Calabi - Yau modularity and Feynman Graphs

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Albrecht Klemm

Bethe & Hausdorff centers Uni Bonn

International Group de Travail on differential equation (in Paris).

with Kilian Bönnisch, Fabian Fischbach

Christoph Nega and Reza Safari

[1] arXiv: 1912.06201

[2] arXiv: 2008.10574

[3] Work in Progress w. Claude Duhr

[4] " " " Janis Dücker

I Feynman graphs and Calabi-Yau motives

There is growing evidence that Feynman integrals are quite generally related to (relative) period integrals, i.e. related to the moduli dependent pairing between parts of the (relative) cohomology

$H^*(W, F)$ and the (relative) homology $H_*(M, L)$ of (degenerate) algebraic varieties $\Rightarrow$ mixed variation of (mixed) Hodge structures or more generally motives

Growing evidence that Calabi-Yau motives play an important role: Preliminary dictionary

Complex moduli of W $\Leftrightarrow$ physical parameters of $\underline{F}\,\underline{G}$ external momenta and masses

dimension of W	$\leftrightarrow$	loop order of Feynman graph
Monodromy groups	$\leftrightarrow$	analytic properties
MHM degenerations	$\leftrightarrow$	high momentum regions
Periods over different $\Omega \subset H^*(W)$	$\leftrightarrow$	different master integrals for a class of FG
Gauss-Manin system	$\leftrightarrow$	differential relations between master integrals
$\hat{\Gamma}$ - classes	$\leftrightarrow$	(multiple) zeta values
Factorisation of Hasse Weil ζ-function	$\leftrightarrow$	special arithmetic values of Feynman integrals

integrality of instanton expansion $\longleftrightarrow$ modularity of Feynman integrals

? $\longleftrightarrow$ dimensional regularisation

$\varepsilon = \frac{1}{2}(D - D_0)$, $D_0 = 4$

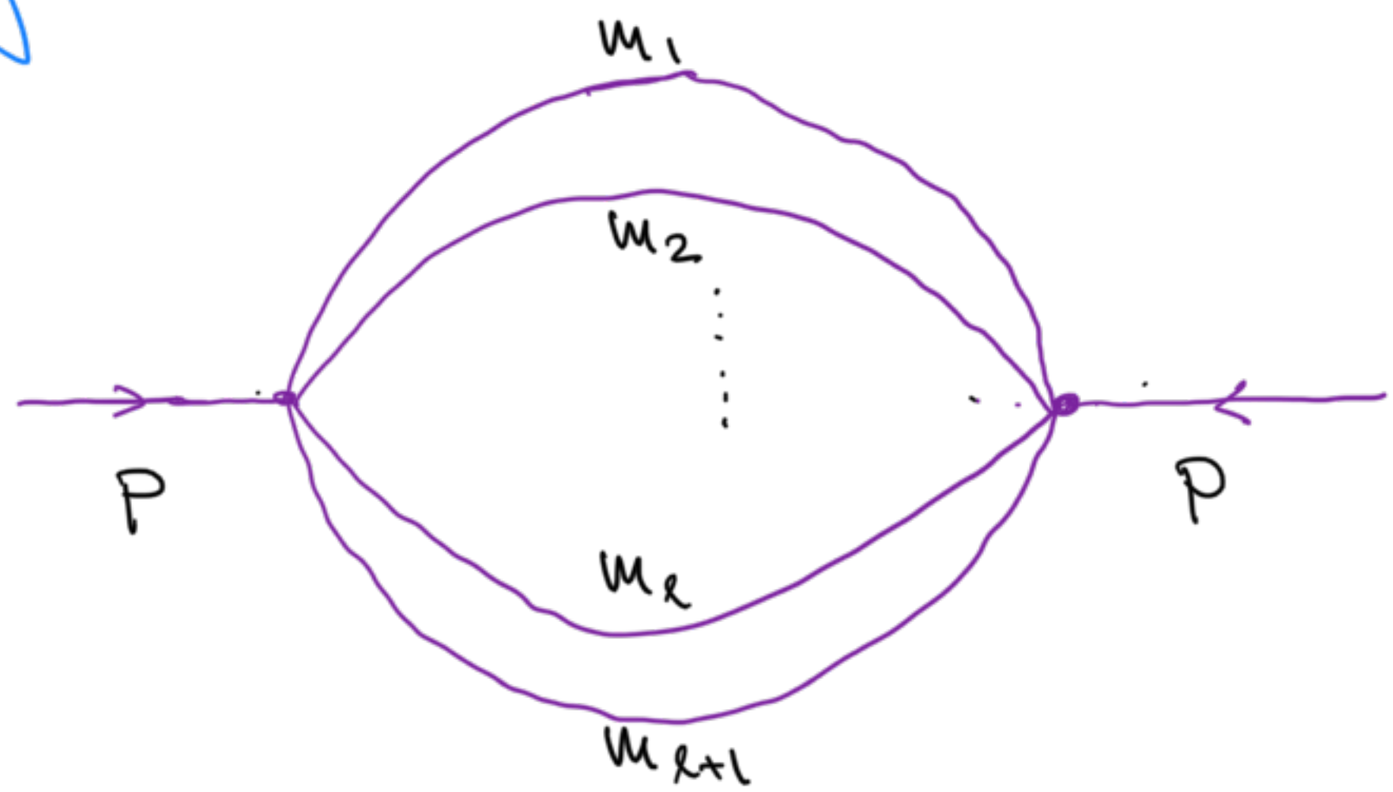
Integrality structure mirror map Instanton structure $\longleftrightarrow$? $l=2$ elliptic Fcts modularity

