Higher Central Charges and Gapped Boundaries

Or: beyond gravitational anomaly

To appear with

Justin Kaidi

Kantaro Ohmori

Sahand Seifnashri

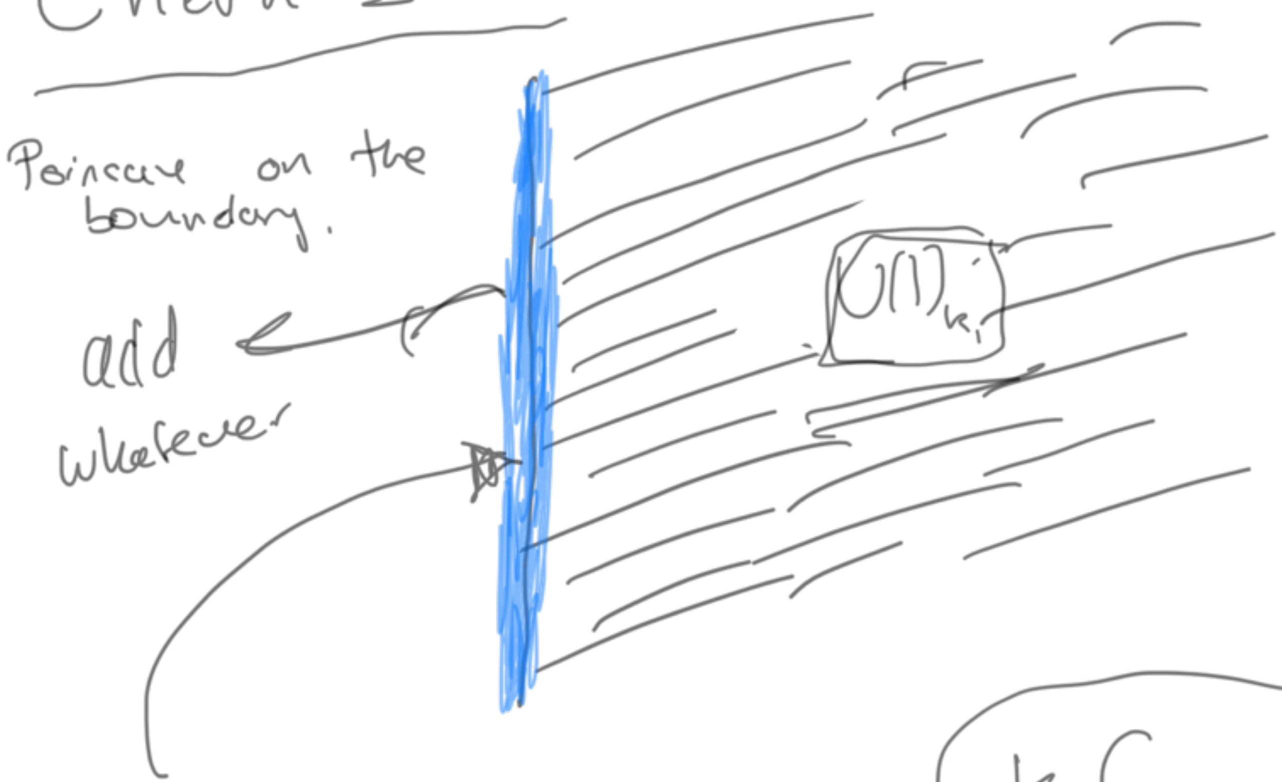
Shu-Heng Shao

$S^2 \to 1$

TFT

This talk is in 2+1 D

It's well known that there are edge modes in many Chern-Simons theories.



