Two tales of networks and quantization

Fei Yan

Rutgers University

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Outlook

I will describe two quantization scenarios.

A. Construction of a link "invariant" (with possible wall-crossing behavior) for links L in a 3-manifold M, where M is a Riemann surface C times a real line. [Neitzke-Y,JHEP09(2020)153], [Neitzke-Y,wip]

- This construction computes familiar link invariants in a new way.
- It unifies that computation with the computation of framed BPS indices counting ground states with spin for line defects in 4d N=2 theories of class-S.
- Certain networks play an important role in the construction.
- Potentially extendable to general 3-manifolds M admitting tetrahedron triangulations.

Outlook

B. Exact WKB method for Schrödinger equations and higher order analogues, arsing as quantization of Seiberg-Witten curves in 4d N=2 theories. Similar networks also play an important part.
(short review plus new results [Y,arXiv:2012.15658],[Y,wip])

I will also briefly sketch a possibility to bridge these two scenarios.
(discussion w/ D. Gaiotto, G. Moore and A. Neitzke)

Line defects in 4d $N = 2$ theories

- Line operators are important tools in studying QFTs: capture global structure of QFTs, affect local dynamics of the theory compactified on S^1, info on bulk spectrum, ...
- Consider 4d $N = 2$ theory, with the insertion of a susy line defect extending along time direction, sitting at the origin of spacial $\mathbb{R}^3$. (susy Wilson-'t Hooft lines and generalizations)

  [Kapustin-Saulina],[Drukker-Morrison-Okuda],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Gaiotto-Kapustin-Seiberg-Willett],[Gaiotto-Kapustin-Komargodski-Seiberg],[Ang-Roumpedakis-Seifnashri],[Agmon-Wang],[Bhardwaj,Hubner,Schafer-Nameki],...
- In abelian gauge theories, they are labeled by electromagnetic charge γ and a parameter $\zeta \in \mathbb{C}^\times$(preserved supercharges):
  Example: 4d $N = 2$ $U(1)$ theory, γ purely electric:

$$\mathbb{L}(\gamma, \zeta) = \exp\left[i\gamma \int_{\mathbb{R}_t} \left(A + \frac{1}{2}\left(\zeta^{-1}\phi + \zeta\bar{\phi}\right)\right)\right]$$

The UV-IR map

4d $N = 2$ theories have a subspace of vacua called the Coulomb branch; the low energy effective field theory is $U(1)^r$ gauge theory. [Seiberg-Witten]
Starting with a susy line defect $\mathbb{L}$ in the UV, deform onto the CB,
$\rightarrow$ superposition of line defects in effective abelian theory. [Gaiotto-Moore-Neitzke]

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An UV-IR map for line defects:

$$\mathbb{L} \rightsquigarrow F(\mathbb{L}) := \sum_{\gamma} \underline{\overline{\Omega}}(\mathbb{L}, \gamma) X_\gamma$$

IR Wilson-'t Hooft lines

$q \to 1$

Defect Hilbert space $\mathcal{H}_{\mathbb{L},u} = \bigoplus_{\gamma} \mathcal{H}_{\mathbb{L},\gamma,u}$

Protected spin character $\underline{\overline{\Omega}}(\mathbb{L}, \gamma, q) := \mathrm{Tr}_{\mathcal{H}_{\mathbb{L},\gamma,u}} (-q)^{2J_3} q^{2I_3} e^{-\beta\{\mathcal{Q},\mathcal{Q}^+\}} \in \mathbb{Z}[q, q^{-1}]$

$SO(3)$ rot. (J_3), $SU(2)_R$ (I_3)

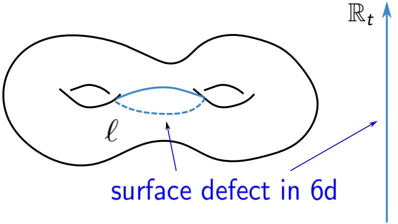
Example — N=2 pure $SU(2)$ SYM, weak-coupling region

$$F(\mathbb{L}_2) = X_{(1,0)} + X_{(-1,0)} + X_{(0,1)}$$

$\mathbb{L}_2$: fund. Wilson line; subscript: (γ_e, γ_m)

Line defects in class-S theory

6d $(2,0)$ $\mathfrak{gl}(N)$ theory on $C \times \mathbb{R}^{3,1}$ (C: Riemann surface) with certain twisting, compactify on $C \rightsquigarrow$ 4d N=2 theory of class S. [Gaiotto],[GMN]



Line defects $\mathbb{L}$ in class-S theory $\leftrightarrow$ "loops" ℓ on C (junctions, laminations)

[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke]...

[Fock-Goncharov],[Sikora],[Le],[Xie],[Saulina],[Coman-Gabella-Teschner],[Tachikawa-Watanabe],[Gabella]...