? $\longleftrightarrow$?

? $\longleftrightarrow$?

Aim of this talk is to show some of these exiting relations for

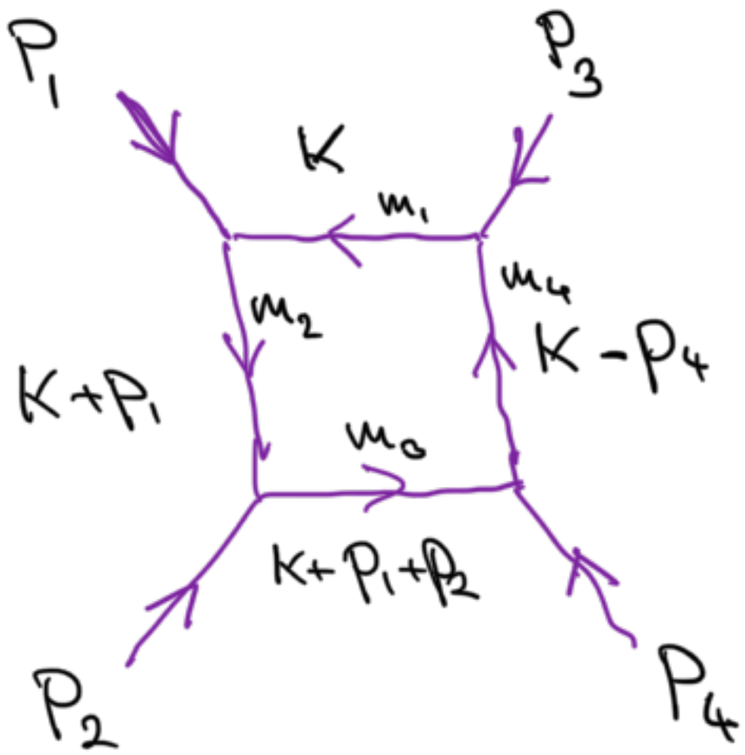
a simple infinite family of higher loop diagrams, the Banana Feynman integrals



ℓ- loop diagram with 2 vertices and $\ell+1$ propagators with different masses m_j $j=1,\dots,\ell+1$ and external momentum p

General Feynman graph evaluation:

Generic Feynman graph evaluation:



$$\sum_i p_i = 0$$

$m_1 m_3 \ll m_2 m_4$

$\rightarrow$ 1-loop bancena

$$I_G = \int \frac{d^4k}{[k^2-m_1][(k+p_1)^2-m_2][(k+p_1+p_2)^2-m_3][(k-p_4)^2-m_4]}$$

needs dimensional regularization

$$D_0 = 4 \rightarrow \underline{D = D_0 - 2\varepsilon}$$

Generall structure

$$= \Gamma\left(\frac{\ell}{\ldots}\right. \quad d^D k \quad \gamma_E \varepsilon\Big) \frac{n}{\ldots} \quad \frac{1}{\ldots}$$

$$I_G = \int \left(\prod_{j=1} \frac{\quad}{(i\pi)^{D/2}} \right. e \qquad \prod_{e \text{ edges}} \frac{\quad}{[q_e^2 - m_e^2]^{\nu_e}}$$

regulated integral

~ propagators

$\nu_e \in \mathbb{N}$

edge multiplicity

$\Leftrightarrow$ different master integrals

$\vdots$

Feynman integral representations

$n = l+1$

$$I_G(\underline{p}, \underline{m}) \sim \int_{\Omega_{n-1}} \mu_{n-1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{G^{\omega}(\underline{p}, \underline{m})} \prod_{i=1}^{n} x_i^{\nu_i - 1} \qquad (1)$$

$$\omega = \sum_{i=1}^{n} \nu_i - \frac{lD}{2} \qquad \#\text{vert} = n - l + \chi - 1$$

