{2} \qquad \beta = \frac{-1-\sqrt{-7}}{2}$$

Claim

$$2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1-\sqrt{-7}}{2}\right] \in B_{\mathbb{Q}}$$

Pf

$$2(-\beta) \wedge \alpha + \frac{1}{\beta} \wedge \frac{\alpha^2}{\beta} = \qquad \text{(cf Zagier paper)}$$

$$\left(\frac{1}{\beta} = \frac{2}{-1-\sqrt{-7}} = \frac{2(1+\sqrt{-7})}{-8} = \frac{-1-\sqrt{-7}}{4}\right.$$

$$2(-\beta)\wedge\alpha + \left(\frac{-1}{\beta}\right)\wedge\left(\frac{\alpha^2}{\beta}\right) = \beta^2\wedge\alpha - \beta\wedge\alpha^2 = 2\beta\wedge\alpha - 2\beta\wedge\alpha = 0$$

(Works b/c $1+\beta=\alpha$ & $1-\beta^{-1}=\alpha^2/\beta$) (This verifies the claim)

K number field, $\mathcal{O}_K$ ring of integers

$$\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$$

$$\begin{tikzcd} \mathcal{O}_K^{*} \arrow[r, hook] \arrow[rr, bend right, dashed, "\ell = \log|\,|,\ \log|\,|^2"'] & (\mathcal{O}_K \otimes \mathbb{R})^{\times} \arrow[r] & \mathbb{R}^{r_1+r_2} \end{tikzcd}$$

$\ell(\mathcal{O}_K^{*})$ is a lattice of rank r_1+r_2-1

$$R_K = \frac{1}{\sqrt{r_1+r_2}} \operatorname{vol}(\ell(\mathcal{O}_K^{*}))$$

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_K \cdot h}{\sqrt{|\text{disc } K|}\ w}$$

(w # of roots of 1 in $\mathcal{O}_K$)

$\exists$ elliptic curve

$$\text{reg}: K_2(E_{\mathbb{C}}) \longrightarrow \mathbb{C} \qquad \text{related to } L(E,2)$$

Alg. K-theory.

This is related to Bloch-Wigner function.

K_n(Alg. variety) = described via symbols s.h. to Bloch gr'p

$$\begin{tikzcd} Ch^*(X) \arrow[r] \arrow[dr] & K(X) \arrow[d] \\ & H^*(X) \end{tikzcd}$$