K3 metrics.

Joint w/ M. Zimet

Parts:

I Motivation [SYZ & Coulomb-branch construction]

3d mirror symmetry

II Hyperkähler quotients [Higgs-branch construction]

III "Metrics"

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explicitly (?)

not a torus.

Q. Find compact $(M, g_{\mu\nu})$ satisfying $R_{\mu\nu} = 0$.

Ricci tensor

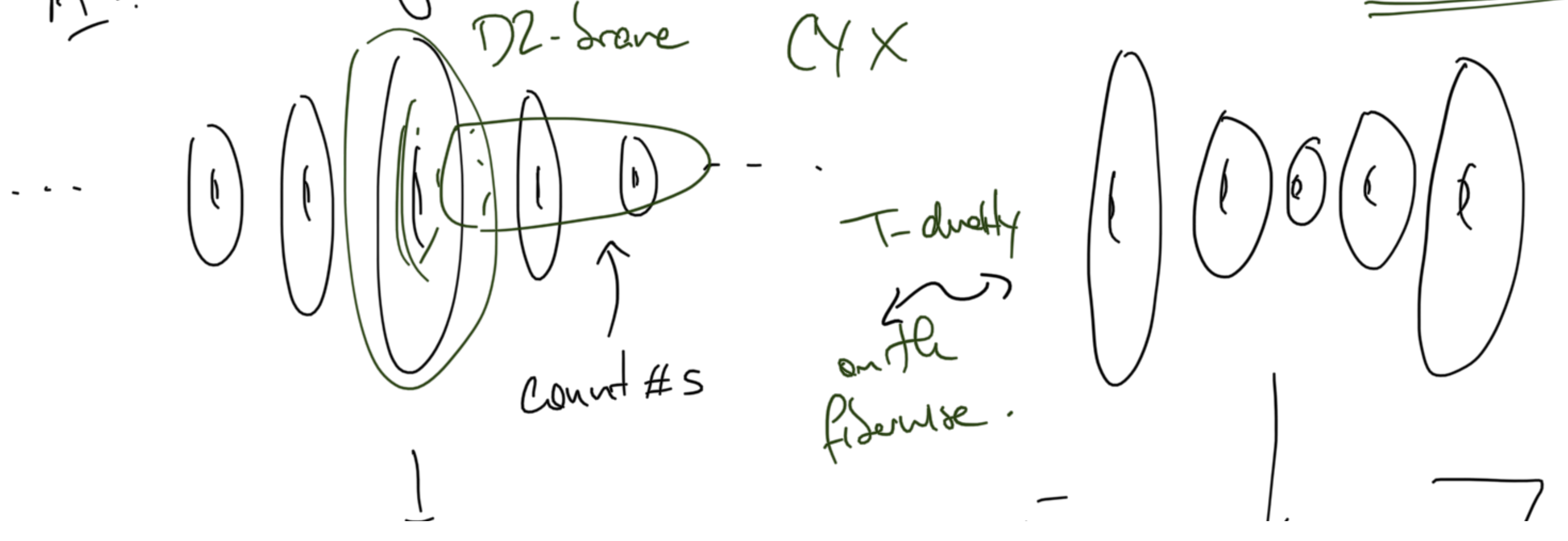
Thm (Yau) M Kähler w/ $c_1(M) = 0$

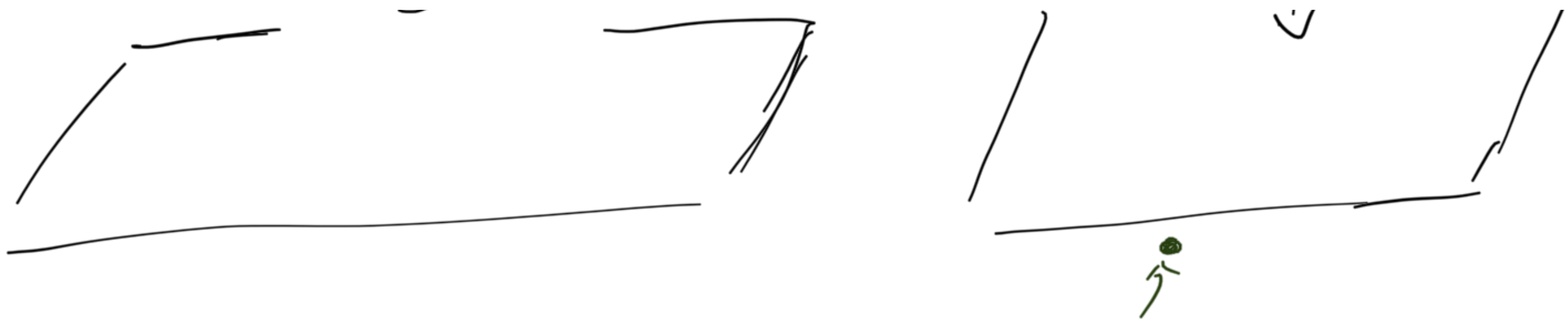
$\Rightarrow \exists$ Ricci-flat metric.