Integration domain:

$$\sigma_{n-1} = \{ [x_1 : \ldots\ldots : x_n] \in \mathbb{P}^{n-1} \mid x_i \in \mathbb{R}_{\geq 0}, \forall i \}$$

Measure: top form on $\mathbb{P}^{n-1}$

$$\mu_{n-1} = \sum_{k=1}^{n} (-1)^{k+1} x_k \, dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n$$

Observation from Gel'fand, Kapranov and Zelevinskiy (1989) that "practically all integrals that arise in perturbative QFT" are related to A-Hypergeometric functions

GKZ - systems

of rational functions in toric varieties

II The family of Banana graphs

For the ℓ-loop Banana Feynman Integral (1) simplifies to

$$I_\ell(p,\underline{m}) \sim \int_{\sigma_\ell} \frac{U_\ell^{\ell+1-\frac{\ell+1}{2}D}}{G_\ell^{\ell+1-\frac{\ell}{2}D}} \mu_\ell$$

with $$U_\ell = \left(\prod_{i=1}^{\ell+1} x_i\right)\left(\sum_{i=1}^{\ell+1} \frac{1}{x_i}\right)$$

$$G_\ell = \left(\prod_{i=1}^{(\ell+1)} x_i\right)\left[\underbrace{t - \left(\sum_{i=1}^{\ell+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{\ell+1} \frac{1}{x_i}\right)}\right]$$

$$t = \frac{p^2}{\mu^2} \qquad \xi_i = \frac{m_i}{\mu}$$

μ mass scale

P_{Δ_ℓ}

Laurent polynomial of a reflexive Newton Polyhedron Δ_ℓ

In particular for $\boxed{D_0 = 2}$ in leading order in ε^0 the integrand becomes

deformation of inner point

$$\frac{\mu_\ell}{P_{\Delta_\ell}} \Rightarrow \oint_{S^1} \frac{u\,\mu_\ell}{P_{\Delta_\ell}} = \Omega \quad (\ell-1)\text{-form}$$

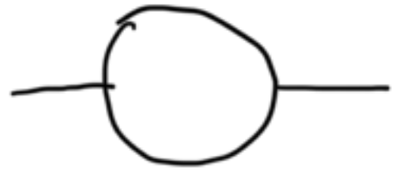
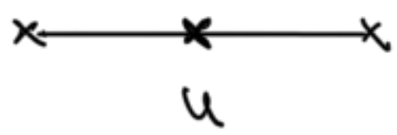
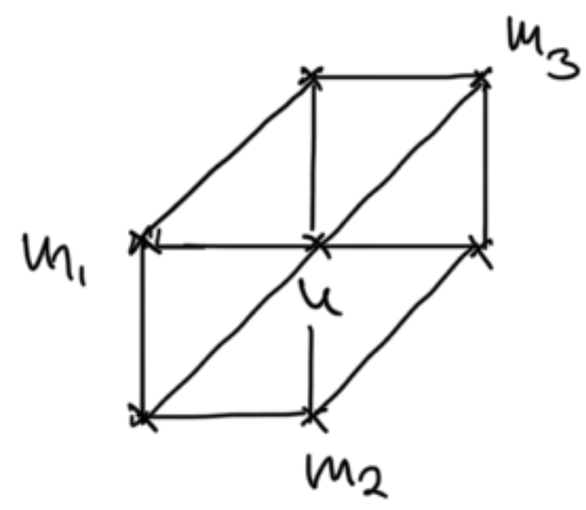
of a Calabi-Yau $(\ell-1)$ fold $W_{\ell-1}$

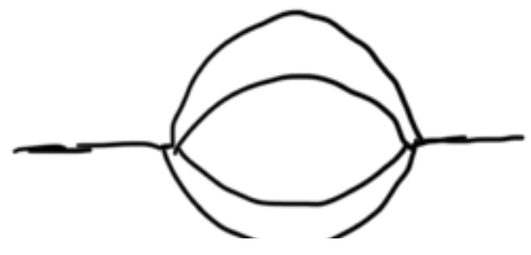
Batyrevs construction for pairs of reflexive polyhedra $(\Delta_\ell, \Delta_\ell^*)$ we

Can construct mirror pairs of CY manifolds

$$W_{\ell-1} = \{ P_{\Delta_\ell} = 0 \subset \mathbb{P}_{\Delta^*_\ell} \} \longleftrightarrow \{ P_{\Delta^*_\ell} = 0 \subset \mathbb{P}_{\Delta^*_\ell} \} = \mathcal{M}_{\ell-1}$$

