Theta-Defects and Non-Invertible Symmetries

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Generalized Symmetry "Revolution"

Generalized Global Symmetries

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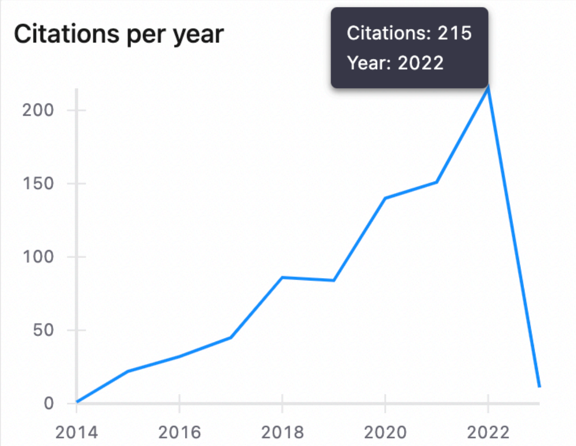


Abstract: (Springer)
A q-form global symmetry is a global symmetry for which the charged operators are of space-time dimension q, e.g. Wilson lines, surface defects, etc., and the charged excitations have q spatial dimensions, e.g. strings, membranes, etc. Many of the properties of ordinary global symmetries (q = 0) apply here. They lead to Ward identities and hence to selection rules on amplitudes. Such global symmetries can be coupled to classical background fields and they can be gauged by summing over these classical fields. These generalized global symmetries can be spontaneously broken (either completely or to a sub-group). They can also have 't Hooft anomalies, which prevent us from gauging them, but lead to 't Hooft anomaly matching conditions. Such anomalies can also lead to anomaly inflow on various defects and exotic Symmetry Protected Topological phases. Our analysis of these symmetries gives a new unified perspective of many known phenomena and uncovers new results.

Note: 49 pages plus appendices. v2: references added

Non-invertible Symmetries in $d > 3$:

In the context of QFTs in $d > 3$ within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng][2]

[Choi, Cordova, Hsin, Lam, Shao]

[Bhardwaj, Bottini, SSN, Tiwari][3]

[Roumpedakis, Seifnashri, Shao]

[Antinucci, Galati, Rizi]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

[Cordova, Ohmori]

[Bhardwaj, SSN, Wu]

[Bartsch, Bullimore, Ferrari, Pearson]

[Bashmakov, del Zotto, Hasan, Kaidi]

[Antinucci,Benini, Copetti, Galati, Rizi]

[Cordova, Hong, Koren, Ohmori]

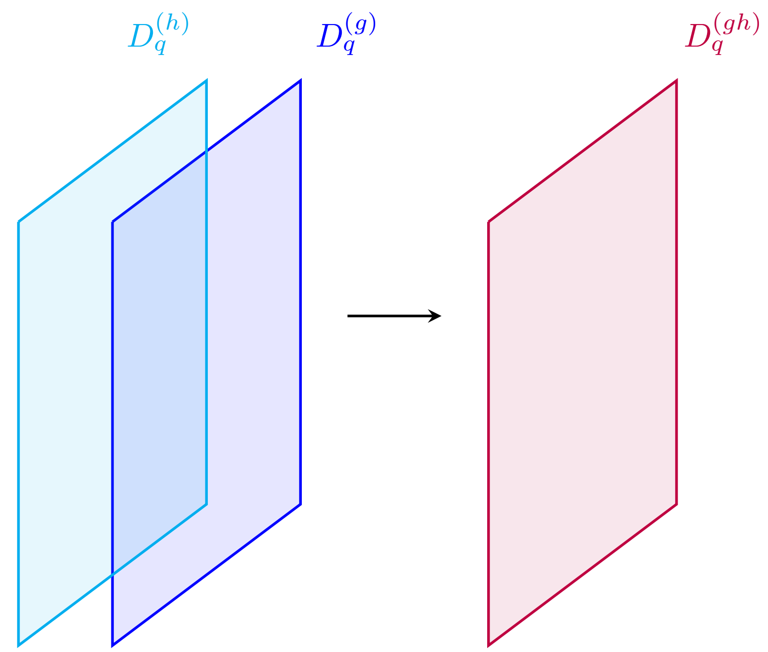
. . .

Symmetries from Topological Operators

Any topological operator in a QFT is a symmetry generator.

