Morrison
GTP
12/08/17
(part III)
Hyp. 3 folds mirror
& mirror symmetry

The Abel–Jacobi map & higher Chow groups

Refs: [redacted]
math.AG/0909116
Kerr, Lewis, Müller-Stach

Let C be a projective curve over $\mathbb{C}$

Period integral can be used to give a map

$$C \longrightarrow \mathrm{Jac}(C) \cong \mathrm{Pic}^0(C)$$

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0$$

$$\mathrm{Jac}(C) \cong (H^{1,0}(C))^\vee / H_1(C,\mathbb{Z})$$

$$\gamma \in H_1(C,\mathbb{Z}) \mapsto \int_\gamma \in H^{1,0}(C)^\vee$$

$H^{1,0}(C)^\vee / H_1(C,\mathbb{Z})$ is a complex torus $\mathbb{C}^g / \mathbb{Z}^{2g}$

in fact, an algebraic variety.

Pick a base point $P_0 \in C$

$$P \mapsto \int_{P_0}^{P_1} \in (H^{1,0}(C))^\vee / H_1(C,\mathbb{Z}),$$

i.e. choose a 1-chain δ s.t. $\partial\delta = P_1 - P_0$ and

$$\int_{P_0}^{P_1} \omega = \int_\delta \omega, \text{ well-defined}$$

up to $\int_\gamma \omega$, $\gamma \in H_1(X,\mathbb{Z})$

Abel's theorem $\Rightarrow$ this is an embedding.

$$\mathrm{Ch}^0(C) = \left\{ \textstyle\sum n_i P_i,\ n_i \geq 0,\ \sum n_i = n \right\} / \sim$$

$$\mathrm{Ch}^0(C)_n \to \mathrm{Jac}(C)$$

$$\int_{nP_0}^{\sum n_i P_i} = \sum n_i \int_{P_0}^{P_i}$$

add in $\mathrm{Jac}(C)$ using group law.

<u>Fact</u> If $\sum n_i P_i$ is linearly equivalent to $\sum m_j P_j$, then

$$\int_{nP_0}^{\sum n_i P_i} = \int_{nP_0}^{\sum m_j P_j}$$

(f = mero fcn on C s.t. $\mathrm{div}(f) = \sum n_i P_i - \sum m_j P_j$)

ask for copy of preprint.

$\mathrm{Sym}^n C/\equiv \longrightarrow \mathrm{Jac}(C)$

If $n \geq g$ then this is an isomorphism,

Chow group : $Ch^p(X)$

X = alg variety

p = dim of cycles

$\Gamma \sim \Gamma'$ if there is a cycle $\Delta \subset X \times \mathbb{P}^1_z$

s.t. $\Delta|_{z=0} = \Gamma, \ \Delta|_{z=\infty} = \Gamma'$

" $\partial\Delta = \Gamma' - \Gamma$ "

$$Ch^p(X) = \{\Sigma\, n_i \Gamma_i,\ \dim \Gamma_i = p\} \big/ \sim$$

rational equivalence

$$Ch^p(X)_0 = \{\Sigma_i n_i \Gamma_i \mid \Sigma_i n_i [\Gamma_i] = 0 \text{ in } H_{2p}(X,\mathbb{Z})\}$$

Griffiths' Abel–Jacobi map :

$Ch^p(X)_0 \ni \Gamma \Rightarrow \Gamma = \partial D$

s.t. $D \in Z_{2p+1}(X, \mathbb{Z})$

$w \mapsto \int_D w$ defines a function on

$$(H^{2p+1,0}(X) \oplus H^{2p,1}(X) \oplus \cdots \oplus H^{p+1,p}(X))^\vee$$

$\int_\Gamma w = \int_D w$ for any D

$$\underbrace{\in (H^{2p+1,0}(X) \oplus \cdots \oplus H^{p+1,p}(X))^\vee \big/ H_{2p+1}(X,\mathbb{Z})}_{\text{complex torus}}$$

" Griffiths intermediate Jacobian "

Application by

Clemens & Griffiths :

X = smooth cubic 3-fold $\in \mathbb{P}^4$

$F(X)$ = Fano variety of lines

$= \{ l \subseteq X \mid l \text{ is a line} \}$

$\dim F(X) = 2$.

choose $l_0 \in F(X)$

$F(X) \xrightarrow{\alpha} J^1(X)$ = abelian variety

image of α generates all of $J^1(X)$.