Mirror - Symmetry

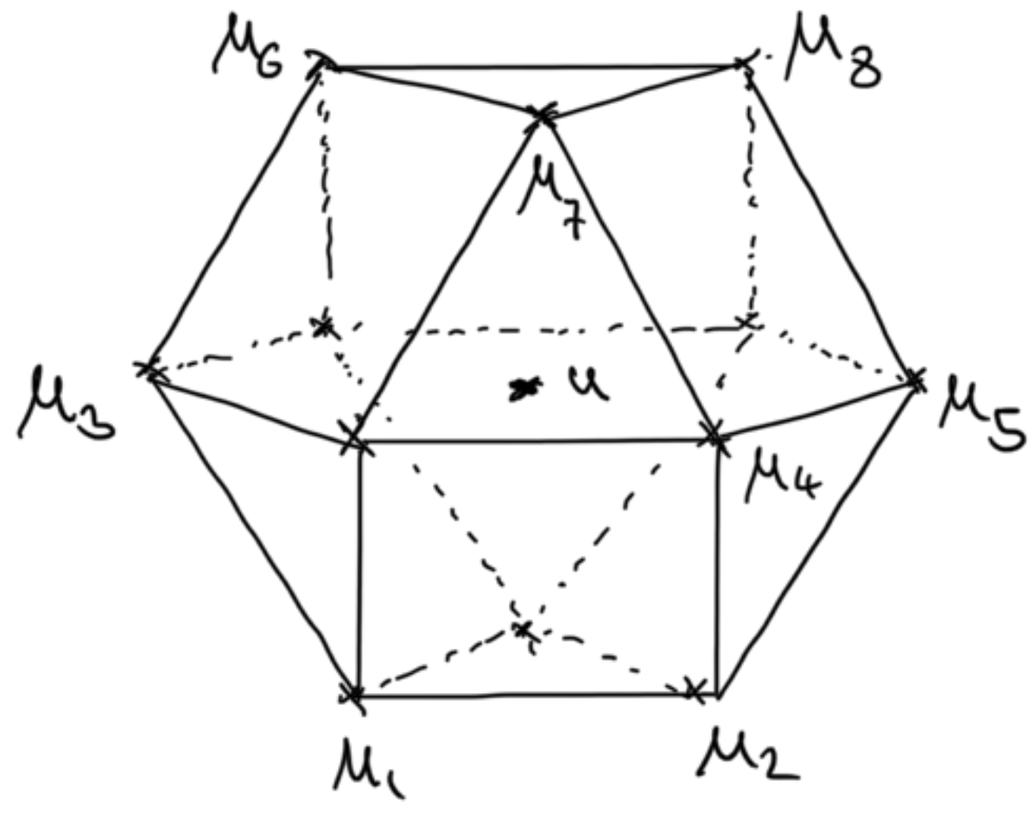
	1-loop	2-loop
Graph		
Δ_ℓ		
$W_{\ell-1}$	2 pt	elliptic curve with 1 modulus 3 residue values

	3-loop
Graph	

$u = t - \bar{\tau} e^2$

$u = t - 4si$

Δ_e



W_2 K3 surface with 9 complex structure def

$l=2$ case S. Bloch, M. Kerr and P. Vanhove (2016)

<u>massive case</u> Griffith reduction method

III Inhomogeneous Gauss-Manin System for the Banana Graphs

[1] [2] We solved the $l=3,4$ loop massive case using the GKZ system

Generally one proceeds with the following steps

<u>Step 1</u>: Determine the Gauss-Manin connection and solve for the periods

$$\Pi_k = \int_{\Gamma_k} \Omega(t, \underline{\xi})$$

Γ_k some basis of $H_{l-1}(W_{l-1}, \mathbb{C})$

$\Omega \in H^{l-1,0}(W_{l-1})$. [1] [2] use started with the GKZ-system and

derived the Picard-Fuchs D-module $\{\hat{\underline{D}}\}$, i.e a complete set of linear differential operators characterizing Π_K by the homogeneous diff eqs

$$\hat{D}_s \Pi_K = 0 \qquad s=1,\dots,\mathrm{rank}(\{\hat{D}\})$$

<u>Step 2</u>: Apply the $\hat{D}_s$ to the chain integral (1) (Note σ_ℓ is not closed) to $u \cdot I_\ell(t,\xi)$ to get the inhomogeneity. The corresponing integrals can be performed at least numerically. $\Rightarrow$

Actual Feynman graph fall fill a system of inhomogeneous diff eqn

(2) $$D_s \, I_\ell(t, \underline{\xi}) = \sum_i{}' \, p_i(\underline{z}) \, q_i(\log(\underline{z})) \qquad s = 1 \ldots \mathrm{Rank}(D)$$

In the <u>large</u> radius coordinates z_j of the Mcem-point, with p_i, q_i polynomials.

