Comments on Lattice vs. Continuum Quantum Field Theory

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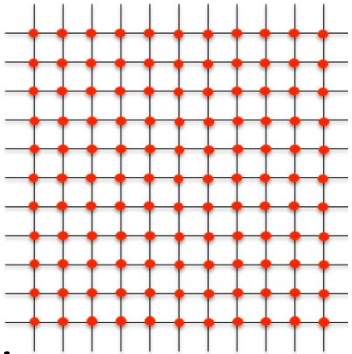
Gorantla, Lam, NS, Shao, 2103.01257, 2108.00020

Lattice vs. continuum QFT

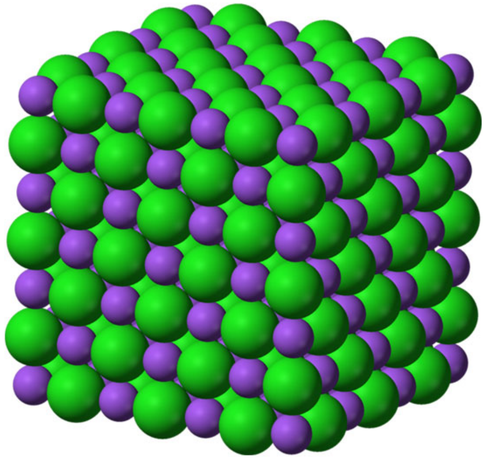
QFT is enormously successful. Yet, it is not mathematically rigorous.



One approach is to regularize it by placing it on a lattice.

- Then, the functional integral is well defined.
- Continuum limit: introduce a lattice spacing a, take $a \to 0$ and the number of sites to infinity holding all the lengths fixed.
  - Compute correlation functions at positions $a \ll x$.

In condensed matter physics, the problem is defined on a lattice and the goal is to find its low-energy/long-distance limit.



- It is expected to be described by an effective continuum field theory.

Challenges in using a lattice to define a given continuum QFT

- Does the limit exist and is it independent of the details of the lattice theory?
- Some continuum theories depend on the topology of field space, which relies on continuity of the fields. How is this captured by the lattice theory?
  - This issue affects various topological terms in the action, certain global symmetries, anomalies, etc. (More below.)
- Some QFTs (e.g., theories with self-dual forms or fermions) do not admit a suitable Euclidean lattice action.
- Some QFTs do not even have a continuum Lagrangian, let alone a lattice version of it.

Challenges in finding a continuum low-energy QFT of a given lattice model

Exotic lattice models, e.g., XY-plaquette model [Paramekanti, Balents, Fisher; …] (see below), fracton models [Chamon; Haah; Vijay, Haah, and Fu; …], do not have a standard continuum limit.

- They are characterized by an exact or emergent subsystem global symmetries […; Emery, Fradkin, Kivelson, Lubensky; …] – separate symmetry group element for different subspaces. (More below.)
  - Observables vary at the lattice scale a, and hence they are discontinuous (and even singular) in the continuum limit.
  - Infinite ground state degeneracy in the continuum limit (sometimes no well defined limit).

UV/IR mixing – long-distance phenomena depend on short-distance details – reminiscent of some string theory constructions.

Exotic continuum QFT [NS, Shao]

Elements in our continuum theories

- Spacetime symmetries (in addition to translations)
  - No Lorentz invariance
  - No rotation symmetry (only discrete rotations)
- Impose exotic global symmetries and then gauge them
- Discontinuous fields and gauge parameters
  - Not as discontinuous as on the lattice – the allowed discontinuities are restricted
  - Universal – independent of most of the details at the lattice scale

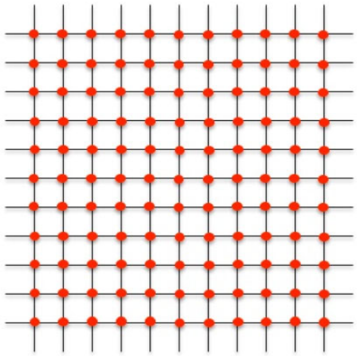
Does this make sense?