Chiral Boson Mode

a U(1) gauge field

$$\boxed{\frac{k}{4\pi}\int a\,da}$$

$\boxed{k \in 2\mathbb{Z}+1}$ spin

$\boxed{k \in 2\mathbb{Z}}$ non-spin

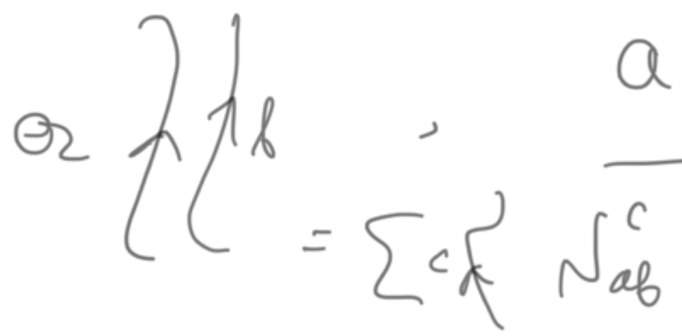
Bosonic

Theories

There are various possible b.c. but they all lead to gapless boundary

This talk: only $\boxed{\text{non-spin}}$ theory.

Suppose we are given some TFT (Topological Field Theory) which is non-spin.



$$a \otimes b = \sum N_{ab}^{c}\, c$$

Is there some choice of b.c. which is gapped?

By b.c. we allow to add dofs on the boundary and couple them.

contact terms

In particular some boundary theories only require an invertible bulk theory. These are 2d theories possibly with 't Hooft anomaly.

$\boxed{\text{chiral bosons moving on } E_8 \text{ lattice.}}$

$c_L - c_R = 8$

This theory has a standard gravitational anomaly

$$\frac{1}{12\pi}\int_{3D} \underline{\mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right)}$$

A.G - W

$\boxed{c_- = 0 \mod 8}$

Doesn't require bulk anyons and it corresponds to a standard 2D Theory perhaps with 't Hooft anomaly (invertible 3D).

[Alvarez-Gaume - Witten]

Therefore, given a <u>theory of anyons</u> $\{a\}$ $*$ 1 is transparent

we can compute

$\boxed{c_L - c_R \equiv c_- \mod 8}$

and if $c_- \neq 0 \mod 8$ then there must be gapless boundary modes due to an unavoidable gravitational anomaly.

MTC

TFT

$$\boxed{e^{2\pi i c_-/8} = \frac{1}{\mathcal{D}} \sum_a' d_a^2 \, \theta_a}$$

$$\mathcal{D} = \sqrt{\sum_a' d_a^2}$$

d_a quantum dim

θ_a spin.

$\boxed{\text{If } c_- = 0 \mod 8}$, it'd seem we can just add chiral bosons on E_8 lattice on the boundary, remove the gravitational anomaly ($c_- = 0$) completely, and hence flow to a gapped boundary.

$T \simeq T'$ $\quad$ $T \times \bar{T}$ $\quad$ $U(1)_2 \times U(1)_{-4}$

$c_- = 0$

Quite ~~amazingly~~,

$$\boxed{\text{There are theories of anyons with } \boxed{c_- = 0 \mod 8} \text{ which have no gapped boundary}}$$

... "anomaly"

c_- mod 8 is the only anomaly that is known to protect boundary modes. But we see it's not enough!