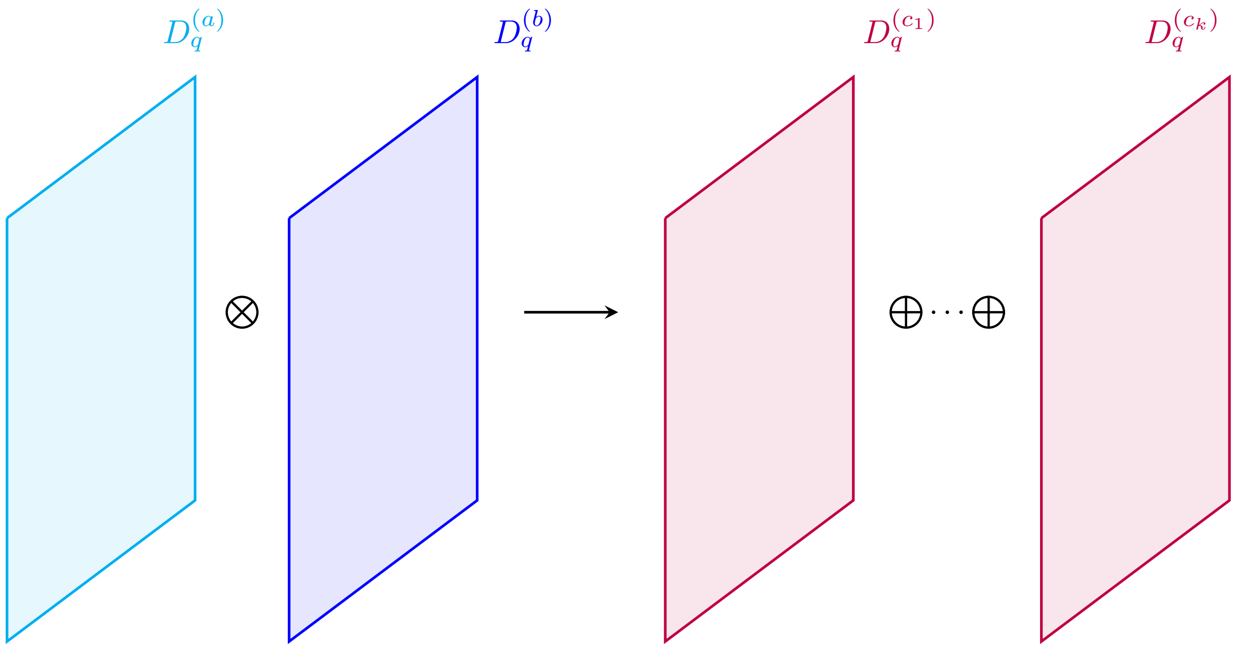
1. Higher-form symmetries $\Gamma^{(p)}$: [Gaiotto, Kapustin, Seiberg, Willett, 2014]
   charged objects are p-dimensional defects, whose charge is measured by codimension $p+1$ topological operators $D^{g}_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$:

$$D_q^g \otimes D_q^h \; = \; D_q^{gh} \,, \qquad g, h \in \Gamma^{(p)}$$



2. Higher-group symmetries:
   $\{p$-form symmetries$\}$ might not form product groups

3. Non-invertible symmetries:
   group ⇒ algebra



Perhaps surprisingly:

These are ubiquitous in higher dimensional QFTs, e.g. 4d pure Yang Mills.

Science-sociological bonus: Provides a really exciting connection between hep-th, hep-ph, cond-mat, and math.

Examples:

2d: Verlinde lines in a 2d rational conformal field theory (RCFT)

$$D_1^i \otimes D_1^j = \bigoplus_k N_k^{ij} D_1^k$$

N_{ij}^k = RCFT fusion coefficients obtained by the Verlinde formula

3d: Modular tensor categories: classification of topological order:

4d: By now many examples and methods of construction: e.g.

(i) $O(2) = U(1) \rtimes \mathbb{Z}_2$ [Heidenreich, McNamara, Montero,Reece, Rudelius, Valenzuela]

(ii) Outer automorphism gauging, e.g. $\mathrm{Pin}^+(4N)$ theory. [Bhardwaj, Bottini, SSN, Tiwari]: Non-invertible 1-form symmetry

(iii) Duality defects: [Choi, Cordova, Hsin, Lam, Shao][Kaidi, Ohmori, Zheng]: Non-invertible 0-form symmetry

Utility of Non-Invertible Symmetries

Many applications – but clearly only scratching the surface:

1. 2d: constraints on existence of gapped phases, and number of vacua
2. Confinement/Deconfinement: in 4d QFT and holography constrained by non-invertible defects in $\mathcal{N} = 1$ pure Yang-Mills
3. Applications to hep-ph: e.g. neutrino mass generation from non-invertible symmetry breaking etc.
4. Swampland/No Global Symmetry conjecture

Non-Invertible to Categorical

Consider more generally a QFT in d-dimensions, with not-necessarily invertible fusion of topological defects of various dimensions.