In order to explore it, let us review a more standard case…

Canonical example of lattice vs. continuum: XY-model in 1+1d

[…; Jose, Kadanoff, Kirkpatrick, Nelson; …]

Use a Euclidean-time, Lagrangian formulation.



On the lattice, phases $e^{i\phi}$ at the sites with the action

$$S = -\beta \sum_{links} \cos(\Delta_\mu \phi)$$

Global $U(1)$ symmetry (momentum)

$$\phi(x,\tau) \to \phi(x,\tau) + \alpha$$

The continuum theory (same β when it is large)

$$S = \frac{\beta}{2} \int d\tau dx (\partial_\mu \phi)^2$$

XY-model in 1+1d – the continuum theory

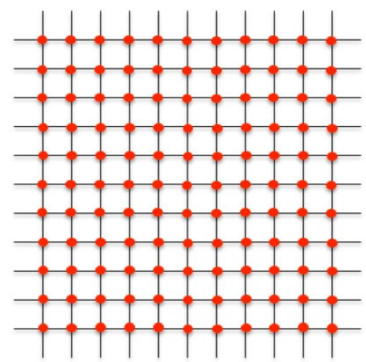
$$S = \frac{\beta}{2}\int d\tau dx\left(\partial_\mu \phi\right)^2$$

- This is the famous $c = 1$ compact boson. Free (quadratic) action.
- Global symmetries $\qquad \partial_\mu j_\mu = 0$
  - $U(1)^m$ momentum $\qquad j_\mu^m = -i\beta\partial_\mu\phi$
  - $U(1)^w$ winding (vorticity), emergent $\qquad j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}\partial_\nu\phi$
  - Mixed 't Hooft anomaly between them
- Exact self-duality (T-duality): exchanging $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$ and $U(1)^m \leftrightarrow U(1)^w$. Not present on the lattice.
- How much of this continuum discussion can be present on the lattice?

XY-model in 1+1d – modify the lattice theory

Following [...; Gross, Klebanov; ...; Sachdev, Park; ...], "suppress the vortices" on the lattice

Use the Villain formulation – replace $\phi \in S^1$ with $\phi \in \mathbb{R}$ coupled to a $\mathbb{Z}$ gauge field n_μ on the links



$$S_{Villain} = \frac{\beta}{2} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2$$

$$\phi \sim \phi + 2\pi\, m$$

$$n_\mu \sim n_\mu + \Delta_\mu m$$

Suppress the vortices by adding the curvature square

$$\kappa \sum_{plaq} (\Delta_\tau n_x - \Delta_x n_\tau)^2$$

XY-model in 1+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$\frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + \kappa \sum_{plaq}(\Delta_\tau n_x - \Delta_x n_\tau)^2$$

For $\kappa \to \infty$, the field strength of the $\mathbb{Z}$ gauge field, $\Delta_\tau n_x - \Delta_x n_\tau$ vanishes – the gauge field is flat.

We can replace the action by the modified Villain action

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i \sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

with a Lagrange multiplier field $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

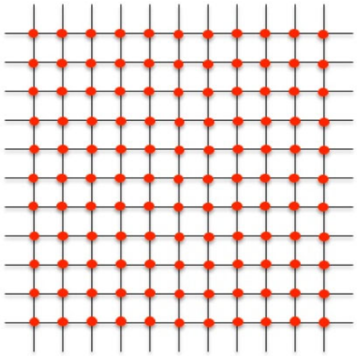
This lattice theory is similar to the continuum theory…

XY-model in 1+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- Free
- Exact global symmetries
  - $U(1)^m$ momentum, $\phi \to \phi + \alpha$, $\quad j_\mu^m = -i\,\beta(\Delta_\mu\phi - 2\pi n_\mu)$
  - $U(1)^w$ winding, $\quad \tilde{\phi} \to \tilde{\phi} + \tilde{\alpha}$, $\quad j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}(\Delta_\nu\phi - 2\pi n_\nu)$
  - 't Hooft anomaly. The symmetries act locally. But the Lagrangian density is not invariant; only e^{-S} is invariant.
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi}$, $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$

An exotic theory: XY-plaquette model in 2+1d [Paramekanti, Balents, Fisher; …]



We will use a Euclidean-time, Lagrangian formulation.