$\boxed{\text{Witt group}}$

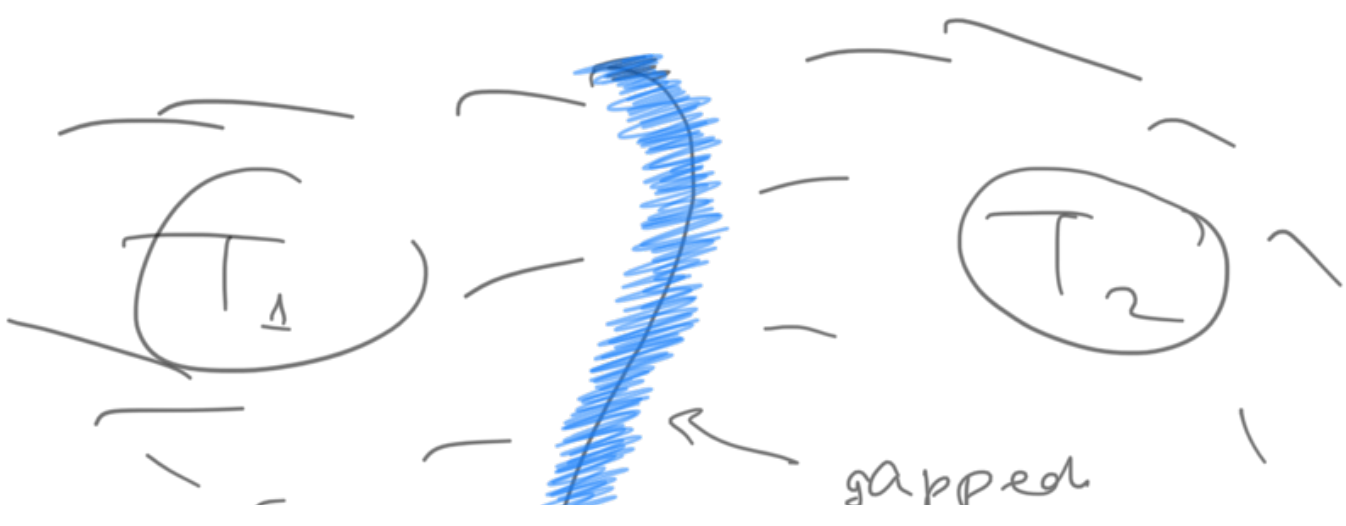
Davydov, Müger, Nikshych, Ostrik,
Davydov, Nikshych, Ostrik
⋮

In fact $c_- = 0$ mod 8 isn't enough even for Abelian theories!

$U(1)_2 \times U(1)_{-4}$	no gapped b.c. $c_- = 0$
$U(1)_2 \times U(1)_{-8}$	$\exists$ gapped b.c. $c_- = 0$
$U(1)_2 \times U(1)_{-2}$	$\exists$ gapped b.c.

Note $\quad T \times \bar{T} \leftarrow$ gapped b.c

$T \simeq T$



A geometric picture:

$\mathbb{R}^3$

cut out a solid torus $\simeq D^2 \times S^1$

put gapped = topological b.c. on the surface $\mathbb{T}^2$.

Since the b.c. is gapped

— top. can shrink the

tube and get

$$\left\langle \boxed{\sum_a Z_{0a}\, a} \right\rangle_{M_3} = \sum_a Z_{0a} \left\langle \right.$$  $$\left. \right\rangle_{M_3}$$

Z_{0a} : non-negative integers

$Z_{00} = 1$ since  $\mathrm{Hom}(1, \sum Z_{0a}\, a) \neq 0$

exists.

On the other hand

$$\left\langle \sum_a Z_{0a}\, a \right\rangle = Z_{M_3 \setminus D^2 \times S^1}$$

$$\Downarrow$$

 $= d_a$

$$\underline{Z_{M_3}} \cdot \underline{\sum Z_{0a}\, d_a} = \underline{Z_{M_3 \setminus D^2 \times S^1}}$$

Z_{S^3}

Take $M_3 = S^3$ then

$$M_3 \setminus \underline{D^2 \times S^1} \approx D^2 \times S^1$$

…ed + trivial

gapp

 $\times S^1 = 1$

Hilbert Space(D^2) = empty
since D^2 has gapped b.c.

$$\Rightarrow \boxed{Z_{S^3} = \frac{1}{\sum' \underline{\underline{Z_{0a}}}\, \underline{\underline{d_a}}} > 0}$$

Related to [Freed-Teleman]

On the other hand

$$Z_{S^3} = \frac{1}{\mathcal{D}^2} \sum d_a^2 \theta_a$$

$$= \frac{1}{\mathcal{D}} e^{2\pi i c_-/8} \quad = 1.$$

$$\Rightarrow \boxed{c_- = 0 \mod 8}$$

For **Abelian** theories, one can repeat this idea and find **many more obstructions.**

$|G| = \#$ of anyons

Take any manifold $\underline{M_3}$ obtained $\underline{\text{from } S^3}$ by surgery with linking matrix $L^A{}_B = L^B{}_A$.