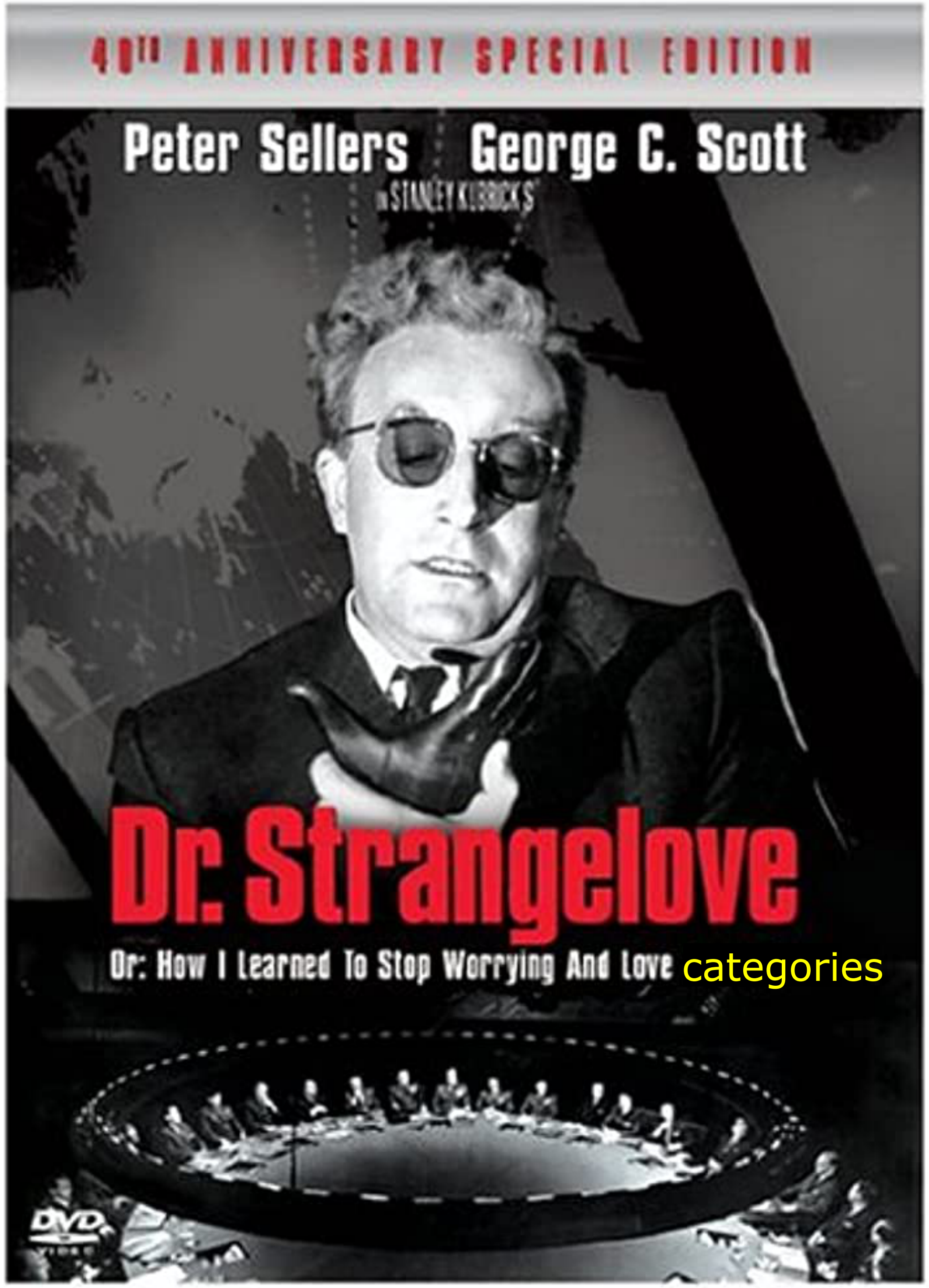
What is the proper framework for characterizing such symmetries?

- For 0-form (and p-form) symmetry groups:
  Obviously, Group Theory and Representations

  Historic note: this was not always so obvious. According to Wigner (1981), Erwin Schrödinger coined the expression "Gruppenpest" and stated it ought to be abandoned.

