Holomorphic Morse theory and moduli spaces of flat connections on 3-manifolds

Pavel Safronov

University of Edinburgh

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Introduction

Consider:

- A Riemannian manifold Y.
- A closed one-form α.

Get supersymmetric quantum mechanics with target Y deformed by a potential specified by α.

Example

4d $\mathcal{N} = 4$ supersymmetric Yang–Mills theory compactified on a 3-manifold M in the Vafa–Witten or Kapustin–Witten twists gives supersymmetric quantum mechanics with target

$$Y = \mathrm{Conn}_G(M),$$

where G is the complexified gauge group and α the differential of the complex Chern–Simons functional $S_{CS}\colon \mathrm{Conn}_G(M) \to \mathbf{C}/\mathbf{Z}$.

Goals of the talk:

- What are different ways to describe spaces of states $\mathcal{H}$ for supersymmetric quantum mechanics?
- Can one make sense of this in the main example?

Twisted cohomology

Y manifold, $\alpha \in \mathrm{H}^1(Y; \mathbf{C})$.
Witten: directly analyzing SUSY quantum mechanics, we get

$$\mathcal{H} = (\Omega^\bullet(Y), d + \alpha \wedge (-)),$$

the twisted de Rham complex. **de Rham theorem**: this is identified with

$$\mathcal{H} = \mathrm{H}^\bullet(Y; \mathbf{C}_\alpha),$$

where α defines a twisted one-dimensional local system.

Question: how does it depend on α?

Theorem (Novikov)

Let Y be a finite CW complex and $\alpha \in \mathrm{H}^1(Y; \mathbf{Z})$ (i.e. $\alpha = df$ for $f\colon Y \to S^1$). Then away from finitely many $\tau \in \mathbf{C}/2\pi i\mathbf{Z}$ the cohomology groups $\mathrm{H}^\bullet(Y; \mathbf{C}_{\tau\alpha})$ are independent of τ.

Later works by Farber, Pajitnov, Sikorav, . . . refining this statement.

We will be interested in the **generic** cohomology $\mathrm{H}^\bullet(Y; \mathbf{C}_{\tau_{gen}\alpha})$.

Morse–Novikov cohomology

Y closed Riemannian manifold, $f\colon Y \to S^1$ a Morse function which can be lifted to a Morse function $F\colon \tilde{Y} \to \mathbf{R}$ on the cover. **Novikov complex**:

- $\mathcal{M}_\bullet$: $\mathbf{C}$-module generated by critical points of F. Deck transformations make it a $\mathbf{C}[t, t^{-1}]$-module.
- $\mathcal{N}_\bullet = \mathcal{M}_\bullet \otimes_{\mathbf{C}[t,t^{-1}]} \mathbf{C}(\!(t)\!)$.
- $d\colon \mathcal{N}_k \to \mathcal{N}_{k-1}$ the differential counting gradient flowlines.

Theorem (Novikov)

There is an isomorphism

$$\mathrm{H}^\bullet(\mathcal{N}_\bullet) \cong \mathrm{H}^\bullet(Y; \mathbf{C}_{\tau_{gen}\alpha}) \otimes_{\mathbf{C}} \mathbf{C}(\!(t)\!).$$

Vanishing cycles

Y complex manifold and $f\colon Y \to \mathbf{C}$ (not necessarily Morse), 0 is the only critical value. **Vanishing cycles**:

- $\mathrm{Crit}(f)$ is the critical locus.
- There is a perverse sheaf ϕ_f of vanishing cycles on $\mathrm{Crit}(f)$. If $\mathrm{Crit}(f)$ is smooth, this is (a shift of) the constant sheaf.

$$\Omega^\bullet_Y(\!(\hbar)\!)_f = (\Omega^\bullet_Y(\!(\hbar)\!), \hbar d + df \wedge (-)).$$

Theorem (Sabbah–Saito)

There is an isomorphism

$$\mathrm{H}^\bullet(Y, \Omega^\bullet_Y(\!(\hbar)\!)_f) \cong \mathrm{H}^\bullet(\mathrm{Crit}(f), \phi_f) \otimes_{\mathbf{C}} \mathbf{C}(\!(\hbar)\!).$$

Let Y be a Kähler manifold and $f : Y \to \mathbf{C}$ a holomorphic function. Then we get $\mathcal{N} = 4$ supersymmetry and so *a family* of TQFTs (parametrized by $\hbar \in \mathbf{C}$) obtained via supersymmetric twisting (see the paper with B. Williams):

- For $\hbar = 0$ (*B-model*) the space of states is $\mathrm{H}^\bullet(\mathrm{Crit}(f), \mathcal{O})$.
- For $\hbar = 1$ (*A-model*) get the usual twisted supersymmetric quantum mechanics.
- The deformation from $\hbar = 0$ to $\hbar = 1$ corresponds to the **Batalin–Vilkovisky quantization** of the critical locus $\mathrm{Crit}(f)$.

Remark

Similar to supersymmetry enhancement of 2d SUSY σ-model with a hyperKähler target. Kapustin has explained its relationship to ordinary deformation quantization.

This gives a way to get good models for twisted spaces of states in many theories with extended supersymmetry.

Critical cohomology

Informal setup:

- Y is an ∞-dimensional Kähler manifold.
- α a closed holomorphic one-form on Y.
- X the zero locus of α which is "finite-dimensional".

Example

Y is the moduli space of G-connections (G a complex Lie group) on a closed oriented 3-manifold M and

$$X = \mathrm{Loc}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G$$

is the **character stack**.

Theory of d-critical structures of Joyce:

- X is a **d-critical locus**.
- **Orientation data** on X: the choice of square root of K_X^{vir}.
- Given an oriented d-critical locus, there is a perverse sheaf ϕ_X (modeled on ϕ_f).