Then $|H_1(M_3)| = |\det L^A{}_B|$.

$H_1(M_3) \simeq \mathbb{Z}^{|L|} / L \cdot \mathbb{Z}^{|L|}$.

Proofs at the End.

If $\boxed{\gcd(\underline{|H_1(M_3)|}, \underline{|G|}) = \underline{\underline{1}}}$

Then $\boxed{\mathrm{Arg}(Z_{M_3}) = \underline{\underline{0}} \mod \underline{\underline{2\pi}}}$

if a gapped boundary exists.

N_{FS} is $\overset{\text{smallest int.}}{\vee}$ s.t. $\Theta_a^{N_{FS}} = \underline{1}$ $\forall a$.

For Abelian Theories N_{FS} divides $2|G|$.

Take $\boxed{L = n}$, s.t.

$\gcd(n, |G|) = \underline{1}$.

$\boxed{L = n}$

$n = 2$

These are the $\underline{\text{Lens spaces}}$

$\underline{L(n,1)}$. $(\underline{L(1,1) = S^3})$

In general

$$Z'_{L(n,1)} = \frac{1}{\mathcal{D}^2} \sum_a d_a^2 \, \Theta_a^n$$

For Abelian Theories ($d_a = 1$)

$$Z'_{L(n,1)} = \frac{1}{|G|} \sum_a \Theta_a^n$$

Write

$$e^{i \xi_n} |Z_{L(n,1)}| = \frac{1}{|G|} \sum_a \Theta_a^n$$

Then

$$\boxed{\text{gapped boundary} \implies \xi_n = 0 \mod 2\pi}$$

$$\xi_1 = 2\pi \frac{c_-}{8}$$

ξ_n

ξ_n for $\boxed{\gcd(n, |G|) = 1}$

are "higher central charges"

[Ng Schopieray Wang
Ng Rowell Wang Zhang]

Example:

$U(1) \times U(1)$,

$U(1)_2 \times U(1)_{-4}$

List of spins (8 anyons)

$$\left\{ 1,\ e^{-i\pi/4} \times 2,\ -1,\ e^{i\pi/2},\ e^{i\pi/4} \times 2,\ e^{-i\pi/2} \right\}$$

$$\underline{Z_{S^3}} = \frac{1}{8}\left(2\cos(\pi/2) + 4\cos(\pi/4)\right)$$

$$= \frac{1}{8}\left(0 + 2\sqrt{2} \right)$$

$$= \frac{\sqrt{2}}{4} = \frac{1}{\sqrt{8}} > 0.$$

$$\Rightarrow c_- = 0$$

$\gcd(3,8) = 1 \quad \xi_3$

$$Z_{L(3,1)} = \frac{1}{8}\left(2\cos\frac{3\pi}{2} + 4\cos\frac{3\pi}{4}\right)$$

$$= \frac{4}{8} \cdot \frac{-\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}$$

$$\operatorname{Arg} Z_{L(3,1)} = \pi \neq 0 \mod 2\pi.$$

Therefore $U(1)_2 \times U(1)_{-4}$ cannot have a gapped b.c.

cannot ...
due to the non-vanishing of ξ_3.

There are theories s.t.
for all $\gcd(n, |G|) = 1$
$\mathrm{Arg}(Z_{L(n,1)}) = 0 \mod 2\pi$
but no gapped b.c. exists.

$\xi_n = 0 \mod 2\pi$

Simplest example:

$$\boxed{Spin(8)_1 \times \left[SU(3)_{-1}\right]^2}$$

Result: We can enlarge the space of higher central charges s.t.