- For non-invertible symmetries: topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion:
  Higher-Fusion Categories

$\Rightarrow$ (Higher) Categorical Symmetries


40TH ANNIVERSARY SPECIAL EDITION
Peter Sellers
George C. Scott
in STANLEY KUBRICK'S
Dr. Strangelove
Or: How I Learned To Stop Worrying And Love categories

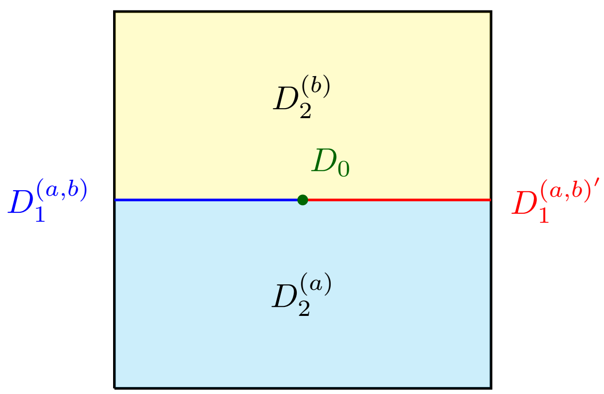
Categorical Symmetries

Higher-categories play a similar role to groups and group representations.

A fusion p-category has the following structure:

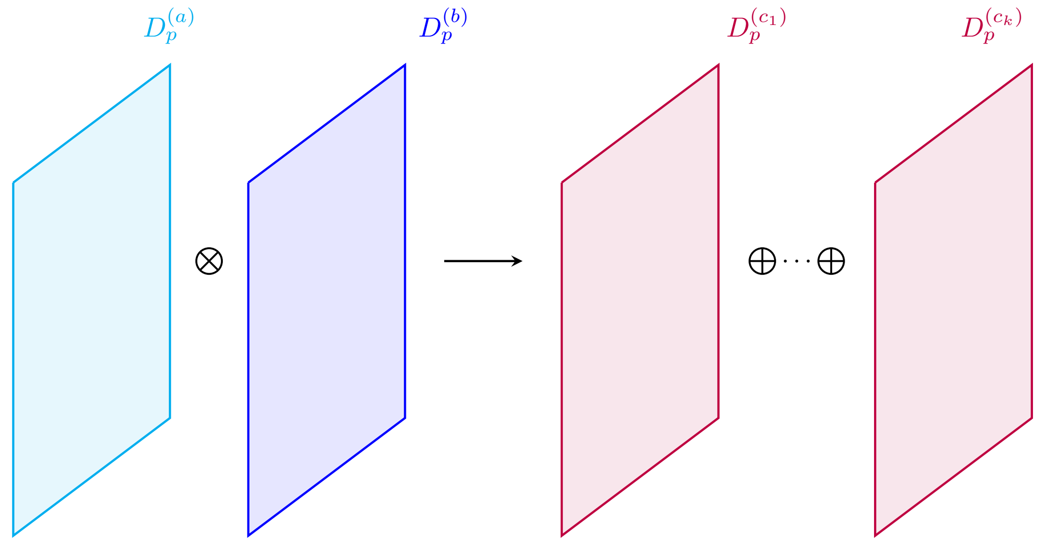
1. A set of objects: D_p of p-dimensional topological defects
2. 1-Morphisms: maps between two defects D_p and D'_p, i.e. D_{p-1} defects.
3. 2-Morphisms: maps between two 1-morphisms, i.e. D_{p-2} defects.
4. $\cdots$

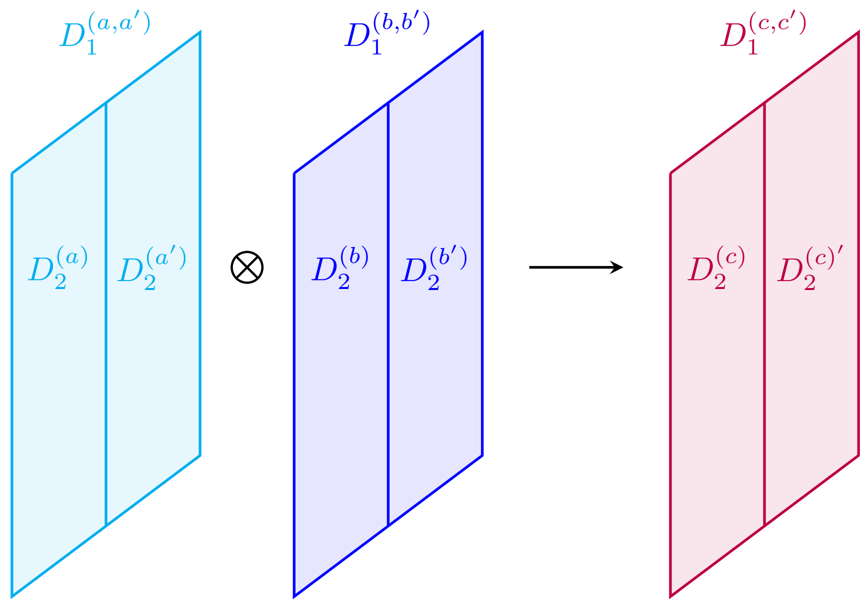
Example: 2-fusion categories [Douglas, Reutter; 2018]