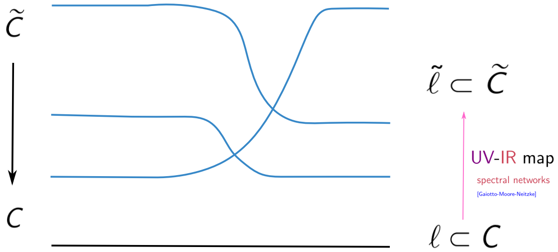
The surface defect carries representation of $\mathfrak{gl}(N)$, consider fundamental representation.

The UV-IR map: geometric picture

A pt. in Coulomb branch $\leftrightarrow$ a N-fold branched covering $\widetilde{C} \to C$,
$\widetilde{C} \subset T^*C$ is the Seiberg-Witten curve.

IR: bulk theory approx. by 6d $(2,0)$ theory of type $\mathfrak{gl}(1)$ on $\widetilde{C} \times \mathbb{R}^{3,1}$.

IR line defects $\leftrightarrow$ loops $\tilde{\ell}$ on $\widetilde{C}$



The UV-IR map as the trace map

Class-S theory on S^1 $\rightarrow$ 3d $N = 4$ sigma model with target M_{Hitchin} [GMN]
M_{Hitchin} hyper-Kähler, cplx. structure labeled by $\zeta \in \mathbb{CP}^1$:
⋆ $\zeta = 0$, cplx integrable system, fibered over Hitchin base (4d CB)
⋆ $\zeta \in \mathbb{C}^\times$, moduli space of flat $GL(N, \mathbb{C})$-connections $\mathcal{A}_\zeta$ on C

- VEV of UV line defect $\mathbb{L}$ wrapping S^1: holomorphic trace functions

$$\langle \mathbb{L}(\zeta, \ell) \rangle = \text{Tr}\left[\text{Hol}_\ell \mathcal{A}_\zeta\right]$$

- VEV of IR line defect X_γ wrapping S^1: Darboux coordinates $\mathcal{X}_\gamma$
- The UV-IR map $\rightarrow$ the trace map:

$$\text{Tr}\left[\text{Hol}_\ell \mathcal{A}_\zeta\right] = \sum_\gamma \overline{\underline{\Omega}}(\mathbb{L}, \gamma) \mathcal{X}_\gamma$$

refined index $\overline{\underline{\Omega}}(\mathbb{L}, \gamma, q) \rightarrow$ quantization of UV-IR map/trace map?

Line defects OPE

algebra of hol. functions on M_{Hitchin} $\leftrightarrow$ line defects operator products:

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta)\rangle = \langle \mathbb{L}_1(\zeta)\rangle\langle \mathbb{L}_2(\zeta)\rangle$$

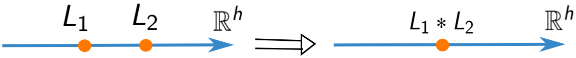
This algebra structure admits a quantization via skein algebras.

[Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],

[Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

Turning on Ω-bkg on a $\mathbb{R}^2$-plane: non-commutative associative OPE $*$

[Nekrasov-Shatashvili],[Gaiotto-Moore-Neitzke],[Ito-Okuda-Taki],[Yagi],[Oh-Yagi],...



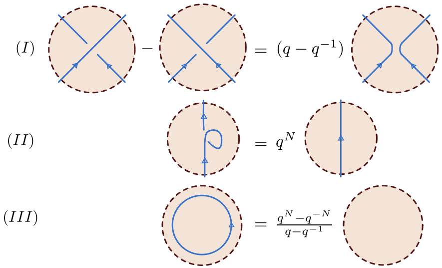
IR: quantum torus algebra $X_{\gamma_1} * X_{\gamma_2} = (-q)^{\langle\gamma_1,\gamma_2\rangle} X_{\gamma_1+\gamma_2}$

quantum UV-IR map: UV skein algebra $\rightarrow$ quantum torus algebra

The UV and IR skein algebras

UV skein algebra: $\mathfrak{gl}(N)$ HOMFLY skein algebra of $M = C \times \mathbb{R}^h$
IR skein algebra: (twisted) $\mathfrak{gl}(1)$ skein algebra of $\widetilde{M} = \widetilde{C} \times \mathbb{R}^h$
algebra structure $\leftrightarrow$ stacking links along $\mathbb{R}^h$

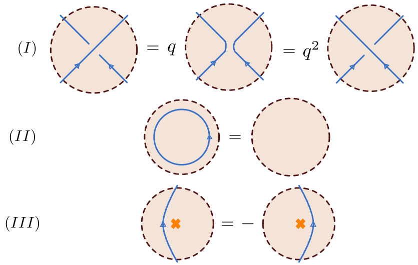
UV: $\mathbb{Z}[q^{\pm 1}]$-module of oriented framed links $L \subset M$



4d $\frac{1}{4}$-BPS line defects: $U(1)$ rot. & $U(1)_R \subset SU(2)_R$

[Witten],[Gaiotto-Witten],[Ooguri-Vafa],[Gukov-Schwarz-Vafa],[Dimofte-Gaiotto-Gukov], [Chun-Gukov-Roggenkamp],[Gukov-Putrov-Vafa].[Gukov-Pei-Putrov-Vafa]. ...