(Calabi-Yau)

"Ans." [Strominger-Yau-Zaslow]

CY $\hat{X}$





$\hat{X}$ "produced by" the moduli of D2-brane probe on X.

D0-branes

some approximate semiflat metric

If we want to "produce" $\hat{X}$, start w/ answer given by low-energy gauge-theoretic approx. on X as corrected by instantons + corrections.

Defn. A hyperkähler manifold of $\dim_{\mathbb{R}} 4n$ is a Riemannian manifold (M, g) w/ hol. $Sp(n)$.

$\rightsquigarrow$ **Rmk.** Such g is automatically Ricci-flat.

Rmk. (M, g) is Kähler w/ 3 integrable almost-cplx structures $I, J, K \in \mathrm{End}(TM)$ such that

$$\mathbb{R}\langle \mathrm{Id}, I, J, K \rangle \simeq \mathbb{H}$$

& hol. symp. Ω

$\Rightarrow$ any pure imaginary unit quaternion $\zeta \in S^2$

or $\mathbb{R}\langle I, J, K\rangle$ of norm 1

lin. comb.

yields a ... $I(\zeta)$, $J(\zeta)$, $\Omega(\zeta)$

of I, J, K ... $\underline{\omega(\xi)}$, $J(\xi)$, $\underline{\Omega(\xi)}$.
w/ norm $\underline{1}$.

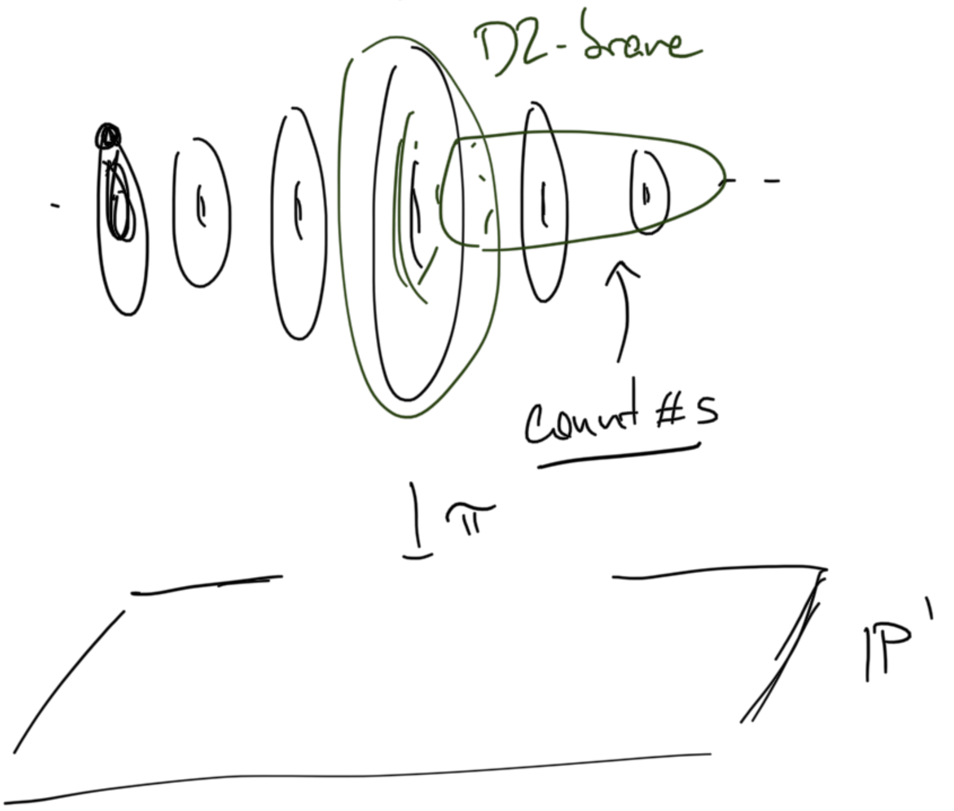
Rmk. $\underline{g_{ij}} = -(\omega_I)_{il} (\omega_J^{-1})^{lk} (\omega_K)_{kj}$.

