Emanant symmetries

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IAS

Meng Cheng and NS, arXiv:2211.12543

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NS and Shu-Heng Shao, to appear

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From the UV to the IR

A common problem in physics is to find the long-distance (IR) behavior of a given short-distance (UV) theory.

One of the main tools is to constrain the possible answers by matching the global symmetries and their ’t Hooft anomalies.

The short-distance theory has a global symmetry group $\mathcal{G}_{UV}$.

Its subgroup $G_{UV} \subset \mathcal{G}_{UV}$ that does not act on the coordinates is the internal symmetry group.

The action of $\mathcal{G}_{UV}$ on the coordinates depends on whether the UV theory is in the continuum or on a lattice (and on the kind of lattice).

It is common that the IR theory is a continuum field theory.

It has a global symmetry $\mathcal{G}_{IR}$ and an internal symmetry group $G_{IR} \subset \mathcal{G}_{IR}$.

From the UV to the IR – comparing G_{UV} and G_{IR}

Often, the UV internal symmetry group G_{UV} differs from the IR internal symmetry group G_{IR}.

Every internal symmetry operator in the UV is mapped to a symmetry operator in the IR (homomorphism)

$$G_{UV} \to G_{IR}$$

Some UV symmetries are trivial in the IR (kernel)

New symmetries in the IR theory.

- Emergent (accidental) symmetries
  - Arise when the IR theory has no relevant, G_{UV}-preserving, but G_{IR}-violating operators (e.g., $B - L$ in the Standard Model, continuous rotation in lattice models).
  - The low-energy effective Lagrangian includes irrelevant operators that violate the emergent symmetries (e.g., proton decay or neutrino masses in the Standard Model).

From the UV to the IR – comparing G_{UV} and G_{IR}

$$G_{UV} \to G_{IR}$$

New symmetries in the IR theory.

- Emanant symmetries emanate from UV space symmetries, typically from UV translations. Unlike emergent symmetries:
  - There can be relevant operators violating the emanant symmetries, but they are not present in the low-energy effective Lagrangian (or Hamiltonian).
  - The low-energy effective Lagrangian does not include even irrelevant operators that violate the emanant symmetries.
  - The emanant symmetries are exact in the low-energy theory.
  - ‘t Hooft anomaly matching for emanant symmetries.

From the UV to the IR – comparing G_{UV} and G_{IR}

- Emanant symmetries
  - Examples in this talk (old wine in a new bottle):
    - Majorana chain
    - $1+1d$ lattice Ising model
    - $1+1d$ system with a global $U(1)$ symmetry with a chemical potential
    - Heisenberg Chain (XXZ model)
  - Many others

Majorana chain [many references]

A lattice with L sites and real periodic fermions χ_ℓ at the sites

$$\chi_\ell = \chi_{\ell+L} \ , \qquad \{\chi_\ell, \chi_{\ell'}\} = 2\delta_{\ell,\ell'}$$

Impose invariance under lattice translation and fermion-parity

$$T: \chi_\ell \to \chi_{\ell+1} \ , \qquad (-1)^F: \chi_\ell \to -\chi_\ell$$

Typical Hamiltonian $\quad H_+ = \frac{i}{2}\sum_{\ell=1}^{L} \chi_{\ell+1}\chi_\ell$

Add a fermion-parity defect (equivalently, use H_+ with anti-periodic boundary conditions) $\quad H_- = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell - \frac{i}{2}\chi_1\chi_L$

Most of our discussion is independent of the details of $H_\pm$.

H_+– periodic boundary conditions

H_- – like H_+ with anti-periodic boundary conditions, equivalently same as H_+ with a $(-1)^F$ defect.

Majorana chain [many references]

$$H_{\pm} = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_{\ell} \pm \frac{i}{2}\chi_1\chi_L$$

Most of our discussion is independent of the details of $H_{\pm}$.

Four fermionic theories:

- Even L. H_- leads in the continuum to the NSNS Majorana CFT and H_+ leads to the RR theory.
- Odd L. H_- leads in the continuum to the RNS theory Majorana CFT and H_+ leads to the NSR theory. (Here the quantization is notoriously confusing. Will comment about it below.)

Majorana chain – even $L = 2N$ [many references]

Typical Hamiltonians

$$H_\pm = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell \pm \frac{i}{2}\chi_1\chi_L$$

Symmetries generated by translation $T_\pm$ and fermion parity $(-1)^F$.