<u>Step 3</u>: Find the actual basis of solutions that corresponds to the Feynman integral in terms of local Frobenius bases ω_k^{loc}

$$I_\ell(t,\underline{\xi}) = \sum_k{}' \lambda_k^{loc} \Theta_k^{loc}(\underline{z}^{loc})$$

$\lambda_k^{loc} \in \mathbb{C}$

local basis of sol to (2)

For this we found the $\hat{\Gamma}$-class of $I_\ell(t,\underline{\xi})$. Was latter confirmed in [G] 2011.0591 math Alg.

This might become generally a very useful tool!

IV A better Calabi-Yau motive

A problem with the naive geometric description $W_{\ell-1} = \{P_{\Lambda_\ell} = 0 \subset \mathbb{P}_{\Lambda^*}\}$

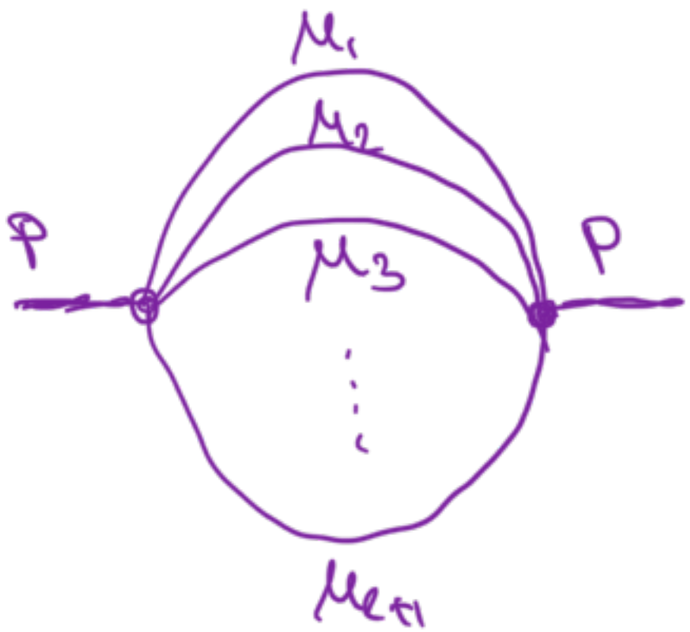
CY Fano

is the <u>great</u> <u>redundancy</u> of the geometric moduli and (co)homology groups

Physical parameters : P^2, M_i $\# = l+1$

complex moduli : $\dim(W_{l-1}, TW_{l-1}) = h^{l-3,1} = l^2$

The better motive for the Banana graphs is the primitive vertical quantum cohomology of a infinite series of Calabi-Yau spaces



$$\Longleftrightarrow \quad W_{\ell-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{\ell+1} & 1 & 1 \end{array} \right) \subset \left(\begin{array}{c||c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{\ell+1} & 1 \end{array} \right) = F_\ell$$

<u>Calabi-Yau</u>

<u>ℓ-1 fold</u>

2 deg (1,...,1) constraints

<u>Fano</u>

1 deg (1,...,1) constraint

In particular in the large volume region the complexified Kähler parameter of the $\ell+1$ $\mathbb{P}^1$'s can be identified with the logarithm of the physical param

$$\frac{\log(Z_K)}{2\pi i} = \frac{\log}{2\pi i}\left(\frac{m_K^2}{p^2}\right) = T^K = \frac{1}{2\pi i}\int_{\mathbb{P}^1_K} (i\omega - b) + \mathcal{O}(e^{-T})$$

b-field

(1,1)
Kähler form

$$Z_K = 0 \iff p^2 \gg m_K^2$$

Mean point

This yields the solution ω_K without redundancy in terms of the GKZ system of the complete intersection CY We-1 analysed in [G] Hosono, Klemm Theisen, Yau 9406055

A-Hypergeometric system with ℓ-vector

$$\ell^{(k)} = \left(\tilde{\ell}^{(k)}_1, \dots, \tilde{\ell}^{(k)}_2 ; \ell^{(k)}_1, \dots, \ell^{(k)}_{\underset{\overset{\shortparallel}{2(\ell+1)}}{c}}\right) \qquad \hat{\ell}^{(k)}_j, \ell^{(k)}_j \in \mathbb{Z}$$

$$\ell^{(1)} = (-1, -1 ; 1\ 1\ 0\ 0\ \dots\dots\ 0\ 0)$$

$$\ddots$$

$$\ell^{(\ell+1)} = (-1, -1 ; 0\ 0\ 0\ 0\ \dots\ 1\ 1)$$

$\Rightarrow$ All solutions from Frobenius deformation method from:

$$\omega_0(\underline{z}, \underline{s}) = \sum_{u_1, \dots, u_{\ell+1} \geq 0}' C(\underline{u} + \underline{s})\, \underline{z}^{\underline{u} + \underline{s}}$$

with

$$C(\underline{u} + \underline{s}) = \frac{\prod_{j=1}^{2} \Gamma\left(-\sum_{k=1}^{\ell+1} \tilde{\ell}^{(k)}_j (u_k + s_k) + 1\right)}{\prod^{2\ell+2} \Gamma\left(\sum_{k=1}^{\ell+1} \ell^{(k)} (u_k + s_k) + 1\right)}$$

$$\prod_{i=1} \left(\sum_{k=1} \lambda_i^{(k)} (u_k + \rho_k) + 1 \right)$$

Structure of solutions :