"I'm studying the metric" = compute $\Omega(\xi)$.

Defn. A $\underline{\text{K3 m'fld}}$ is a hyperkähler m'fld of $\dim_{\mathbb{R}} 4$.

Return to the SYZ picture:

Start w/ an elliptic K3 ; hyperkähler rotate
(pass to an orthogonal $\xi \in S^2$)



$\Longrightarrow$ expect interesting counts $\underline{N_\gamma(u)}$, "open GW invts"

$u \in B' \simeq \mathbb{P}^1 \setminus \{\text{sing. pts}\}$,

$$\gamma \in \overline{H_2(K3, \pi^{-1}(u); \mathbb{Z})}$$

$$\Gamma_u$$

$$\underbrace{0 \to \overset{\text{rank } 20}{H_2(K3;\mathbb{Z})/(b,f)}} \to \underbrace{H_2(K3, \pi^{-1}(u); \mathbb{Z})/(b)} \to \underbrace{\overset{\text{rank } 2}{H_1(\pi^{-1}(u); \mathbb{Z})}} \to 0$$

$$\left(0 \to \Gamma_f \to \Gamma \to \Gamma_g \to 0 \right)$$

SYZ-esque: knowing these $N_\gamma(u)$'s allows one to construct the mirror as a complex manifold

... as a Riemannian / hyperkahler m'fld.

Defn. $\Gamma_u := H_2(K3, \pi^{-1}(u); \mathbb{Z})/(b)$ a local system on $\mathbb{P}^1 \setminus \{(24) \text{ sing pts}\}$.

$Z : \Gamma_u \to \mathbb{C}$ hol. (2,0) vol form on K3

$$\gamma \mapsto \int_\gamma \Omega \qquad \Omega \in H^2(K3; \mathbb{C})$$

Observe $\Omega|_{\pi^{-1}(u)} = 0$.

Conj. There exist $N_{\vec{\gamma}}(\vec{u}) \in \mathbb{Z}$ "a slightly renamed version" of the open genus 0 GW invts that "wall-cross appropriately". (satisfying KSWCF)

(i.e. $N_\gamma(u)$ are locally constant as functions of $u \in B'$.

They have discontinuities at

$$\longrightarrow \quad W_{\gamma_1, \gamma_2} = \{ Z_{\gamma_1}(u) \,||\, Z_{\gamma_2}(u) \} \subset B'$$

Defn. z_1 & $z_2 \in \mathbb{C}^\times$ are parallel if $z_1/z_2 \in \mathbb{R}^+$.

[Y S Lin].

Conj. [Gaiotto - Moore - Neitzke, Kachru - T - Zimet 18]
08, 09

has hol. functions

In the situation above, we ...

$$\mathcal{X}_\gamma(u) = \mathcal{X}_\gamma^{sf}(u) \times \exp\left(-\frac{1}{4\pi i}\sum_{\gamma' \in \Gamma_u} N_{\gamma'}(u)\langle\gamma,\gamma'\rangle \int_{\rho \| Z_\gamma(u)} \frac{d\rho'}{\rho'}\frac{\rho'+\rho}{\rho'-\rho}\log(1-\mathcal{X}_{\gamma}(\rho'))\right)$$

prescribed by iterating the above integral equation starting from the "zeroeth order guess" $\mathcal{X}_\gamma^{sf}(\rho)$

These functions $d\log\mathcal{X}_1(\rho) \wedge d\log\mathcal{X}_2(\rho)$? are the hol. symp. forms $\Omega(\rho)$.

How might one interpret these counts $N_\gamma(u)$ "in the K3 situation"?

6d (1,0) LST [het. 5-brane] / T^3 ⇝ moduli space is a K3.

[Intriligator]

BPS state counts in

Exponential growth – LST entropy growth.

Goal: compute these $N_\gamma(u)$!

Why? → You'll provide a Ricci-flat metric on K3s!

→ Study a dual frame to the above construction!

"Apply 3d mirror symmetry"?

II. Hyperkähler quotients

$G \curvearrowright (M, \omega)$ $\quad M \xrightarrow{\mu} \mathfrak{g}^\vee$ $\quad \xi \in (\mathfrak{g}^\vee)^G$

Symplectic quotient $\quad M /\!\!/^{\xi} G := \mu^{-1}(\xi) / G$.