With appropriate phases in their definitions:

For H_-

$$T_-^L = (-1)^F$$
$$T_-(-1)^F = (-1)^F T_-$$

For H_+

$$T_+^L = 1$$
$$T_+(-1)^F = -(-1)^F T_+$$

[Rahmani, Zhu, Franz, Affleck; Hsieh, Hal´asz, Grover]

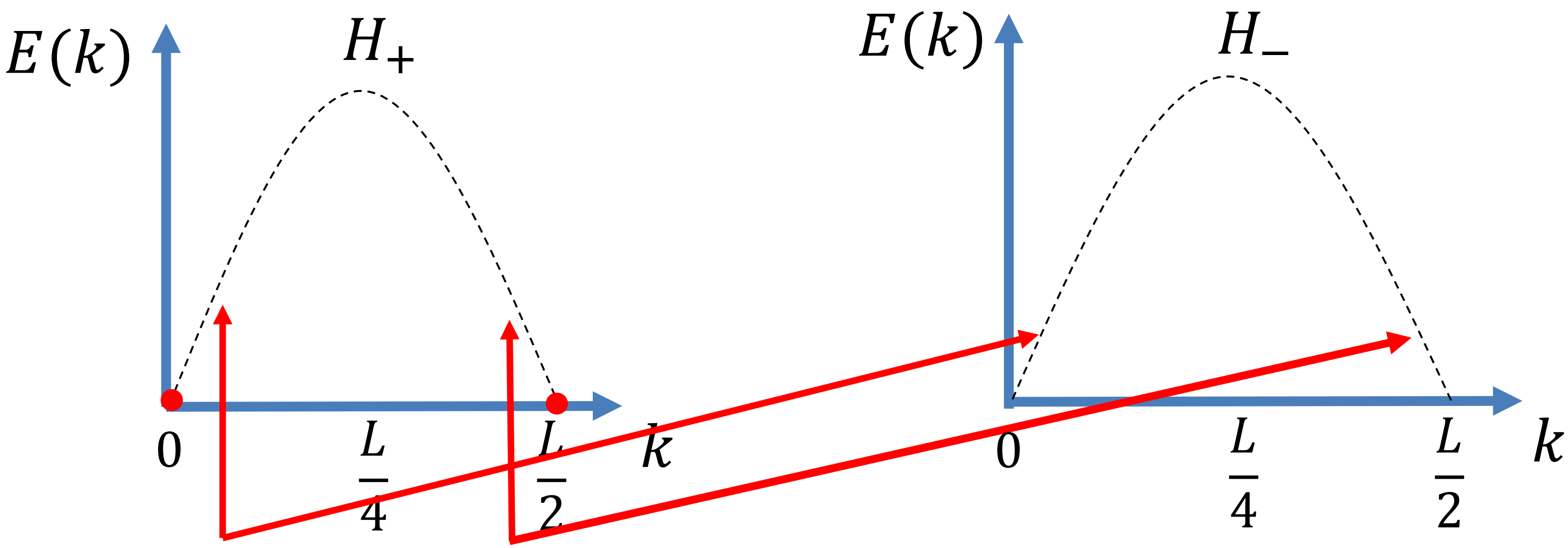
The minus sign reflects an anomaly between fermion-parity and lattice-translation.

In the continuum, no anomaly involving translations. How is this UV anomaly realized at low energies?

Majorana chain – even $L = 2N$ [many references]

$$H_\pm = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell \pm \frac{i}{2}\chi_1\chi_L$$

For the specific $H_\pm$, normal mode expansion leading to free fermions:



- Right-movers and left-movers from the two ends of the spectrum
- H_+ leads to the RR theory. H_- leads to the NSNS theory.
- On the lattice, only $(-1)^F$; no $(-1)^{F_L}$, $(-1)^{F_R}$.
- Without a chiral symmetry, why is the fermion massless?