$l \mapsto \int_{l_0}^{l}$,

one can show $J^1(X) \cong \mathrm{Jac}(C)$

for any curve C.

<u>Cor</u> X is not birational to $\mathbb{P}^3$.

$$0 \to \mathrm{Jac}(C) \to \mathrm{Pic}(C) \to \mathbb{Z} \to 0$$

$$0 \to J^p(X) \to H_D^{2p}(X,\mathbb{Z}) \xrightarrow{\pi} H^{2p}(X,\mathbb{Z}) \to 0$$

if $\pi(\alpha) = \pi(\beta)$, then

$\alpha - \beta \in J^p(X)$.

$$Ch^p(X) \to H_D^{2p}(X, D)$$

- use a quasi-projective X

  $\Rightarrow H^*(X)$ has a "mixed" Hodge structure.

- consider a degeneration

  X_t, $t \in \Delta$

  $t \searrow x_0$

  $\Rightarrow H^*(X_t) \circlearrowleft$ monodromy

  has a "limiting mixed Hodge structure"

e.g. $g(C) = 1$

$$H^2(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$$

$C_t \to C_0 =$ 

limiting MHS $H^{1,0} \to H^0$

$H^{0,1} \to H^2$

$$H_D^{2p}(X, \mathbb{Z}(n))$$

"Tate twist"

number $\left(\text{put } n\ 2\pi i\right)$

<u>S. Bloch</u>:

$$Ch^p(X, n) \to H_D^{2p}(X, \mathbb{Z}(n))$$

What kind of objects belong to $Ch^p(X,n)$?

<u>Motivation</u>:

$$\begin{tikzcd} \bigoplus_p Ch^p(X) \arrow[r] \arrow[d] & \bigoplus_p H^{2p}(X,\mathbb{Z}) \\ K(X) \arrow[ur] & \end{tikzcd}$$

Quillen had defined higher (algebraic) K-theory $K(X,n)$

$\left(Ch^p(X) = Z^p(X) / \mathrm{Im}\, \partial \right.$

2 variants :

$\Delta^n = \operatorname{Spec} \dfrac{\mathbb{C}[t_1, \dots, t_n]}{(1 - \sum t_j)} \cong \mathbb{C}^n$

① Bloch's original : "alg. geometers simplex"

② "alg. geometers cube":

$$\square^n = (\mathbb{P}^1 \setminus \{1\})^n$$

Faces set some variables z_i to 0 or ∞

$$Z^p(X, n) := \{ \zeta \in Z^p(X \times \square^n) \}$$

$$\partial_i : Z^p(X, n) \to Z^p(X, n-1)$$

using i^{th} face

$$\delta = \sum (-1)^i \partial_i$$

Fact $\delta^2 = 0$

Def: $\mathcal{C}^p(X, n) = \ker : Z^p(X, n) \to Z^p(X, n-1)$ / im $Z^p(X, n+1) \to Z^p(X, n)$

KMS : express the AJ map for homologically trivial cycles in terms of integrals

$$\omega \in H^{k,l}(X)$$

working on $X \times \square^n$

with

$$\frac{1}{(2\pi i)^n} \pi_X^* \omega \wedge \pi_\square^* (d\log z_1 \, d\log z_2 \wedge \dots \wedge d\log z_n + \dots)$$

($CH^p(\operatorname{Spec} \mathbb{F}) = p^{th}$ Milnor K group of $\mathbb{F}$)