1 hdom solution

$$\Theta_0(\underline{z}) = \omega_0(\underline{z}, \underline{\rho})\Big|_{\underline{\rho}=0} \qquad s = \dim\left(H_{kk}^{prim}\right)$$

k-th order logarithmic solution $k = 1, \dots, l-1$

$$\Theta_k^{(s)} = \sum_{\{i_1 \dots i_k\}} a_{i_1 \dots i_k}^{(s)} \frac{\partial^k}{\partial \rho_{i_1} \dots \partial \rho_{i_k}} \omega_0(\underline{z}, \underline{\rho})\Big|_{\underline{\rho}=0}$$

- $a_{i_1 \dots i_k}$ are easily determined by the classical intersection numbers of the order k chow ring elements

generating the primitive homology of $W_{\ell-1} \subset F_\ell$. In addition one has an inhomogeneous solution of order ℓ in $\log(z)$ ... that we call $\Theta_\ell^{(1)}$

The $\hat{\Gamma}$-class of the Feynman integral

To fix the actual basis of the above solution we need some geometric understanding of the Feynman Integral. The relevant real part of the Feynman integral is

determined by the $\hat{\Gamma}$-Class

$$(3) \quad I_\ell(T(p,m)) = \int_{F_\ell} e^{\omega T} \frac{\Gamma(1-c_1)}{\Gamma(1+c_1)} \cos(\pi c_1) + \mathcal{O}(e^T)$$

The imaginary part of the Feynman integral corresponds precisely to the linear combination with corresponds to the vanishing cycle at the nearest conifold

$$(4) \quad \nu_\ell(T(p,\underline{m})) = \int_{W_{\ell-1}} e^{\omega T} \hat{\Gamma}(TW_{\ell-1}) + \mathcal{O}(e^T)$$

From this we can find the precise linear combination

$$I_\ell(p,m) = \sum_{k=0}^{\ell-1} \sum_{s=1}^{h^{1,1}_{vert}-1} \lambda_{k,s} \, \Theta^{(s)}_k$$

For example one finds

ℓ	$\lambda^{(\ell)}_{0,1}$	$\lambda^{(\ell)}_{1s}$	$\lambda^{(\ell)}_{2s}$	$\lambda^{(\ell)}_{3s}$	$\lambda^{(\ell)}_{4s}$
2	$18\zeta(2)$	$2\pi i$	1		
3	Euler number of K3 $\downarrow$ $-16\zeta(3)+24\pi i\zeta(3)$	$-18\zeta(2)$ $-18\zeta(2)$ $-18\zeta(2)$	$-2\pi i$	1	

		$-18\zeta(2$			
		$16\zeta(3)-24\pi i\zeta(2)$	$6\zeta(2)$		
		"	$6\zeta(2)$		
4	$-450\zeta(4)-80\pi\zeta(3)i$	"	$6\zeta(2)$	$2\pi i$	
		"	$6\zeta(2)$		
		"	$6\zeta(2)$		

Euler number of Barth-Nieto quintic

The reason that we calculate so simply is that all Chern classes c_i have to calculate

$$\hat{\Gamma}(\mathcal{F}) = \exp\Big(\sum_{k\geq 2}{}' (-1)^k (k-1)!\, \zeta(k)\, ch_k(\mathcal{F})\Big)$$

$$ch_k = (-1)^{(k+1)} k \left[\log\left(1 + \sum_{i=1}^{\infty} c_i x^i\right) \right]_k$$

take x^k coefficient

are by the adjunction formula very simple

$$c_k(W_{\ell-1}) = (-1)^k k! \sum_{j=0}^{k-1} \frac{(-2)^j (k+1-j)}{j!} S_k(\underline{H})$$

$$c_k(F_\ell) = (-1)^k k! \sum_{j=0}^{k-1} \frac{(-2)^j}{j!} S_k(\underline{H})$$

symmetric k-order polynomial of Hyperplane classes in $\mathbb{P}^1$

<u>Remark</u>: I) The topological data

of the underlying Calabi-Yau model appear in the leading logarithms of the high energy limit of the Feynman graph!

<u>Remark II</u>) This gives the full analytic structure of this family of l-loop integrals.