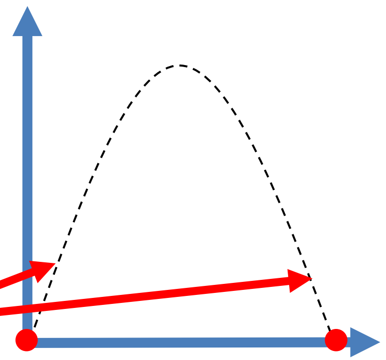
Majorana chain – even $L = 2N$

Consider H_+. On the lattice, no $(-1)^{F_L}$. In the IR, it emanates from T_+.

$$T_+^L = 1$$

$$T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$$

$$e^{2\pi i P_+} = 1$$



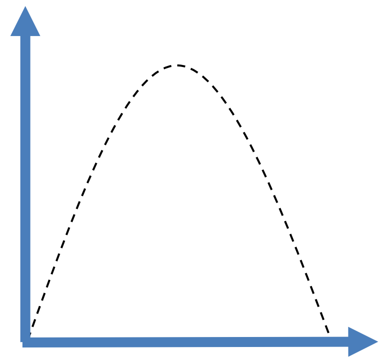
- P_+ is the momentum of the continuum RR theory.
- On the lattice, only T_+ is well-defined. In the continuum, $(-1)^{F_L}$ and P_+ are separately meaningful exact symmetries.
- The relation $T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$ is exact, without finite L corrections. Otherwise, cannot satisfy $T_+^L = 1$.
- Anomaly in the continuum RR theory [...; Delmastro, Gaiotto, Gomis; ...]

$$(-1)^F (-1)^{F_L} = -(-1)^{F_L} (-1)^F$$

It matches the UV fermion-parity/lattice-translation anomaly.

Similarly for H_-, except $e^{2\pi i P_-} = T_-^L = (-1)^F$

Majorana chain – even $L = 2N$



The $(-1)^{F_L}$ symmetry of the continuum theory is exact.

- It is not an emergent symmetry.
- It emanates from lattice translation.
- It explains why the fermion is massless without a UV chiral symmetry (a fermion mass is $(-1)^F$ invariant, but it violates lattice-translation and hence the emanant $(-1)^{F_L}$).
- The UV fermion-parity/lattice-translation anomaly is matched in the IR with an anomaly involving the internal symmetry $(-1)^F$ and the emanant $(-1)^{F_L}$.

Majorana chain – odd $L = 2N + 1$

Odd number of real fermions χ_ℓ.

An irreducible representation of the Clifford algebra has no $(-1)^F$.

Alternatives:

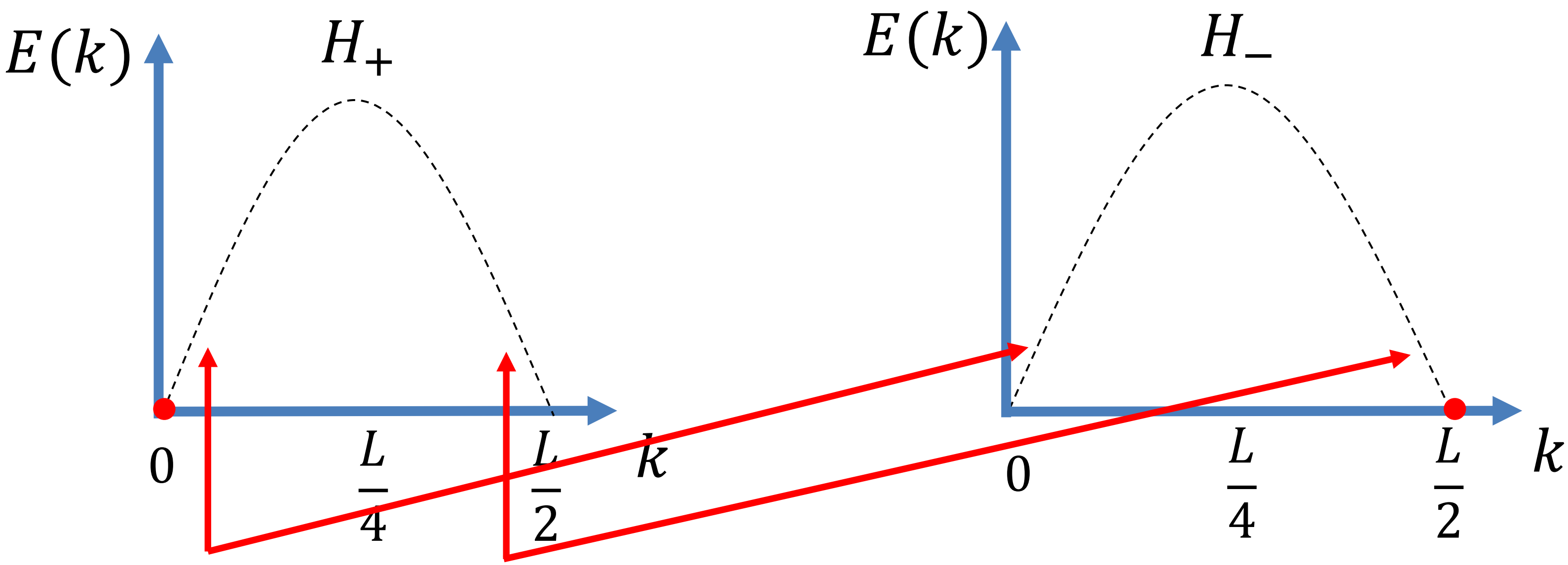
- Declare the problem inconsistent and ignore it.
- Canonical quantization
  - Give up on $(-1)^F$ and then the Hilbert space is not graded. We will follow this approach.
  - Add a decoupled fermion such that the total number of fermions is even and then define $(-1)^F$.
  - Either way, tensor products of such systems are confusing.
- Assume a standard tensor product and use the path integral to find that there is no Hilbert space interpretation, e.g., $\mathrm{Tr}\, 1 = \sqrt{2}$ [...; Delmastro, Gaiotto, Gomis; Witten; ...]