On the lattice, phases $e^{i\phi}$ at the sites with the action

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos(\Delta_x \Delta_y \phi)$$

Global $U(1)$ subsystem (momentum) symmetry

$$\phi(x,y,\tau) \rightarrow \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$$

Related continuum theory 2+1d ϕ-theory

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right)$$

2+1d ϕ-theory [NS, Shao]

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right)$$

- Free
- Because of the derivative structure, some discontinuous field configurations are not suppressed, e.g., $\phi(x,\tau)$ discontinuous in x.
- Subsystem global symmetries $\partial_\tau j_\tau = \partial_x \partial_y j_{xy}$
  - $U(1)^m$ momentum $j_\tau^m = i\mu_0 \partial_\tau \phi\ ,\ \ j_{xy}^m = \frac{i}{\mu} \partial_x \partial_y \phi$
  - $U(1)^w$ winding (vorticity), emergent

    $$j_\tau^w = \frac{1}{2\pi} \partial_x \partial_y \phi\ ,\ \ j_{xy}^w = \frac{1}{2\pi} \partial_\tau \phi$$
  - Mixed 't Hooft anomaly between them

2+1d ϕ-theory [NS, Shao]

- Exact self-duality – T-duality (not present on the lattice)
  - $\mu_0 \leftrightarrow \frac{\mu}{(2\pi)^2}$
  - $U(1)^m \leftrightarrow U(1)^w$

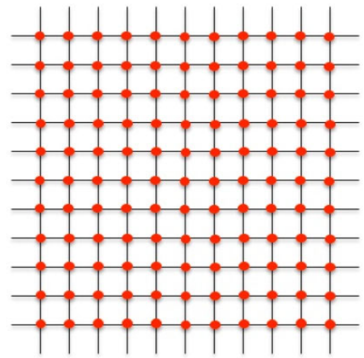
Many questions:

- How should we treat more precisely such a continuum field theory with discontinuous fields and other peculiarities? Make the treatment more rigorous.
- How much of that depends on the continuum limit? Can we find these phenomena (winding subsystem symmetry, 't Hooft anomaly, self-duality, etc.) on the lattice?

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

Repeat the discussion of the 1+1d XY-model for this model.

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos\left(\Delta_x \Delta_y \phi\right)$$



Use the Villain form

$$S_{Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$

Here $\phi \in \mathbb{R}$, $n_\tau, n_{xy} \in \mathbb{Z}$ with the $\mathbb{Z}$ tensor gauge symmetry

$$\phi \sim \phi + 2\pi\, m$$
$$n_\tau \sim n_\tau + \Delta_\tau m$$
$$n_{xy} \sim n_{xy} + \Delta_x \Delta_y m$$

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

- Add to the action the gauge invariant term

$$\kappa \sum_{cubes} \left(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau\right)^2$$

- For $\kappa \to \infty$ the field strength of the $\mathbb{Z}$ tensor gauge field (n_τ, n_{xy}), $\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau$ vanishes. We can replace the action by the modified Villain action

$$S_{mod.Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2 + i \sum_{cubes} \tilde{\phi}(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

with a Lagrange multiplier $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$S_{mod.Villain}$$
$$= \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$
$$+ i \sum_{cubes} \tilde{\phi}(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

Similar to the continuum version of the XY-plaquette model:

- Free
- Exact subsystem global symmetries with mixed 't Hooft anomaly
  - $U(1)^m$ momentum $\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$
  - $U(1)^w$ winding $\tilde{\phi}(x,y,\tau) \to \tilde{\phi}(x,y,\tau) + \tilde{\alpha}_x(x) + \tilde{\alpha}_y(y)$
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi},\ \beta_0 \leftrightarrow \frac{1}{(2\pi)^2 \beta}$

XY-plaquette lattice model and the continuum ϕ-theory in 2+1d

[Gorantla, Lam, NS, Shao]

Now, we can ask more questions about the theory

- Spectrum
- Correlation functions

We can study them either on the lattice (in its modified Villain form) or in the continuum.

We'll present it in the continuum, but the same answers are obtained on the lattice.

As a preliminary, let us review briefly the spectrum of the 1+1d compact boson.