This due an 30 year study of the Banana-graph in the low energy limit

Broadhurst, Laporta and Remiddi,

Bauberger, Schuster, Weinzierl, Bogner, Verdipp

Kowalski Saleur, Weinzierl, Bogner, Vanhove

$$I_\ell(t,\underline{m}) = 2^\ell \int_0^\infty z\, I_0(\sqrt{t}\, z) \prod_{k=1}^{\ell+1} K_0(\xi_k z)\, dz$$

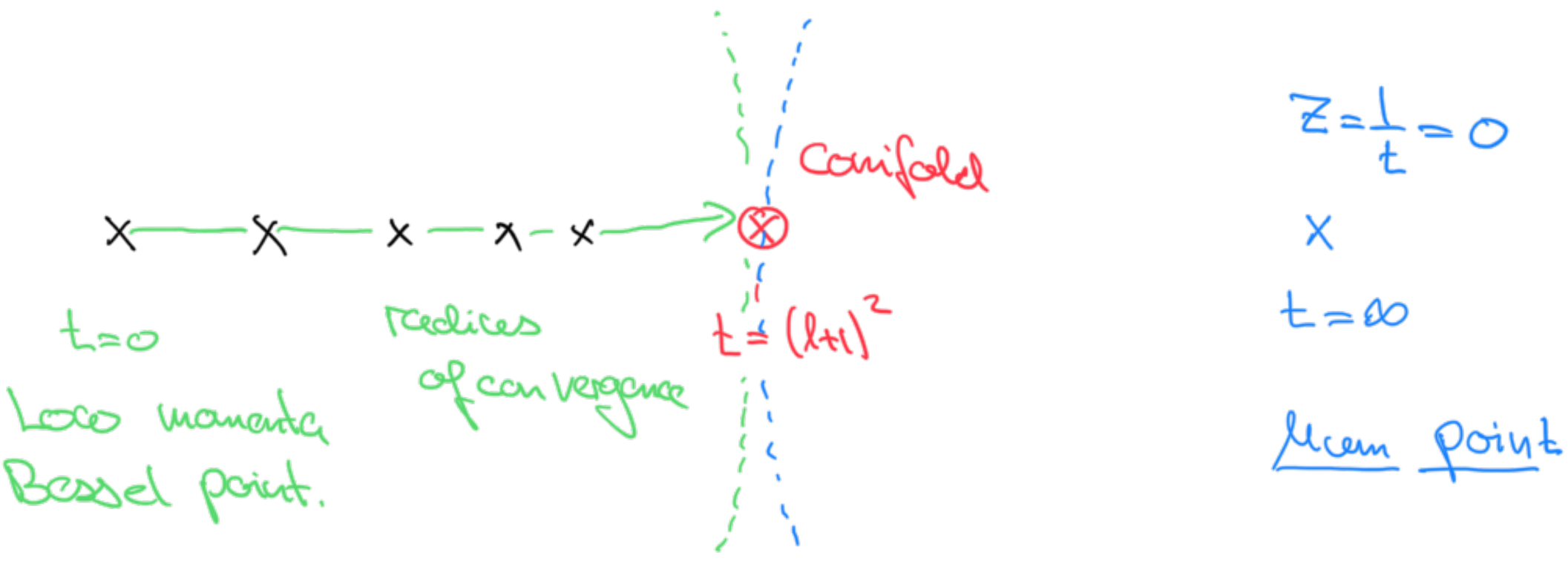
$$\sqrt{t} < \sum_{k=1}^{\ell+1} \xi_k$$

Bessel function

Analytic structure : $m_k = \mu \quad \forall k \Rightarrow$

$\xi_k = 1$

One modulus case :



New analytic

Exact analytic

expression of vanishing cycle at conifold

solutions in the large momentum region

<u>Remark III</u> We had calculated in $D_0 = 2$. How is this relevant to 4-dim quantum field theory?

General Fact: There is a difference relation between $D_0 \to D_0 \pm 2$ that one can use if one knows all master integrals

$$I_3(D-2,z) = \frac{4}{(D-6)^2(z+1)(z+9)}\Big[(D-3)\big[3(D-2)+2(D-3)z\big]$$

$$\cdot\, I_3(D,z) - 2(D-3)\frac{d}{dz}S(D,z) + \frac{3(z+3)}{4(D-4)^2}\Big]$$

Masterintegral

shift of the ε-poles

Here the differential is determined by the U-polynomial and one needs the full set of master integrals [3] Böhmisch, Duhr, Fischbach

Nega, A.K.