Majorana chain – odd $L = 2N + 1$

$$H_\pm = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell \pm \frac{i}{2}\chi_1\chi_L$$

Take an irreducible representation of the Clifford algebra.

No $(-1)^{F_L}$, $(-1)^{F_R}$, $(-1)^F$. Only lattice translation $T_\pm$, with an anomaly $T_\pm^L = e^{\mp\frac{2\pi i}{16}}$



- Right-movers and left-movers from the two ends of the spectrum
- H_+ leads to the NSR theory. H_- leads to the RNS theory.

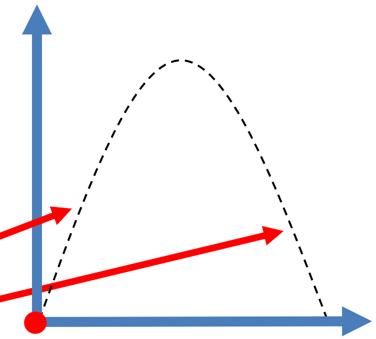
Majorana chain – odd $L = 2N + 1$

- No $(-1)^{F_L}$, $(-1)^{F_R}$, $(-1)^{F}$ on the lattice.
- Consider H_+. In the IR, $(-1)^{F_L}$ emanates from T_+

$$T_+^L = e^{-\frac{2\pi i}{16}}$$

$$T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$$

$$e^{2\pi i P_+} = (-1)^{F_L} e^{-\frac{2\pi i}{16}}$$



  - P_+ is the momentum of the continuum NSR theory.
  - On the lattice, only T_+ is well-defined. In the continuum, $(-1)^{F_L}$ and P_+ are separately meaningful exact symmetries.
  - The relation $T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$ is exact, without finite L corrections.
- For H_-, change $T_+ \to T_-$, $e^{-\frac{2\pi i}{16}} \to e^{\frac{2\pi i}{16}}$, $P_+ \to P_-$, $F_L \to F_R$. This is the RNS theory.

From the Majorana chain to the Ising model – GSO on the lattice

Sum over the "spin structures" by first doubling the Hilbert space (related work in [Baake, Chaselon, Schlottmann; Grimm, Schutz; Grimm; ...])

$$\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$$

with the Hamiltonian $$\widetilde{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

(H_+ corresponds to fermions with periodic boundary conditions. H_- corresponds to fermions with antiperiodic boundary conditions.)

Translation symmetry $$\tilde{T} = \begin{pmatrix} T_- & 0 \\ 0 & T_+ \end{pmatrix}$$

Because of the doubling of the Hilbert space, a quantum $\mathbb{Z}_2$ symmetry

$$\tilde{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the Majorana chain to the Ising model – even $L = 2N$

Some operators in the doubled Hilbert space $\widetilde{\mathcal{H}}$ are nonlocal. So imitating the continuum, we project:

- $\tilde{\eta}(-1)^F = +1$ leads to the Ising model $\widetilde{\mathcal{H}}|_{Ising} = \mathcal{H}_{Ising}$

  Using a Jordan-Wigner transformation in $\mathcal{H}_{Ising}$,

$$H_{Ising} = \widetilde{H}\Big|_{Ising} = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N} X_j X_{j+1}$$

  (X_j, Z_j are Pauli matrices at the site $j = 1, \cdots, N$)

- $\tilde{\eta}(-1)^F = -1$ leads to the $\mathbb{Z}_2$-twisted Ising model

  $H_{twisted\ Ising} = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N-1} X_j X_{j+1} + \frac{1}{2} X_N X_1$

From the Majorana chain to the Ising model – even $L = 2N$

Because of the anomaly, T_+ does not commute with $(-1)^F$.