IR: $\mathbb{Z}[q^{\pm 1}]$-module of oriented framed links $\widetilde{L} \subset \widetilde{M}$

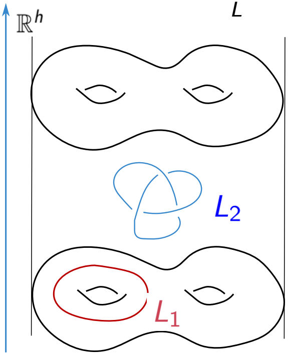


iso. to quantum torus

The quantum UV-IR map

The quantum UV-IR map sends $L \subset M$ to combinations of $\widetilde{L} \subset \widetilde{M}$:

$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L}, \quad \alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}]$$



$F(L_1)$: gen. function for framed BPS index

$\overline{\underline{\Omega}}(\mathbb{L}, \gamma, q)$

$F(L_2) = q^{Nw(L_2)} P_{\text{HOMFLY}}(L_2, a = q^N, z = q - q^{-1})$

self-linking number

N=2: vertex model

[Witten],[Gaiotto-Witten]

See also [Bonahon-Wong],[Goncharov-Shen],[Douglas-Sun],...

The quantum UV-IR map: construction

Given $L \subset M$, enumerate all possible $\widetilde{L} \subset \widetilde{M}$, assoc. with factor $\alpha(\widetilde{L})$.

Physics: twisted and Ω-deformed 5d $N = 2$ U(N) super Yang-Mills on $M \times \mathbb{R}^2_\epsilon$, w/ fund. Wilson line insertion along $L \subset M$, in a background that generically breaks $U(N) \to U(1)^N$ labeled by $\widetilde{M} \to M$.
"Expand" the partition function via partition function of IR effective theory, with IR Wilson line insertions.

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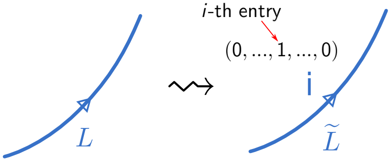
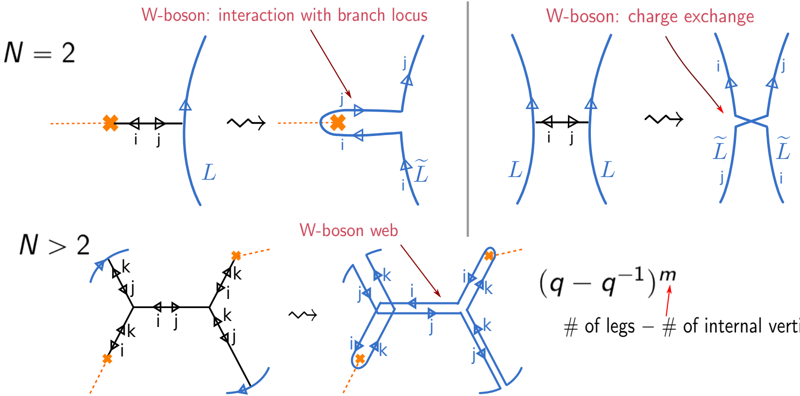
This hints at the strategy of enumerating $\widetilde{L}$.

- locally away from branch points:



The quantum UV-IR map: network of webs

- corrections from massive W-bosons ↔ networks of webs on C (BPS ij-trajectories introduced by [Gaiotto-Moore-Neitzke])



$(q - q^{-1})^m$

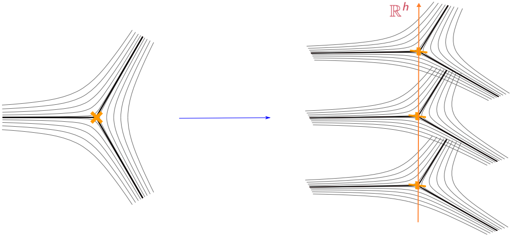
# of legs − # of internal vertices

The quantum UV-IR map: BPS leaves

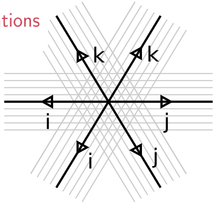
Given a point in CB, labeled by $\widetilde{C} : \lambda^N + p_1(z)\lambda^{N-1}(z) + \ldots + p_N(z) = 0$

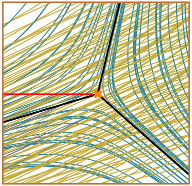
BPS *ij*-leaves: 1-dim leaves on C, $\mathrm{Im}\left[e^{-i\theta}(\lambda_i - \lambda_j)\right] = 0$ (mutually BPS)