1+1d compact boson – the spectrum

$$S = \int d\tau dx \frac{\beta}{2} (\partial_\mu \phi)^2$$

Denote the circumference of space by ℓ

$$\phi(x,t) = \phi_0(t) + 2\pi W \frac{x}{\ell} + \sum_{k \in \mathbb{Z}_{\neq 0}} a_k(t) e^{2\pi i \frac{kx}{\ell}}$$

- Plane waves (oscillators) $E \sim 1/\ell$
- States charged under the momentum symmetry $E \sim 1/\beta\ell$
- States charged under the winding symmetry $E \sim \beta/\ell$

The spectrum is gapless – the energies of all these states vanish as $\ell \to \infty$.

The energies of all these states are of order $1/\ell$ and hence they are equally important.

2+1d ϕ-theory – spectrum [NS, Shao]

Analyze the continuum theory in Lorentzian signature. (Can do it on the lattice.) For simplicity, $\ell_x = \ell_y = \ell$

$$\phi(x,y,t)$$

$$= \phi_x(x,t) + \phi_y(y,t) + \sum_{k_x,k_y \in \mathbb{Z}_{\neq 0}} a_{(k_x,k_y)}(t) e^{2\pi i \left(\frac{k_x x}{\ell} + \frac{k_y y}{\ell}\right)}$$

Plane waves (oscillators) with $\omega^2 = (2\pi)^4 \frac{k_x^2 k_y^2}{\mu \mu_0 \ell^4}$. Because of this dispersion relation:

- $\omega \sim 1/\ell^2$ (and not $1/\ell$, as in more standard systems)
- For large ℓ, can have low ω with large $p_x = k_x/\ell$, provided $p_y = k_y/\ell$ is sufficiently small – high momentum with low energy. This leads to UV/IR mixing. (More below.)

2+1d ϕ-theory – spectrum

$$\phi(x, y, t)$$
$$= \phi_x(x, t) + \phi_y(y, t) + \sum_{k_x, k_y \in \mathbb{Z}_{\neq 0}} a_{(k_x, k_y)}(t) e^{2\pi i \left(\frac{k_x x}{\ell} + \frac{k_y y}{\ell}\right)}$$

States charged under the momentum subsystem symmetry:

- The modes $\phi_x(x, t)$, $\phi_y(y, t)$ can be thought of as associated with the spontaneous breaking of this symmetry. We will soon see that this is not the case in the quantum theory.
- They include the standard winding modes $\phi = \frac{2\pi}{\ell}\left(W_x x + W_y y\right)$ and hence these should not be considered separately.

2+1d ϕ-theory – spectrum

- For simplicity, ignore the common zero mode of $\phi_x(x,t)$ and $\phi_y(y,t)$. Then, $\phi_x(x,t)$ and $\phi_y(y,t)$ are independent rotors at different positions:

$$S = \frac{\ell\mu_0}{2}\int dt\left(\oint dx\,\left(\partial_t\phi_x(x,t)\right)^2 + \oint dy\left(\partial_t\phi_y(y,t)\right)^2\right)$$

Like 1+1d free fields without the spatial derivatives (pointwise periodic).

Going back to the lattice with lattice spacing a,

$$H = \frac{1}{2\ell\mu_0\, a}\left(\sum_{\hat{x}} n_x(\hat{x})^2 + \sum_{\hat{y}} n_y(\hat{y})^2\right)\ , \qquad n_x(\hat{x}), n_y(\hat{y}) \in \mathbb{Z}$$

Their energies diverge $E \sim \frac{1}{\mu_0\ell a} \to \infty$.

2+1d ϕ-theory – spectrum

$$H = \frac{1}{2\ell\mu_0\, a}\left(\sum_{\hat{x}} n_x(\hat{x})^2 + \sum_{\hat{y}} n_y(\hat{y})^2\right)\ , \qquad n_x(\hat{x}), n_y(\hat{y}) \in \mathbb{Z}$$

Their energies diverge $E \sim \frac{1}{\mu_0 \ell a} \to \infty$.

- The momentum subsystem symmetry was spontaneously broken in the classical theory, but it is restored in the quantum theory.