Therefore, $\tilde{T} = \begin{pmatrix} T_- & 0 \\ 0 & T_+ \end{pmatrix}$ does not commute with the $\tilde{\eta}(-1)^F = +1$ projection and does not act in the projected Hilbert space $\tilde{\mathcal{H}}|_{Ising}$.

It is not a symmetry.

$\tilde{T}^2$ and $\tilde{\eta}$ act in $\tilde{\mathcal{H}}|_{Ising}$. Standard symmetries of the Ising model

$$T_{Ising} = \tilde{T}^2\Big|_{Ising}, \qquad \eta = \tilde{\eta}\Big|_{Ising}$$

Lattice-translation $\quad T_{Ising}^N = 1$

$\mathbb{Z}_2$ Ising symmetry $\quad \eta^2 = 1$

From the Majorana chain to the Ising model – even $L = 2N$

$\begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}$ commutes with the $\tilde{\eta}(-1)^F = +1$ projection and hence it acts in $\widetilde{\mathcal{H}}|_{Ising}$.

New symmetry of the lattice Ising model (related to Kramers–Wannier duality)

$$D = \begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}\Big|_{Ising}$$

It is a partial isometry (a unitary symmetry on the orthogonal complement of its kernel) – a noninvertible symmetry

$$D^2 = \frac{1}{2}(1+\eta)T_{Ising}$$

Can express D in terms of the local operators X_j, Y_j, Z_j and check explicitly that it commutes with the Hamiltonian.

From the Majorana chain to the Ising model – even $L = 2N$

The noninvertible lattice symmetry $D = \begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}|_{Ising}$ flows to a noninvertible symmetry of the continuum theory $\mathcal{D}$ [Oshikawa, Affleck; Petkova, Zuber; Frohlich, Fuchs, Runkel, Schweigert; Chang, Lin, Shao, Wang, Yin]

$$D = \frac{1}{\sqrt{2}}\mathcal{D}e^{\frac{2\pi i P}{2N}}$$

$$\mathcal{D}^2 = 1 + \eta \quad , \quad \eta^2 = 1 \quad , \quad \eta\mathcal{D} = \mathcal{D}\eta = \mathcal{D} \quad , \quad e^{2\pi i P} = 1$$

D and $\mathcal{D}$ satisfy different algebras, $D^2 = \frac{1}{2}(1+\eta)T_{Ising}$.

$\mathcal{D}$ is an emanant noninvertible symmetry. It is exact in the IR effective theory. (Not violated even by irrelevant operators.)

On the lattice, only T_{Ising} and D. In the continuum, P and $\mathcal{D}$.

The relation $D = \frac{1}{\sqrt{2}}\mathcal{D}e^{\frac{2\pi i P}{2N}}$ is exact. No finite N corrections.

From the Majorana chain to the Ising model – odd $L = 2N + 1$

In this case, no $(-1)^F$, and hence, no projection is needed.

A Jordan-Wigner transformation in the doubled Hilbert space $\widetilde{\mathcal{H}}$ leads to the Ising model with a D defect [Schutz; Grimm, Schutz; Grimm; Ho, Cincio, Moradi, Gaiotto, Vidal; Hauru, Evenbly, Ho, Gaiotto, Vidal; Aasen, Mong, Fendley]

$$H = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N} X_j X_{j+1} - \frac{1}{2} X_1 Y_{N+1}$$

It flows in the IR to the continuum Ising CFT with a noninvertible defect $\mathcal{D}$ [Oshikawa, Affleck; Petkova, Zuber; Frohlich, Fuchs, Runkel, Schweigert; Chang, Lin, Shao, Wang, Yin].