$\gcd(n, |G|) = 1$
N_{FS}, $\Theta^{N_{FS}} = 1$
$N_{FS} \mid 2|G|$ Then

$$\boxed{\gcd\left(n, \frac{N_{FS}}{\gcd(n, N_{FS})}\right) = 1}$$

$\mathrm{Arg}\, Z_{L(n,1)} = 0 \mod 2\pi$

iff There exists a gapped b.c.

General Result: An Abelian theory has gapped b.c. iff it's DW theory.

Some facts about Non-Abelian theories.

$\gcd(n, N_{FS}) = 1$ still implies $\Theta^{N_{FS}} = 1$

... mod 2π if

$$\text{Arg}\, Z_{L(n,1)} = 0$$

there's a gapped b.c.

If $c_- = 0 \mod 8$ then

$$\text{Arg}\, Z_{L(n,1)} = 0 \text{ or } \pi \mod 2\pi.$$

for $\gcd(n, N_{FS}) = 1$.

We don't have a complete list of higher central charges.

2D RCFT

Suppose we start with C-S theory with the standard Dirichlet b.c. leading to 2D current algebra.

current algebra b.c.

$\times\ S^1$

Elitzur-M-S - Schwimmer

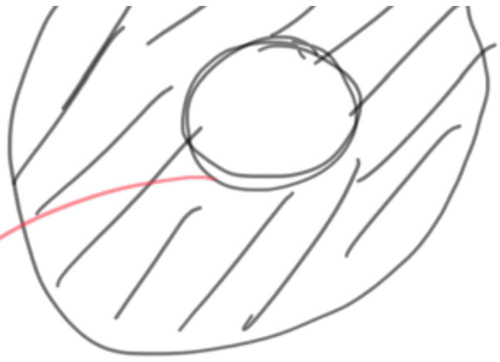
[Moore-Seiberg]

$$Z_a(D^2 \times S^1) = \boxed{\chi_a(\tau)}$$

Now consider the annulus

Current algebra b.c.

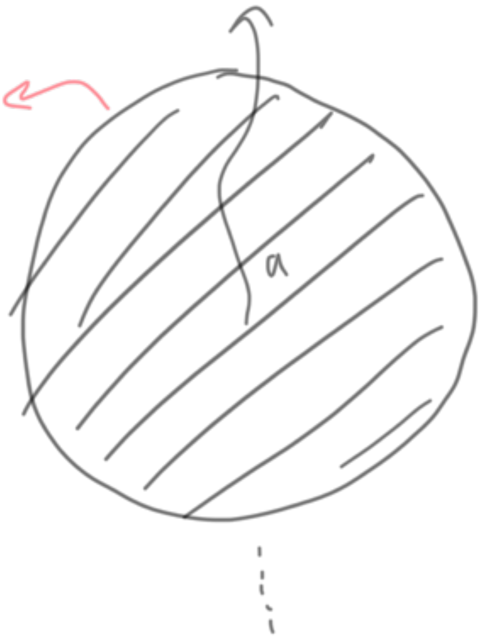
gapped b.c.



$\times \; D$

Since the gapped b.c. is topological can write

current alg. b.c.

$= \sum_a{}'$  $\boxed{Z_{0a}} \times S^1$

$$= \sum{}' Z_{0a} \chi_a(\tau)$$

But since it's an annulus partition fun. it's modular invariant. (holomorphic)

$\Longrightarrow$ Chiral alg can be extended to <u>one</u> character.

$$\boxed{\sum_b S_{ab} Z_{0b} = Z_{0a}}$$

in particular: $Z_{00} = 1 \Longrightarrow$

$$\sum_a Z_{0a} d_a = \mathcal{D} = \sqrt{\sum_a d_a^2}$$

[For Abelian theories this is the statement that
a Lagrangian subgroup should be present
= that a non-anomalous one-form symmetry group of $\dim = \sqrt{|G|}$
exists] .
see [Kapustin-Saulina]

Physical interpretation of higher central
charges : [Harvey-Hu-Wu]

Take $\chi_a(\tau)$, act with Hecke operators,
measure the central charge of new
characters. It's 4 mod 8 or 0 mod 8
if the original theory has $c_- = 0 \mod 8$,

If one of the new central charges is
4 mod 8 then no gapped b.c. exists.