$N = 2$

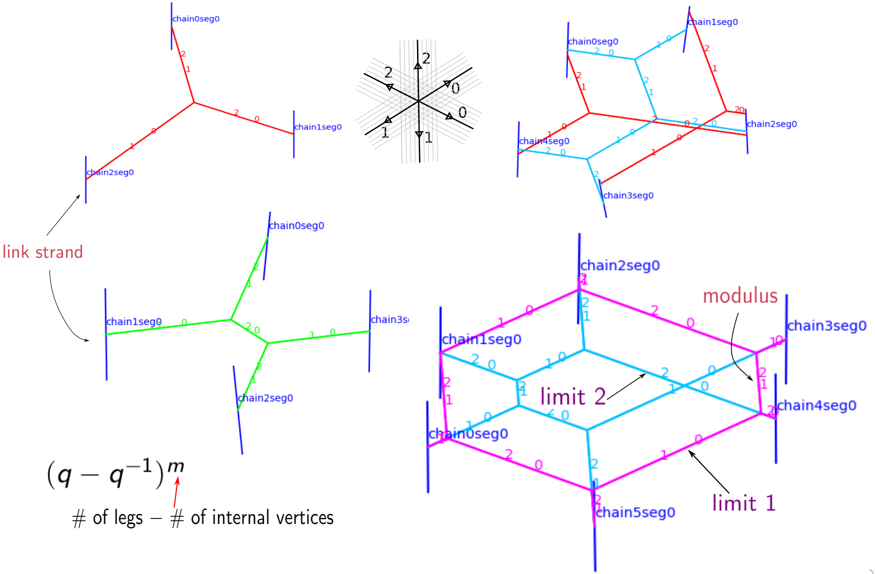


$N > 2$: $\binom{N}{2}$ distinct foliations





Example of W-boson webs: C=$\mathbb{R}^2$, M=$\mathbb{R}^2 \times \mathbb{R}^h$, $N = 3$



$$(q - q^{-1})^m$$

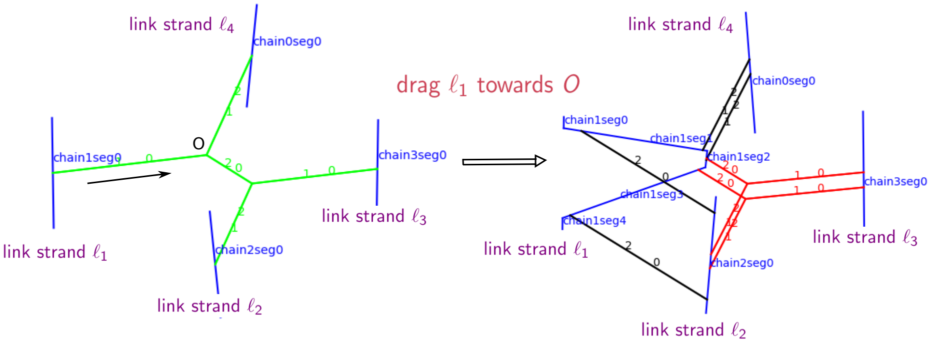
# of legs − # of internal vertices

The quantum UV-IR map: bootstrapping $\alpha(\widetilde{L})$

Prescriptions for $\alpha(\widetilde{L})$ determined by bootstrap-like method:

the quantum UV-IR map F preserves skein relations, is isotopy invariant.

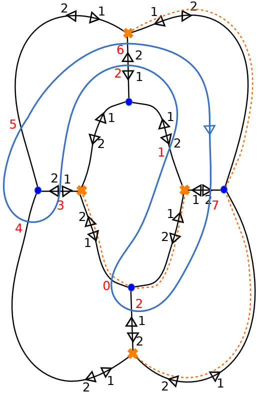
e.g. weights associated with webs determined in a recursive way:



Example of protected spin character: $SU(2)$ with $N_f = 4$

Take $N = 2$, $M = C \times \mathbb{R}^h$ where C is a four-punctured sphere,

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2} dz^2.$$



$$\begin{aligned}
&X_{\gamma_1+\mu_1-\mu_3} + X_{\gamma_2+\mu_1-\mu_3} + X_{\gamma_1+\gamma_2+\mu_1-\mu_3} + X_{-\gamma_2-\mu_1+\mu_3} + X_{\gamma_1+\mu_1+\mu_3} \\
&+ X_{\gamma_1-\gamma_2+\mu_1+\mu_3} + X_{2\gamma_1-3\gamma_2-\mu_2+2\mu_3-3\mu_4} \underline{- (q + q^{-1})} X_{2\gamma_1-2\gamma_2-\mu_2+2\mu_3-3\mu_4} \\
&+ X_{2\gamma_1-\gamma_2-\mu_2+2\mu_3-3\mu_4} + X_{\gamma_1-2\gamma_2-\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\gamma_2-\mu_1+\mu_3-2\mu_4} \\
&+ X_{\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{- (q + q^{-1})} X_{2\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{- (q + q^{-1})} X_{2\gamma_1-2\gamma_2+\mu_1+\mu_3-2\mu_4} \\
&+ X_{\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + \underline{(2 + q^2 + q^{-2})} X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\mu_2-\mu_4} \\
&+ X_{\gamma_1-\gamma_2-\mu_2-\mu_4} + X_{\gamma_1+\mu_2-\mu_4} + X_{\gamma_1-\gamma_2+\mu_2-\mu_4} + X_{\gamma_1+2\mu_1+\mu_2-\mu_4} \\
&\underline{- (q + q^{-1})} X_{2\gamma_1+2\mu_1+\mu_2-\mu_4} + X_{2\gamma_1-\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} \\
&+ X_{2\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+2\mu_3-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+2\mu_3-\mu_4}.
\end{aligned}$$

Framed wall-crossing

The quantum UV-IR map F depends on the covering $\widetilde{M} \to M$. Moving on the Coulomb branch, the framed BPS index could change discontinuously. This is called (framed) wall-crossing, controlled by the BPS spectrum of bulk 4d theory. [Kontsevich-Soibelman],[Fock-Goncharov], [Gaiotto-Moore-Neitzke], [Dimofte-Gukov-Soibelman],...

For example, across a wall corres. to a BPS hypermultiplet with charge γ:

$$F(L) \to E_q(X_\gamma) F(L) E_q(X_\gamma)^{-1},$$

where $E_q(x)$ is the quantum dilogarithm.

By studying $F(L)$ before and after the jump, we obtain information about the bulk BPS spectrum with spin.
(motivic Donaldson-Thomas invariants)

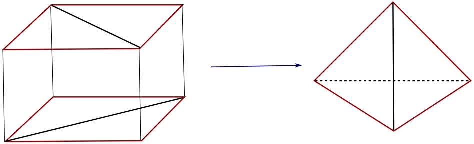
General three-manifold M

We have taken $M = C \times \mathbb{R}$. What about more general three-manifold M?

Reducing the 6d theory (with surface defect) on M gives a 3d $N = 2$ theory with line defect insertion. One could also make a perturbation, labeled by $\widetilde{M} \to M$, under which the bulk theory flows to a Lagrangian theory in the IR. [Dimofte-Gaiotto-Gukov],[Cecotti-Córdova-Vafa],[Dimofte-Gaiotto-van der Veen],...

Q: how does a UV line defect decompose in terms of IR line defects? Here the quantum UV-IR map should go from the $\mathfrak{gl}(N)$ skein module of M to the (twisted) $\mathfrak{gl}(1)$ skein module of $\widetilde{M}$.

Extend the construction to 3-mfds admitting tetrahedron triangulations?



$\widetilde{M}$ has an new kind of singularity, one within each tetrahedron.
[Dimofte-Gaiotto-van der Veen],[Cecotti-Córdova-Vafa],[Freed-Neitzke]

Summary A

- The quantum trace map/quantum UV-IR map, embedding $\mathfrak{gl}(N)$ HOMFLY skein algebra into the quantum torus algebra.
- A new computation of HOMFLY polynomial, unified with computation of refined framed BPS index for line defects in class-S theories.
- Interesting to consider more general 3-mfd M and the UV-IR map for line defects in 3d N=2 theories, as a map between UV and IR skein modules.
- Certain networks of webs play an important role in the construction. A subset of networks (spectral networks [GMN]) also appear in a seemly different setup.

The Schrödinger equation

There has been many interesting developments in application of resurgence theory and exact WKB analysis to quantum mechanics.

[Alvarez-Casares],[Zinn-Justin,Jentschura],[Dunne-Ünsal],[Basar-Dunne-Ünsal],[Behtash-Dunne-Schäfer-Sulejmanpasic-Ünsal],

[Misumi-Nitta-Sakai],[Fujimori-Kamata-Misumi-Nitta-Sakai],[Sueishi-Kamata-Misumi-Ünsal] ...

The (complex) Schrödinger equation:

$$\left[\partial_z^2 + \hbar^{-2} P(z)\right] \psi(z) = 0,$$

$P(z)$ is holomorphic or meromorphic.

Geometric approach to study exact quantization conditions (e.g. for bound states), Stokes data etc.