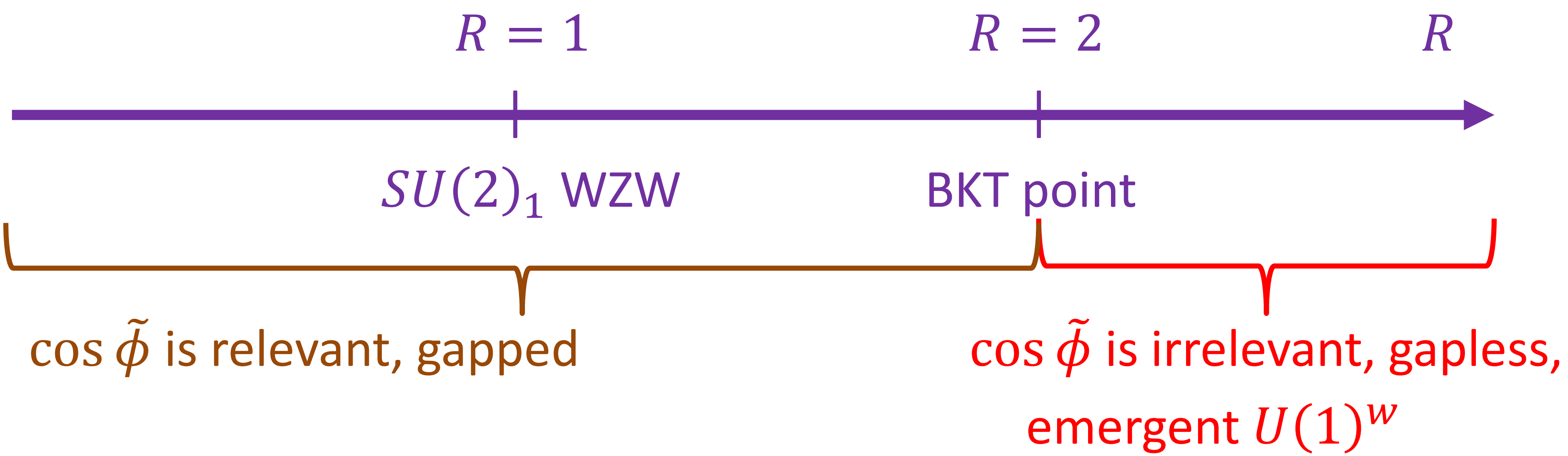
Explicit breaking of winding symmetry

Consider the $c = 1$ compact boson and use the dual field $\tilde{\phi} \sim \tilde{\phi} + 2\pi$

$$S = \int dt dx \left(\frac{1}{4\pi R^2} \left(\partial_\mu \tilde{\phi} \right)^2 + \lambda \cos \tilde{\phi} \right)$$

Only $U(1)^m$ (and discrete symmetries). No winding symmetry $U(1)^w$.

For $\lambda = 0$,



The emergent $U(1)^w$ symmetry is violated by irrelevant operators. (It is not an emanant symmetry.)

Add a chemical potential for $U(1)^m$

(related to [Haldane (1980)])

The charge is $Q^m = \frac{R^2}{2\pi}\int dx \partial_t \phi = \frac{1}{2\pi}\int dx \partial_x \tilde{\phi}$ and hence,

$$S = \int dtdx \left(\frac{1}{4\pi R^2}\left(\partial_t \tilde{\phi}\right)^2 - \frac{1}{4\pi R^2}\left(\partial_x \tilde{\phi} + \frac{2\pi q}{\boldsymbol{L}}\right)^2 + \lambda \cos \tilde{\phi} \right)$$

Space is a circle $x \sim x + \boldsymbol{L}$. The chemical potential is such that the total charge is near $q \in \mathbb{Z}$.

In terms of $\hat{\phi} = \tilde{\phi} + \frac{2\pi q x}{\boldsymbol{L}}$

$$S = \int dtdx \left(\frac{1}{4\pi R^2}\left(\partial_t \hat{\phi}\right)^2 - \frac{1}{4\pi R^2}\left(\partial_x \hat{\phi}\right)^2 + \lambda \cos\left(\hat{\phi} - \frac{2\pi q x}{\boldsymbol{L}}\right) \right)$$

For $\lambda = 0$, same as without the chemical potential (spectral flow).

For nonzero λ, the winding operator can be relevant or irrelevant. Either way, for large q, it oscillates rapidly in space. Hence, it does not act in the low-energy theory.

