Supersymmetry & Geometry

Sources: • Notes on supersymmetry ~ Bernstein
• Superhomework ~ Witten
• Supersolutions ~ Deligne-Fried
• The Sign Manifesto (Deligne)

(Quantum Fields & Strings: A Guide for Mathematicians (by Deligne, DRM, —)

Algebraic Structures:

Start w/ a structure $\rightsquigarrow$ make a $\mathbb{Z}_2$ graded superstructure

e.g. V vector space $\longrightarrow$ supervector space $V = \underbrace{V_0}_{\text{even}} \oplus \underbrace{V_1}_{\text{odd}}$

Lie alg $\longrightarrow$ Super Lie alg.

We can form $\mathrm{Hom}(V_0 \oplus V_1, V_0 \oplus V_1)$, etc.

$X_1, \dots, X_k \in V_0$ ~ commuting elements

$\psi_1, \dots, \psi_\ell \in V_1$ ~ anticommuting elements.

Functions are going to be:

$$\mathrm{Sym}\, V_0 \otimes \Lambda^{\cdot} V_1$$

in the u.s. situation.

<u>Issues</u>

- Signs are difficult to deal w/
- Trace/det. need to be tweaked, e.g. we use

$$\mathrm{str}\left[\begin{array}{ccc|ccc} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_n & & & \\ \hline & & & b_1 & & \\ & 0 & & & \ddots & \\ & & & & & b_m \end{array}\right] = \sum a_i - \sum b_j \qquad \text{(Supertrace)}$$

instead of the usual trace.

- Berezinian replaces det (only defined on invertible endomorph)

$$\mathrm{Ber}\begin{bmatrix} a_1 & & & & * & \\ & \ddots & a_n & & & \\ & d & & b & & \\ & & & & \ddots & b_m \end{bmatrix} = \mathrm{Ber}\left[\begin{array}{c|c} A & * \\ \hline * & B \end{array}\right] = \frac{\det(A)}{\det(B)} \quad \text{if } *=0$$

$$\left(\text{i.e. } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

Example

Two supersymmetry gens $d\Psi_1, d\Psi_2$.

$$\int dx_1 \cdots dx_k \, d\Psi_1 \, d\Psi_2 \, S(\;) = \int dx_1 \cdots dx_k \frac{\partial^2}{\partial t_1 \partial t_2} S(\;)$$

$\overline{\mathbb{R}^{k-1,1}} \longrightarrow X$ - to write Lagrangian, want X to have a metric g_{ij}

$$\int dx_1 \cdots dx_k \, g_{ij} \, \Phi^i \, \Phi^j + \cdots$$

If X is a Kähler manifold, (use coord. z_α, $\bar{z}_\alpha$),

$$g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \mathcal{K}$$

Kähler potential

$$\mathcal{K} \longmapsto \mathcal{K} + f(z) + \overline{g(z)}$$

Objectives

- Define Super Minkowski Space $M^{k-1,1}$ ($\mathbb{R}^k$ w/ metric of signature $(-,+\cdots+)$)
- Study spinor ~~bundle~~ on $M^{k-1,1}$

- $S = \{\text{spinors}\}$ a vector space

  Rep of $\mathrm{Spin}(\mathbb{R}^{k-1,1})$

ΠS = an odd vector space $\cong S$ as a nongraded v.s.

$$\underset{\substack{\psi \\ x}}{M^{k-1,1}} \times \underset{\substack{\psi \\ \psi}}{\Pi S} \sim \int dx\, d\psi\, f(x,\psi) = \int dx\, \underline{g(x,\overset{\text{even}}{\psi})}$$

(Don't eliminate ψ's completely, but remaining ones come in pairs)

· Super Poincaré group:

Classically, we had:

Minkowski space ; Lorentz group ; $O(k-1,1)$

Poincaré: $O(k-1,1) \ltimes \mathbb{R}^{k-1,1}$

choose forward light cone . . .

Euclidean

Orthog. $O(k)$

~~Euclid~~

$O(k) \ltimes \mathbb{R}^k$

Super version

Poincaré algebra $SO(k-1,1) \ltimes \mathbb{R}^{k-1,1} = \mathfrak{g}_0$ ← Even part

$\parallel$

alg. of $Pin(k-1,1)$

& $Spin(k-1,1)$

$$\mathfrak{g}_1 = S_+^{\oplus r_1} \oplus S_-^{\oplus r_-}$$

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Super Poincaré algebra

A particle is a rep of Poincaré:

- Trivial rep gives a scalar field. (e.g. Higgs)
- Spinor representation ⟶ Fermion (e.g. Electron)
- "Standard" rep ⟶ Spin 1 boson (e.g. photon)

[Also S^2 Standard ⟶ Spin 2 boson (e.g. graviton)]
Spin 3/2 ⟶ (Rarita-Schwinger field)
[Higher spin?]

Representations of Super Poincaré:

Take reps of super poincaré;

restract to Poincaré, get a sum of reps
only a few w/ spin ≤ 2.

This is a "multiplet".

$M^{0,1|1}$ $\sim$ $\varphi^{-2} + \psi\psi$

$\int d\psi$ 

$M^{2,1|2}$ $\sim$

$\uparrow$

Seiberg-Witten Theory

Complex manifolds, Kähler metrics, ...

$M^{3,1|4}$ Hyperkähler manifold

$M^{5,1|(8,0)}$ Locally symmetric spaces