$\Updownarrow$

not too hard

choosing of $W_{\ell-1}$

Remark IV Arithmetic properties and Calabi-Yau modularity

(i) Detect modular properties in the instanton expansion at large radius

$q = e^{2\pi i \tau}$

$\xi_k = 1$

$$I_3(T) = \text{r-class-term} + \sum_{k=1}^{\infty} n_k \, Li_3(e^{kT})$$

Corresponds to Fano F_3

$n_k = \{48, 36, 32, 48, 48, 24, 48, 48, 32, 36, 48, 32$

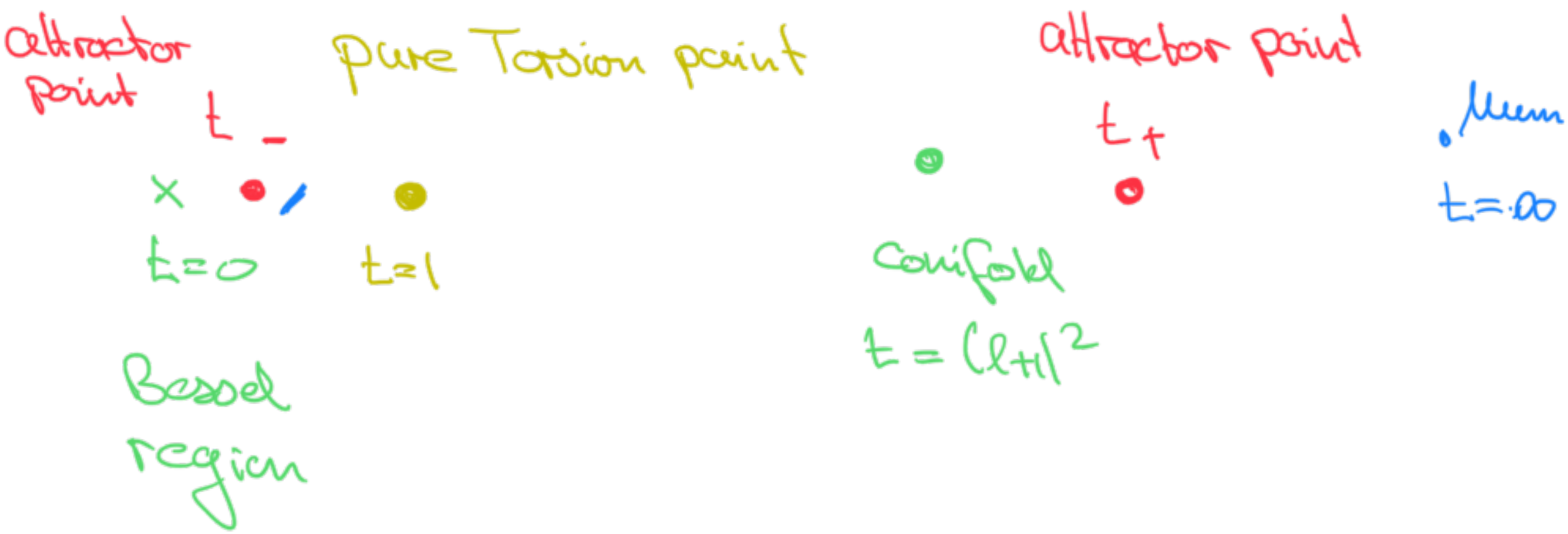
$+ \text{periodic} \}$

$$\Rightarrow \left(\frac{d}{dt}\right)^3 I_3 = \frac{1}{5}\Big(121 + E_4(q) - 2E_4(q^2) - 9E_4(q^3)$$

$$+ 16E_4(q^4) + 18E_4(q^6) - 144E_4(q^{12})\Big)$$

$k=4$ form of $\Gamma_0(12)$ .....

(2) Arithmetic (equal mass case)



$$\left.\begin{array}{l} t=1 \\ t_\pm = (33 \pm 8\sqrt{17}) \\ \quad (\ell=4) \end{array}\right\}$$

are algebraic values in the moduli where the Hasse Weil Zeta function factorizes

For example at the attractor point the numerator of the Hasse Weil Zeta function becomes [7] Candelas, de la Ossa Elmi, van Straten 1912.06146

$$P_3(\mathcal{M}/\mathbb{F}_p, T) = (1 - a_p(pT) + p(pT)^2)(1 - b_p pT + p^3T^3)$$

$a_p, b_p \in \mathbb{Z}$ and according to the Weil conjectures $|a_p| \leq 2p$ $|b_p| \leq 2p^{3/2}$ moreover one finds that the a_p b_p coefficients are weight 2 and 4 Hecke eigenforms in $S_*^{new}(\Gamma_0(34), \chi_{43}(33,\cdot))$

In [2] we checked that the

modular periods $\omega^\pm$ $\omega^\pm$

modular quasi periods $\eta^\pm$ $\eta^\pm$

determine all maximal cut master integrals exactly!