4D $\mathcal{N}=2$ Supersymmetric $SU(2)$ gauge theory

w/ no matter (following Seiberg-Witten hep-th/9407087)
(hypermultiplet)
(8 supercharges)

$\mathcal{N}=2$ multiplets:

$\mathcal{N}=2$ (chiral) vector: gauge field A_μ
2 Weyl fermions λ, ψ
Scalar ϕ ~ complex

transform in adjoint rep of G
$G = SU(2)$

$\mathcal{N}=2$ hypermultiplet:
2 Weyl fermions $\psi_q, \psi_{\tilde{q}}^\dagger$
2 $\mathbb{C}$ Scalars $q, \tilde{q}^\dagger$

transform in a rep of G

(In $\mathcal{N}=1$ terms:

$\mathcal{N}=2$ vector is $\mathcal{N}=1$ vector + $\mathcal{N}=1$ chiral

$\mathcal{N}=2$ hyper is 2 $\mathcal{N}=1$ chirals)

Classical Potential for $\mathcal{N}=2$ theory

$$V(\varphi) = \frac{1}{g^2} \operatorname{Tr}[\phi, \phi^+]^2$$ (g = coupling constant)

(Note: φ takes values in $su(2)$)

$V(\varphi) = 0$ can happen if $[\varphi, \varphi^+] = 0$

leading to a space of solutions (vacuum degeneracy)

$G = SU(2) \rightsquigarrow$ φ is conjugate to $\frac{1}{2} a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$u = \frac{1}{2} a^2 = \operatorname{Tr} \phi^2$ is the natural parameter

- If $a \neq 0$, $SU(2)$ broken to commutat of $\frac{1}{2} a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is $U(1)$

Weyl$(SU(2))$: $a \longmapsto -a$; thus the a^2 in the formula earlier instead of a

Selberg's idea: Promote the scalar a to a field

$A =$ $\mathcal{N}=2$ vector multiplet whose scalar component is

a

Terms in SUSY low energy theory involve

a holomorphic function $\mathcal{F}(A)$

<u>In $\mathcal{N}=1$ language</u>:

Lagrangian has the form $\frac{1}{4\pi}\mathrm{Im}\left[\int \partial^4\theta \, \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A}\right.$

$$\left. + \int \partial^2\theta \, \frac{1}{2}\frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W_a W^a \right]$$

The structure of the theory is encoded in

$\mathcal{F}(A)$.

$$\tau_{cl} = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$$

There is a Kähler potential for metric in space of scalars

$$K = \operatorname{Im}\left(\frac{\partial \mathcal{F}}{\partial A}\bar{A}\right)$$

$$ds^2 = \operatorname{Im}\left(\frac{\partial^2 \mathcal{F}(a)}{\partial a^2}\, da\, d\bar{a}\right) = \text{Kähler metric in the space of vacua}$$

Features of the Quantum Theory

1) If $|a| \gg 0$, the theory is asymptotically free, so we can think of it as weakly coupled

2) Vacuum degeneracy cannot be removed quantum mechanically, since $\mathcal{N}=2$ SUSY does not permit an $\mathcal{N}=1$ superpotential

3) Classical value of $\mathcal{F}$ is

$$\mathcal{F}(A) = \frac{1}{2} \tau A^2$$

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

$$\mathcal{F}_{\text{one-loop}} = i \frac{1}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}$$

(Λ = dynamical scale)

($\mathcal{N}=2 \Rightarrow$ 1-loop exact)

$$\mathcal{F} = i \cdot \frac{1}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} \underbrace{\mathcal{F}_k \left(\frac{\Lambda}{A}\right)^k A^2}_{\text{instanton contribs}}$$

$\mathcal{F}_1 \neq 0 \ldots$

Duality

$$ds^2 = \operatorname{Im} \tau(a)\, da\, d\bar{a}$$

$$\tau(a) \approx i \frac{\left(\ln\left(\frac{a^2}{\Lambda^2}\right) + 3\right)}{\pi}$$

- Multivalued, $-ds^2$ not pos def for all a

$\rightsquigarrow$ should be other local descriptions

Let $a_D = \frac{\partial \mathcal{F}}{\partial a}$; $ds^2 = \operatorname{Im}(da_D\, d\bar{a}) = \frac{-i}{2}(da_D\, d\bar{a} - da\, d\bar{a}_D)$

Observation

$a \longleftrightarrow -a_D$ gives another description of the same theory

Note:

$a_D \mapsto a_D + ba$
$a \mapsto a$

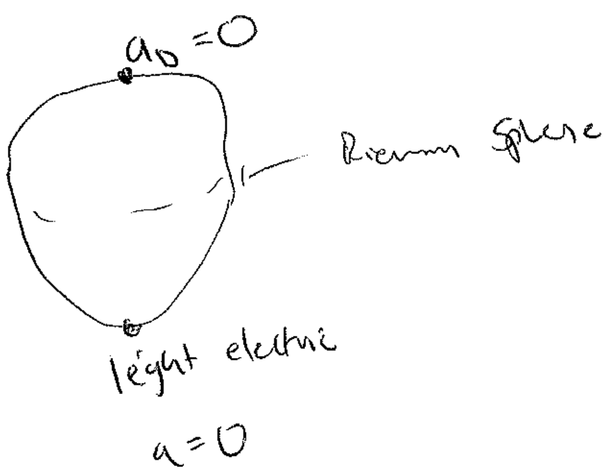
leaves the theory invariant, $\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}\right]$ Electric-Magnetic duality

$\rightsquigarrow$ Invariance under $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\rightsquigarrow$ Invariance under $SL(2,\mathbb{R})$

Broken to $SL(2,\mathbb{Z})$ by BPS particles

$Z = a n_e + a_D n_m$ — Central Charge



$a \mapsto -a$ from $Weyl(SU(2))$