→ $G \curvearrowright (M, \text{hyperkähler } \omega(\xi))$ $\quad M \xrightarrow{\mu} (\mathfrak{g}^\vee)^3$

[HKLR] $\quad M /\!\!/\!\!/^{\xi} G := \mu^{-1}(\xi) / G$

hyperkähler m'fld. $\quad \xi \in ((\mathfrak{g}^\vee)^G)^3$

Ex. [Kronheimer] ALE m'flds as HKQuots

$\Gamma \curvearrowright \mathbb{C}^2$ preserving its HK structure, i.e. $\Gamma \subset SU(2)$ finite

finite

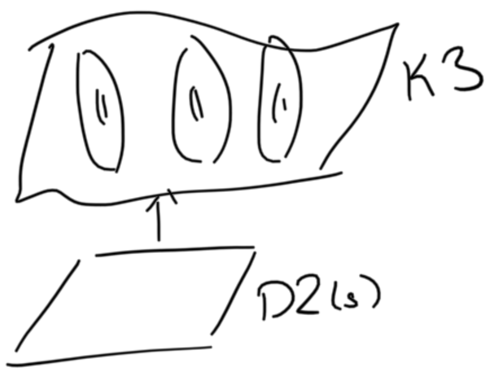
[D2 probing orbifold singularity $\mathbb{C}^2/\Gamma$ – Douglas-Moore]

$R(\Gamma)$ - regular representation of Γ

$$G = Z_{U(R_\Gamma)}(\Gamma) \subset U(R_\Gamma)^{\Gamma}$$

$$A = (\mathfrak{u}(R_\Gamma) \otimes_{\mathbb{R}} \mathbb{C}^2)^{\Gamma}$$

Thm. [K] $A /\!/\!/_{\xi} G$, $\xi \in (\mathfrak{g}^\vee)^G$ is an interesting family of HK m'folds deforming $\mathbb{C}^2/\Gamma$.



So, K3s.

Observation:

(some) K3s are $T^4/\mathbb{Z}_2 \simeq \mathbb{C}^2/\underbrace{(\mathbb{Z}^4 \rtimes \mathbb{Z}/2)}_{\Gamma}$.

Idea: perform Kronheimer/Douglas-Moore for Γ.

Halt! Easier to work in a T-dual frame w/ D6 wrapping $T^4/\mathbb{Z}_2$

Easier setting: let's take $\Gamma = \mathbb{Z}$

study $G = Z_{U(R_T)}(\Gamma)$

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & & \\ & a_{00} & a_{01} & \\ & a_{10} & a_{11} & \\ & & a_{21} & \end{pmatrix} \leftarrow a_{ij}, \quad i,j \in \mathbb{Z}.$$

$$a_{ij} = a_{i+1,j+1}$$

an $\mathbb{Z}$-labeled list of $\mathbb{C}$

$\ell^2(\mathbb{Z}) \simeq \mathrm{Maps}(S^1, \mathbb{C})$.

$\{\text{infinite matrices}\} \simeq \mathrm{Maps}(S^1, U(1))$.

Thm. 1.P [T-Zimet]. Let $\mathcal{A} := \{ \mathbb{Z}/_2$-eq connections on a trivial principal $su(2)$-bundle / $\tilde{T}^4 \}$

& $\mathcal{G} := \mathrm{Maps}_{\mathbb{Z}/2}(T^4, SU(2))$.

Then, for generic $\xi \in \left(\left(\mathrm{Lie}\,\mathcal{G}\right)^{\vee}\right)^3$

$\mathrm{Maps}_{\mathbb{Z}/2}(T^4, \mathfrak{su}(2))$

$\mathrm{Dist}_{\mathbb{Z}/2}(T^4, \mathfrak{su}(2))$

16 real-dim'l.

$$\mathcal{A} /\!/\!/_{\xi}\, \mathcal{G}$$ is a smooth K3 manifold.

Remark. [Hitchin] K3s may be constructed as HK quots of flat spaces.

Remark. [B-field]

Remark. [Other orbifolds]

Remark. [Donaldson]

Moment maps: $A \longmapsto F^{SD}$

So, what are the moment map eq'ns?

$$\{F^{SD} = \delta\} / \mathcal{G}$$ = moduli spaces of deformed equivariant instantons.

on $\hat{T}^4$

III. "Metrics"

Thm [T-Zimet]. There exist explicit solutions to the moment map equations

$$F^{SD} = \zeta \qquad (*)$$

in powers of ξ.

Pf. Find a Green's function for an elliptic operator & iteratively keep applying it to RHS of $(*)$

Q. Does the series so produced possibly converge?