A more physical interpretation would
be nice. Also a more complete set of
higher central charges in non-Abelian theory.

A note on the relation to the 3D-3D
correspondence. → [Dimofte, Gaiotto, Gukov]

Consider an Abelian theory with k

matrix $\frac{K_{IJ}}{4\pi}\int a_I \wedge da_J$ Belov-Moore.

and compute Z on a M_3 with

Surgery link L_A^B.

Assume: $\gcd(|H_1|, |\det K|) = 1$

$\gcd(|\det L|, |\det K|) = 1$

Then $Z_{k}(L) = \omega \sqrt{\frac{|\det L|}{|\det K|}}\, Z_{L}(-k)$

* $\omega^8 = 1$ can be determined using k, L.
* such a symmetry between $K \longleftrightarrow L$ is reminiscent of 3D-3D correspondence.
* due to $\gcd(|\det K|, |\det L|) = 1$ this formula has many ramifications for the problem of higher central charges .....

$R_A^{AA} = 1$ $\quad \Theta = 0$ $\quad \mathcal{A} = \sum Z_{0a}\, a$

$F_A^{AAA} = 1$ $\quad \ldots$

Z_{0a} NN integers

$d_{\mathcal{A}} = \sqrt{\sum d_a^2}$

$(SU(3)_1)^2 \times (\mathrm{Spin}\, 8)_1$, $\mathbb{Z}_2$ (Fibonacci)^{20}

$\mathbb{Z}_2$

e

m

THANK

YOU

Addendum:

In the text we quoted
a result that if

$$\gcd(|H_1|, |G|) = 1$$

Then $\operatorname{Arg} Z_{M_3}$ is an obstruction
to gapped b.c.

I'll sketch 2 proofs:

★ A Lemma that we didn't discuss
is that a gapped b.c. exists
~~iff~~ $\exists$ one-form symmetry
which is non-anom. of order
$\sqrt{|G|}$.

Gauging it we get a trivial

theory.

This is called Abelian Anyon condensation.

We need to introduce flat 2-form gauge fields to do the gauging.

But $\gcd(|H_1|, |G|) = 1$

$\Rightarrow \gcd(|H_1|, \sqrt{|G|}) = 1$

and one can show that no nontrivial 2 form gauge fields exist.

$\Rightarrow$ gauging a one-form symm. doesn't change the phase of Z.

* Another proof

Following [Belov-Moore] and Refs within every Abelian theory is realizable with a K-matrix.

Then

$$Z_K[L] \sim \sum_{m \in \substack{\text{Abelian} \\ \text{group}} = \mathbb{Z}^{|K|}\times\mathbb{Z}^{|L|} / K\cdot\mathbb{Z}^{|K|}\times\mathbb{Z}^{|L|}} e^{i\, m K^{-1}\otimes L\, m}$$

Possible to show that for

$$\gcd(|H_1|, |G|) = 1$$

there's a new $\tilde{K}$ s.t.

$$Z_{\tilde{K}}[S^3] = Z_K[L]$$

and $\tilde{K}$ has a one form symmetry of order $\sqrt{|\det \tilde{K}|}$ with no anomaly $\Rightarrow$ $C_-[\tilde{K}] = 0$ mod 8

$$\Rightarrow \operatorname{Arg} Z_K[L] = 0 \quad 2\pi.$$

 $\rightarrow \sum$  $\cdot\, Z_{0a}$