Schrödinger equations as quantum Seiberg-Witten curves

- Consider the Seiberg-Witten curve of certain 4d $N = 2$ theory:

$$\widetilde{C}: \; y^2 + P(z) = 0,$$

Promoting y (momentum) and z (position) to Heisenberg operators, in position representation:

$$\left[\partial_z^2 + \hbar^{-2} P(z)\right] \psi(z) = 0$$

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- Polynomial potential ↔ Argyres-Douglas theories
(Modified) Mathieu potential $cosh(z)$,$cos(z)$ ↔ pure $SU(2)$ SYM
Certain polynomials in $e^{\pm iz}$ ↔ $SU(2)$ with fundamental matter

[Gaiotto],[Dunne-Ünsal], [Basar-Dunne-Ünsal],[Huang],[Grassi-Hatsuda-Mariño],[Codesido-Grassi-Mariño],

[Grassi-Mariño],[Ito-Mariño-Shu],[Kashani-Poor,Troost],[Ashok-Jatkar-John-Raman-Troost],[Ito-Kanno-Okubo],

[Hollands-Neitzke],[Grassi-Gu-Mariño],[Coman-Longhi-Teschner],[Dumas-Neitzke],[Imaizumi],[Grassi-Hao-Neitzke],...

Warning: subtleties regarding choices of quantization

4d $N = 2$ theories and quantum mechanical systems

- The gauge/Bethe correspondence: [Nekrasov-Shatashvili],[Nekrasov-Rosly-Shatashvili]
  4d $N = 2$ gauge theories in the NS limit ($\epsilon_1 = \hbar, \epsilon_2 \to 0$) of Ω-bkg
  $\to$ quantization of the underlying SW integrable systems.
- The TS/ST correspondence: [Grassi-Hatsuda-Mariño],[Mariño],[Codesido-Grassi-Mariño]
  Association of a quantum mechanical operator to a toric Calabi-Yau, explicit spectrum computable via topological string free energy.
- The conformal limit: [Gaiotto]
  Class-S theory on $S^1_R \to$ 3d $N = 4$ sigma model with target M_{Hitchin}.
  scaling limit: $R \to 0, \zeta \to 0$, $\hbar = \zeta/R$ fixed. (ζ: cplx. structure)
  Hitchin section $\longrightarrow$ variety of opers
  Rk: geometric reformulation of exact WKB via abelianization.

  [Gaiotto-Moore-Neitzke],[Hollands-Neitzke],...

  [Voros],[Silverstone],[Delabaere-Dillinger-Pham],[Kawai-Takei],[Iwaki-Nakanishi],...

Exact WKB for Schrödinger equations

WKB ansatz: $\psi(z) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda(z')dz'\right) \rightarrow [\partial_z^2 + \hbar^{-2}P(z)]\psi(z) = 0$

$\lambda(z)$ obeys the Ricatti equation

$$\lambda(z)^2 + P(z) + \hbar\partial_z\lambda(z) = 0.$$

Build a formal series solution λ^{formal} in powers of $\hbar$,

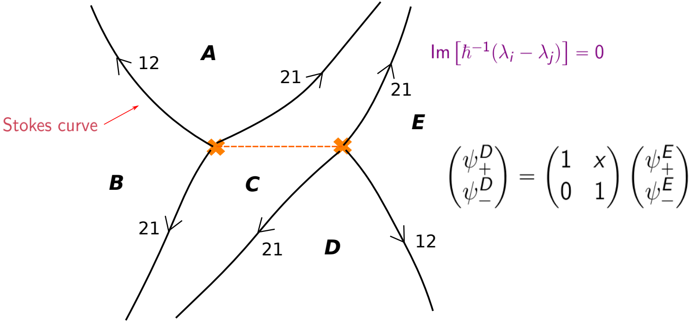
$$\text{order-}\hbar^0 : \; y^2 + p(z) = 0, \quad \text{classical SW curve}$$

Choose a branch labeled by $i \in \{\pm\}$:

$$\lambda_i^{\text{formal}} = y_i - \hbar\frac{P'}{4P} + \hbar^2 y_i \frac{5P'^2 - 4PP''}{32P^3} + \ldots$$

$\longrightarrow$ Two formal solutions $\psi_\pm^{\text{formal}}(z,\hbar)$ as series in $\hbar$.

Exact WKB for Schrödinger equations



- $\exists$ two actual solutions $\psi_\pm(z)$ within each region, where the solutions jump across a Stokes curve.
- In class-S theory context: Stokes curves $\leftrightarrow$ spectral networks [GMN] corresponds to the critical networks in part A of the talk

The Voros symbol

The Voros symbol: $\mathcal{X}_\gamma(\hbar) \in \mathbb{C}^\times$, $\gamma \leftrightarrow$ 1-cycles of Seiberg-Witten curve

- $\mathcal{X}_\gamma(\hbar)$ captures the Borel resummed quantum periods:

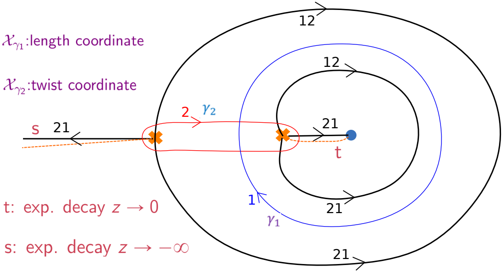
$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar) dz = \sum_{n=0}^{\infty} \Pi_\gamma^{(n)} \hbar^n$$

- $\mathcal{X}_\gamma(\hbar)$ expressed as Wronskians of distinguished local solutions:
  - ⋆ asymptotically decaying solutions as z approaches a singularity
  - ⋆ eigenvectors of the monodromy around a loop
- $\mathcal{X}_\gamma(\hbar)$ could be identified as spectral coordinates on a moduli space of flat $SL(2, \mathbb{C})$-connections.
  (conformal limit of Darboux coordinate $\mathcal{X}_\gamma(\zeta)$ in part A of the talk)

The Voros symbol: modified Mathieu operator

$$[-\hbar^2\partial_x^2 + 2\cosh(x) - 2E]\psi(x) = 0 \quad (E > 1)$$

$$z = -e^{-x} \to \left[\hbar^2\partial_z^2 + \left(\frac{1}{z^3} + \frac{1}{z} + \frac{2E + 0.25\hbar^2}{z^2}\right)\right]\tilde{\psi}(z) = 0.$$ SU(2) SYM



bound states: s prop. to $t \to \mathcal{X}_{\gamma_2} = 1$ (exact quantization condition)

[Mironov-Morozov],[He-Miao],[Basar-Dunne],[Dunne-Ünsal],[Codesido-Marino-Schiappa],[Hollands-Neitzke],...

The Voros symbol: 4d $N = 2$ theory perspective

- $\mathcal{X}_\gamma(\hbar)$ obey TBA-like integral equation: [Gaiotto],[Gaiotto-Moore-Neitzke],[Ito-Mariño-Shu]

Classical period

$$\mathcal{X}_\gamma(\hbar) = \exp\left[\frac{Z_\gamma}{\hbar} + \frac{1}{4\pi i}\sum_\mu \Omega(\mu)\langle\gamma,\mu\rangle \int_{\hbar' \in \mathbb{R}_- Z_\mu} \frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1+\mathcal{X}_\mu(\hbar'))\right]$$

BPS index

- Instanton calculus resums special quantum periods $\{a_i(\hbar), a_D^i(\hbar)\}$

Nekrasov-Shatashvili free energy

$$a_D^i(a_1,\ldots,a_r;\hbar) = \frac{\partial F_{\mathrm{NS}}(a_1,\ldots,a_r;\hbar)}{\partial a_i}, \quad i = 1,\ldots,r$$

rank

$a(\hbar), a_D(\hbar) \leftrightarrow$ special Voros symbols: Fenchel-Nielsen coordinates

[Nekrasov-Rosly-Shatashvili],[Hollands-Kidwai],[Jeong-Nekrasov],...

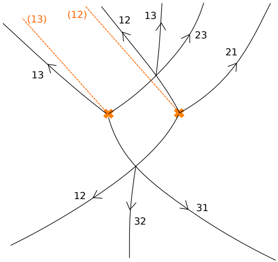
Higher rank generalization

Interesting and challenging to generalize the story to higher order Schrödinger-like equations

[Aoki-Kawai-Takei],[Hollands-Neitzke],[Jeong-Nekrasov],[Haouzi-Oh],[Dumas-Neitzke],[Ito-Kondo-Kuroda-Shu],...

$$\left[\partial_z^N + P_2(z,\hbar)\partial_z^{N-2} + \dots P_N(z,\hbar)\right]\psi(z) = 0.$$

Network of Stokes curves (spectral networks [GMN]) becomes complicated:

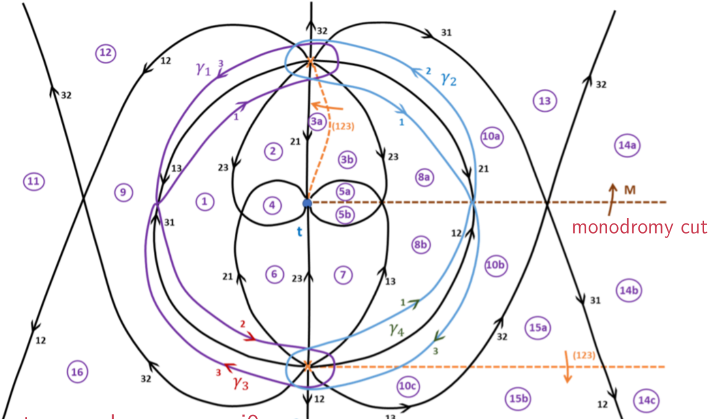


pure SU(3) SYM
[Y, 2012.15658] and wip

$$\left[\partial_z^3 + \hbar^{-2}\frac{u_1+\hbar^2}{z^2}\partial_z + \left(\hbar^{-3}\left(\frac{\Lambda}{z^4} + \frac{u_2}{z^3} + \frac{\Lambda}{z^2}\right) - \hbar^{-2}\frac{u_1+\hbar^2}{z^3}\right)\right]\psi(z) = 0,$$

Pure $SU(3)$ SYM: strong coupling region

Strong-coupling region: 12 BPS states



t: exp. decay $z \to -i0$

s: exp. decay $z \to -i\infty$

Pure $SU(3)$ SYM: strong coupling region

The Voros symbols $\mathcal{X}_\gamma(\hbar)$ expressed via special solutions $s, t, M^{\pm}s, M^{\pm}t$.