2+1d ϕ-theory – spectrum

What about states charged under the winding subsystem symmetry?

To be periodic modulo 2π and carry the winding charge,

$$\phi = \frac{2\pi}{\ell}\left(x\Theta(y-y_0) + y\Theta(x-x_0) - \frac{xy}{\ell}\right) \qquad 0 \le x, y < \ell$$

$$j_\tau^w = \frac{1}{2\pi}\partial_x\partial_y\phi = \frac{1}{\ell}\left(\delta(y-y_0) + \delta(x-x_0) - \frac{1}{\ell}\right)$$

$$Q^x(x) = \oint dy\, j_\tau^w = \delta(x-x_0),$$

$$Q^y(y) = \oint dx\, j_\tau^w = \delta(y-y_0)$$

These configurations have infinite energy. Restoring the lattice spacing a, their energy is $\sim \frac{(2\pi)^2}{\mu\ell a}$.

2+1d ϕ-theory – spectrum

To summarize:

- Plane waves (oscillators), created by $\partial_x \partial_y \phi$, $\partial_\tau \phi$, etc.

$$E \sim \frac{1}{\sqrt{\mu\mu_0}\,\ell^2}$$

- States charged under the momentum subsystem symmetry, created by $\exp(i\phi)$

$$E^m \sim \frac{1}{\mu_0 \ell a}$$

- States charged under the winding subsystem symmetry, created by $\exp(i\tilde{\phi})$

$$E^w \sim \frac{1}{\mu \ell a}$$

Only the plane waves are present in the spectrum of the continuum theory.

2+1d ϕ-theory – spectrum

The main surprising result of the analysis of the spectrum is that the states charged under the momentum and winding subsystem symmetries have high energy – infinite in the continuum limit.

- The momentum and winding states exist in the Hilbert space of the lattice theory (in its modified Villain form), but they are not dynamical excitations in the continuum theory – they are not present in the Hilbert space of the continuum theory.
- Since they carry conserved charges, they are well-defined defects that can be added to the continuum theory. Note that they are exchanged by the self-duality.

2+1d ϕ-theory – UV/IR mixing

Go back to the lattice with $L_x = L_y = L$ sites and $\ell = aL$

Plane waves $\qquad E \sim \frac{1}{\ell^2} = \frac{1}{L^2 a^2}$

Momentum and winding states $\qquad E \sim \frac{1}{\ell a} = \frac{1}{L a^2}$

We are interested in $L \to \infty$.

Above, we took $a \to 0$ with fixed $\ell = La$. This kept the plane waves and pushed the charged states to infinity.

Alternatively, if we hold a fixed, i.e., $\ell \to \infty$, all these states approach zero energy.

We see that

$$[\ell \to \infty, a \to 0] \neq 0$$

UV/IR mixing.

2+1d ϕ-theory – correlation functions

Consider the lattice theory in its modified Villain form. Fix a gauge and then all the correlation functions are determined by the Green's function (propagator)

$$\langle \phi\phi \rangle = \text{explicit but complicated expression}$$

Study correlation functions of "good" local operators like $\exp(i\phi)$, $\Delta_\tau \phi$, etc. and then take $L_x = L_y = L \to \infty$. This can be done in two different ways

- Continuum limit: $a \to 0$, with $\ell = aL$ fixed. Operators at fixed positions in space are separated by many (infinite in the limit) lattice sites. Can later take $\ell \to \infty$.
- Thermodynamic limit: fixed a, i.e., $\ell = aL \to \infty$. Can later separate operators to be many lattice spacings apart.

$\langle \partial_\tau \phi \; \partial_\tau \phi \rangle$

Take the continuum limit and then infinite volume – i.e., $\ell \to \infty$
(For simplicity, set $\mu = \mu_0 = 1$ and drop constants of order one.)

$$\langle \partial_\tau \phi(0,0,0) \partial_\tau \phi(x,y,\tau) \rangle \sim \begin{cases} -\dfrac{1}{(xy)^2} & |\tau| \ll |xy| \\ -\dfrac{1}{\tau^2} \log \dfrac{|\tau|}{|xy|} & |\tau| \gg |xy| \end{cases}$$

UV divergence as $xy \to 0$.