Finding an emanant symmetry

$$S = \int dtdx \left(\frac{1}{4\pi R^2} (\partial_t \hat{\phi})^2 - \frac{1}{4\pi R^2} (\partial_x \hat{\phi})^2 + \lambda \cos\left(\hat{\phi} - \frac{2\pi q x}{\boldsymbol{L}} \right) \right)$$

Now, the $U(1)^w$ winding symmetry is an emanant symmetry. It emanates from the UV translation symmetry:

- Operators that violate it, like $\cos\hat{\phi}$, are not invariant under translations of the UV theory. Hence, they are absent in the low-energy effective action.
- Operators that violate it, like $\cos\left(\hat{\phi} - \frac{2\pi q x}{\boldsymbol{L}}\right)$, are invariant under UV translations, but not invariant under translation of the IR theory. Hence, they have vanishing matrix elements between low-energy states.
- The emanant symmetry is not violated by relevant or irrelevant operators in the low-energy theory. It is exact. (Not an emergent (accidental) symmetry.)

XXZ model – Heisenberg chain [many references]

$$H_{XXZ} = 2\sum_{j=1}^{N}(X_jX_{j+1} + Y_jY_{j+1} + \lambda_z Z_j Z_{j+1})$$

X_j, Y_j, Z_j Pauli-matrices at site number $j \sim j + N$.

The global internal symmetry is $G_{UV} = O(2) \subset SO(3)$. For $-1 < \lambda_z \leq 1$, it is known to flow to the free compact boson with radius $R(\lambda_z) \geq 1$.

$\lambda_z = 1$ corresponds to the Heisenberg chain with $G_{UV} = SO(3)$. i.e., $R(1) = 1$.

The winding operator $\cos\tilde{\phi}$ preserves G_{UV} and depending on λ_z, it can be relevant. Why isn't it present in the low-energy Hamiltonian and gaps the system? For example, for $\lambda_z = 1$, there is an $SO(3)$-preserving relevant operator.

Why are these models robust (stable)?

The low-energy limit of the XXZ Chain

The model flows to

$$S = \int dtdx \left(\frac{1}{4\pi R^2} \left(\partial_\mu \tilde{\phi}\right)^2 + \lambda \cos\left(2\tilde{\phi}\right) \right)$$

with an emanant (not emergent) $\mathbb{Z}_2$ symmetry.

No term of the form $\cos(n\tilde{\phi})$ with odd n.

This emanant $\mathbb{Z}_2$ symmetry arises from lattice translation as

$$T = Ce^{2\pi i \frac{P}{N}}$$

P is the momentum of the continuum theory.

C generates a $\mathbb{Z}_2$ $e^{i\pi(Q^m + Q^w)}$ symmetry of the continuum theory.

On the lattice, only T is well-defined, but at energies of order $\frac{1}{N}$ (and therefore also in the continuum), P and C are separately well-defined.

The relation $T = Ce^{2\pi i \frac{P}{N}}$ is exact without finite N corrections.

Summary

- Microscopic translation (e.g., lattice translation) can lead to an emanant symmetry.
  - Unlike an emergent symmetry, it is exact at low energies – not violated by relevant or irrelevant operators.
- Four versions of the lattice Majorana chain flow to four continuum models, NSNS, RR, NSR, and RNS.
  - In each case, $(-1)^{F_L}$ (or $(-1)^{F_R}$) emanates from microscopic translation T. The microscopic $T/(-1)^F$ anomaly is matched in the IR by $(-1)^{F_L}/(-1)^F$ (or $(-1)^{F_R}/(-1)^F$) anomaly.
- Summing over the spin structures on the lattice leads to three lattice models: Ising, twisted Ising, and Ising with a D defect.
  - These models flow to the three continuum Ising models with defects. The noninvertible duality symmetry $\mathcal{D}$ of the CFT emanates from D.

Summary

- A system with a $U(1)$ global symmetry with chemical potential can have an emanant symmetry.
- Various lattice spin models, including the Heisenberg chain, lead to a $\mathbb{Z}_2$ emanant symmetry.
- Anomalies involving lattice translations are matched in the IR by anomalies in internal emanant symmetries. We exhibited it in:
  - the Majorana chain
  - Luttinger theorem and filling constraints (the example with a chemical potential)
  - Lieb-Schultz-Mattis type theorems (the example with the Heisenberg chain)

Thank you