The **critical cohomology** $\mathrm{H}^\bullet(X, \phi_f)$ has a mixed Hodge structure and a monodromy automorphism.

Remark

It is a categorified version of the theory of virtual fundamental classes on moduli spaces. The analogous structure to orientation data is an orientation of the determinant line.

Orienting character stacks

M a finite CW complex and G a complex algebraic group. Then $\mathrm{Loc}_G(M)$ has a natural derived enhancement $\mathbf{R}\mathrm{Loc}_G(M)$. A *volume form* (trivialization of K^{vir}) on $\mathbf{R}\mathrm{Loc}_G(M)$ induces an orientation data on $\mathrm{Loc}_G(M)$ (a square root of K^{vir}).

Theorem (Naef–S)

Suppose G is a unimodular algebraic group. Then $\mathbf{R}\mathrm{Loc}_G(M)$ has a natural volume form.

- The value at some point in $\mathbf{R}\mathrm{Loc}_G(M)$ is given by the adjoint Reidemeister torsion.
- For M a closed oriented surface, it coincides with the symplectic volume form (observation of Witten, using Poincaré duality for torsion).

In particular, the critical cohomology

$$\mathrm{H}^\bullet(\mathrm{Loc}_G(M), \phi_{\mathrm{Loc}_G(M)})$$

is well-defined. What is it?

Remark

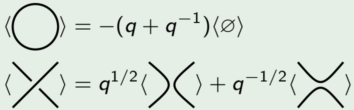
For $G = \mathrm{SL}_2(\mathbf{C})$ it is closely related to the complexified instanton Floer homology of Abouzaid–Manolescu.

Skein modules

M a closed oriented 3-manifold, G a connected complex semisimple algebraic group. The **skein module** $\mathrm{Sk}_G(M)$: the space $\mathbf{C}(\!(\hbar)\!)$ counting graphs ("Wilson lines") embedded in M with edges labeled by G-representations modulo local relations coming from quantum groups.

Example

The **Kauffman bracket skein module** (Przytycki, Turaev) is the $\mathbf{C}(\!(\hbar)\!)$-vector space $\mathrm{Sk}_{\mathrm{SL}_2}(M)$ spanned by isotopy classes of framed unoriented links in M modulo the relations



with $q = \exp(\hbar)$.

Conjecture (Proof in progress, with S. Gunningham)

There is an isomorphism

$$\mathrm{Sk}_G(M) \cong \mathrm{H}^0_{ren}(\mathrm{Loc}_G(M), \phi_{\mathrm{Loc}_G(M)}) \otimes_{\mathbf{C}} \mathbf{C}(\!(\hbar)\!).$$

Critical cohomology and deformation quantization

Given a Heegaard splitting $M = N_1 \cup_\Sigma N_2$ one can identify

$$\mathrm{Loc}_G(M) \cong (L_1 \cap L_2)/G,$$

where $L_1, L_2 \subset S$ are (G-equivariant) holomorphic Lagrangians in a holomorphic symplectic manifold $S \subset G^{2g}$.

Proposition (Gunningham–Jordan–S)

There is an isomorphism

$$\mathrm{Sk}_G(M) \cong \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2,$$

where $\mathcal{A}$ is an (algebraic) deformation quantization of S and $\mathcal{M}_1, \mathcal{M}_2$ are deformation quantizations of L_1 and L_2.

Question: what is the relationship between critical cohomology and deformation quantization?

Critical cohomology and deformation quantization

Kashiwara, Polesello, Schapira, D'Agnolo:

- For a holomorphic symplectic manifold S there is a canonical deformation quantization (algebroid) $\mathcal{W}_S$.
- For a holomorphic Lagrangian $L \subset S$ equipped with a choice of $K_L^{1/2}$ there is a canonical deformation quantization $\mathcal{W}_S$-module $\mathcal{M}_L$.
- For a pair of holomorphic Lagrangians $L_1, L_2 \subset S$ the derived tensor product

$$\mathcal{M}_{L_1} \otimes^{\mathbf{L}}_{\mathcal{W}_S} \mathcal{M}_{L_2}$$

  is a perverse sheaf on $L_1 \cap L_2$.

Remark

The corresponding category of holonomic $\mathcal{W}_S$-modules is related to microlocal perverse sheaves of Waschkies and Côté–Kuo–Nadler–Shende (Riemann–Hilbert correspondence) and the wrapped Fukaya category (Ganatra–Pardon–Shende).

But: $L_1 \cap L_2$ is a d-critical locus and it has an orientation if L_1, L_2 have a choice of $K_L^{1/2}$.

Theorem (Gunningham–S)

There is an isomorphism of perverse sheaves

$$\mathcal{M}_{L_1} \otimes^{\mathbf{L}}_{\mathcal{W}_S} \mathcal{M}_{L_2} \cong \phi_{L_1 \cap L_2} \otimes_{\mathbf{C}} \mathbf{C}(\!(\hbar)\!).$$

Questions

Conjecture (Proof in progress, with S. Gunningham)

There is an isomorphism

$$\mathrm{Sk}_G(M) \cong \mathrm{H}^0_{ren}(\mathrm{Loc}_G(M), \phi_{\mathrm{Loc}_G(M)}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar)).$$

Open questions:

- What is the meaning of the monodromy automorphism on $\phi_{\mathrm{Loc}_G(M)}$ in terms of $\mathrm{Sk}_G(M)$? (Defines an $\hbar$-connection, a shadow of the Langlands duality of 4d $\mathcal{N} = 4$ super Yang–Mills.)
- What is the meaning of the mixed Hodge structure on $\phi_{\mathrm{Loc}_G(M)}$ in terms of $\mathrm{Sk}_G(M)$? (Evaluation at q a root of unity?)