This singularity is not present on the lattice with nonzero a.

As $x \to 0$ with fixed y, it is associated with large momenta p_x.

Similarly for $y \to 0$ with fixed x.

Because of the dispersion relation $\omega^2 \sim \left(p_x p_y\right)^2$, we can have large p_x with finite ω, provided p_y is small enough.

$\langle \partial_\tau \phi \; \partial_\tau \phi \rangle$

$$\langle \partial_\tau \phi(0,0,0) \partial_\tau \phi(x,y,\tau) \rangle \sim \begin{cases} -\dfrac{1}{(xy)^2} & |\tau| \ll |xy| \\ -\dfrac{1}{\tau^2} \log \dfrac{|\tau|}{|xy|} & |\tau| \gg |xy| \end{cases}$$

Regularize the IR by restoring finite ℓ, then, $|p_y| \geq \frac{1}{\ell}$. The singularity as $x \to 0$ becomes $-\frac{1}{\tau^2} \log \frac{\ell}{|y|}$.

(If both x and y are small, it becomes $-\frac{1}{\tau^2} \log \frac{\ell^2}{|\tau|}$.)

This reflects the UV/IR mixing in the spectrum of plane waves.

$\langle \exp(i\phi) \exp(-i\phi) \rangle$

The subsystem symmetry forces the two operators to be at the same spatial position (otherwise, the correlation function vanishes)

$$\langle \exp\big(i\phi(0,0,0)\big) \exp\big(-i\phi(0,0,\tau)\big) \rangle$$

As we take the continuum limit,

$$\langle \exp\big(i\phi(0,0,0)\big) \exp\big(-i\phi(0,0,\tau)\big) \rangle \sim \exp\left(-\frac{|\tau|}{\ell a}\right) \to 0$$

The exponent represents the energy of the lowest momentum state $\quad E \sim 1/\ell a \to \infty$.

This demonstrates our statement above that these exponential operators vanish in the continuum limit – they are infinitely irrelevant – "redundant operators."

$\langle \exp(i\phi) \exp(-i\phi) \rangle$

In the thermodynamic limit (finite a) [Paramekanti, Balents, Fisher],

$$\langle \exp\big(i\phi(0,0,0)\big) \exp\big(-i\phi(0,0,\tau)\big) \rangle \sim \exp\left(- \left(\log\left(\frac{|\tau|}{a^2} \right) \right)^2 \right)$$

For large τ, it decays faster than any power, but is not exponentially suppressed (as in the continuum limit). The a dependence cannot be absorbed in wave-function renormalization.

Consistent with our claim that these operators are not part of a scale invariant continuum theory.

Identical analysis for the winding operators $\langle \exp\big(i\tilde{\phi}\big) \exp\big(-i\tilde{\phi}\big) \rangle$.

This reflects the UV/IR mixing in the spectrum of the momentum and winding modes – their energies go to zero as $L \to \infty$, but slower than the plane waves.

Many other models

- Gapped models with $\mathbb{Z}_N$ subsystem symmetries
- Gauge theories of subsystem symmetries
- More possible subsystem symmetries in 3+1d
- A certain 3+1d $\mathbb{Z}_N$ gauge theory of a subsystem symmetry describes the long-distance behavior of one of the most celebrated fracton models, the X-cube model [Vijay, Haah, Fu].

All these models have a modified Villain version and a corresponding continuum description.

They exhibit even more peculiar UV/IR mixing.

For example, the ground state degeneracy of the X-cube model depends on the number of sites:

$$N^{2(L_x+L_y+L_z)-3}$$

Summary

- The low-energy limit of a lattice theory is expected to be a continuum quantum field theory.
- Exotic lattice models are challenging counter-examples because
  - Subsystem global symmetry
  - UV/IR mixing
  - Large ground state degeneracy (infinite in the continuum limit)
  - Discontinuous and even singular observables in the continuum limit
  - Defects with restricted mobility
- Some peculiar continuum theories can capture these facts. They involve discontinuous fields. They can be made rigorous using modified Villain lattice models.

Thank you

Stay healthy