$\hbar \to 0$:

$$\log\left(\mathcal{X}_\gamma(\hbar)\right) \sim \frac{1}{\hbar}\Pi_\gamma(\hbar)$$

quantum periods

	$\hbar = \frac{1}{2}e^{i\pi/3}$	
	evaluation of (3.4)	$\frac{1}{\hbar}\Pi_\gamma(\hbar)$ at $o(\hbar^6)$
$\log\mathcal{X}_{\gamma_1}$	-11.21119	-11.21120
$\log\mathcal{X}_{\gamma_2}$	-11.21119	-11.21120
$\log\mathcal{X}_{\gamma_3}$	$5.60559 + 2.71805i$	$5.60560 + 2.71808i$
$\log\mathcal{X}_{\gamma_4}$	$5.60559 + 2.71805i$	$5.60560 + 2.71808i$

Pure $SU(3)$ SYM: strong coupling region

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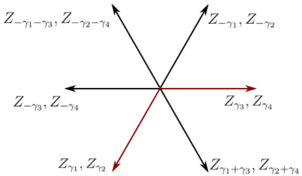
$\log\mathcal{X}_{\gamma_4}$	$5.60559 + 2.71805i$	$5.60560 + 2.71808i$

$\mathcal{X}_\gamma(\hbar)$ also computable via integral equations [Gaiotto],[Gaiotto-Moore-Neitzke]

$$\mathcal{X}_\gamma(\hbar) = \exp\left[\frac{Z_\gamma}{\hbar} + \frac{1}{4\pi i}\sum_\mu \Omega(\mu)\langle\gamma,\mu\rangle \int_{\hbar' \in \mathbb{R}_- Z_\mu} \frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1+\mathcal{X}_\mu(\hbar'))\right]$$

Classical period

BPS index



	$\hbar = e^{i\pi/3}$	
	evaluation of (3.4)	integral equation
$\log\mathcal{X}_{\gamma_1}$	-5.486	-5.492
$\log\mathcal{X}_{\gamma_2}$	-5.486	-5.492
$\log\mathcal{X}_{\gamma_3}$	$2.743 + 1.252i$	$2.748 + 1.243i$
$\log\mathcal{X}_{\gamma_4}$	$2.743 + 1.252i$	$2.748 + 1.243i$

Summary

- Two different quantization scenarios:

  ⋆ quantum UV-IR map: a new computation of HOMFLY polynomial, unified with computation of refined framed BPS indices for line defects in class-S theories

  ⋆ Exact WKB method for Schrödinger-like equations, as quantum Seiberg-Witten curves of 4d $N = 2$ theories

- Both involve certain networks on Riemann surface C

- Possibility to unite these two scenarios?
  Turning on the full Ω-background?

  I will briefly sketch a different possibility. (discussions w/ Gaiotto-Moore-Neitzke)

q-deformed integral equations

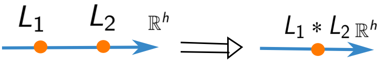
The VEV of IR line defects $X_\gamma(\zeta)$ around S^1_R obeys integral equations [GMN]:

central charge, electromagnetic holonomy

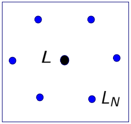
$$\mathcal{X}_\gamma(\zeta) = \exp\left[\frac{R}{\zeta}Z_\gamma + R\zeta\bar{Z}_\gamma + i\theta_\gamma + \frac{1}{4\pi i}\sum_\mu \Omega(\mu)\langle\gamma,\mu\rangle \int_{\mathbb{R}_- Z_\mu} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1+\mathcal{X}_\mu(\zeta'))\right]$$

preserved supercharge, circle radius, BPS index

Turning on half Omega background, OPE becomes non-commutative.



$$X_{\gamma_1} * X_{\gamma_2} = (-q)^{\langle\gamma_1,\gamma_2\rangle} X_{\gamma_1+\gamma_2}$$



q-deformed integral equations: operator equation respecting quantum torus, jumps consists of quantum dilogarithm. Solution to q-TBA solve q-deformed RH problem. [Barbieri-Bridgeland-Stoppa]

Simplification happens if q is N-th root of unity.

Thank You and Stay Healthy!