Objects: topological surface operators
1-morphisms: topological lines
2-morphisms: local operators

Monoidal structure: loosely speaking, there is a fusion between objects, 1-morphisms, etc:





$$D_1^{(c,c')} = D_1^{(a,a')} \otimes D_1^{(b,b')}\,, \qquad D_2^{(a)} \otimes D_2^{(b)} \supset D_2^{(c)} \text{ etc}$$

Theta-Defects:
Universal Construction of Non-Invertibles

[Bhardwaj, SSN, Wu][Bhardwaj, SSN, Tiwari]

Theta-Defects

Lets start with 4d Maxwell:

$$\mathcal{L}_{U(1)} = \frac{1}{g^2}\int F \wedge \star F + \theta \int F \wedge F$$

We can think of this theory as follows:
Consider a 4d trivial theory with a trivial $U(1)$ global symmetry, background A, but a symmetry protected phase (SPT)

$$\mathcal{L}_T = \mathcal{L}_{\text{trivial}} + \text{SPT}\,, \qquad \text{SPT} = \theta \int F \wedge F$$

Gauging $U(1)$ we obtain Maxwell and the SPT becomes the theta-angle

$$\mathcal{L}_{T/U(1)} = \frac{1}{g^2}\int F \wedge \star F + \theta \int F \wedge F$$

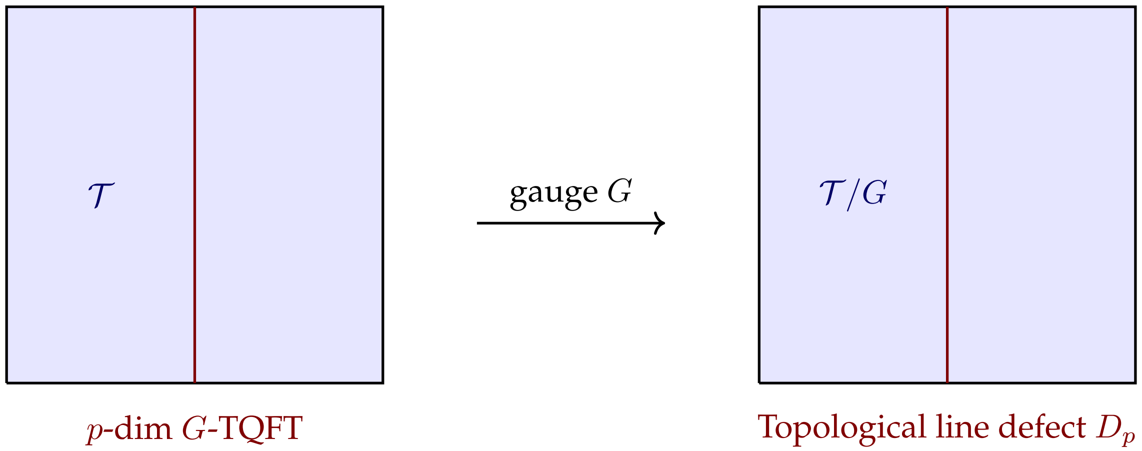
This can be generalized to any theory with $U(1)$ global symmetry that can be gauged:

Stacking with $U(1)$-SPT and gauging adds a θ-angle.

Theta Defects

[Bhardwaj, SSN, Wu][Bhardwaj, SSN, Tiwari]

Let $\mathcal{T}$ be a d-dim QFT with a $G^{(0)}$ symmetry.
Consider a G-symmetric p-dimensional TQFT. Gauge the diagonal G:



In the gauged theory, the TQFT is now a topological defect of the theory.
Generically: the fusion of these defects is non-invertible.

Warmup: Gauging in 2d QFTs (Orbifolding)

Let $\mathcal{T}$ be a 2d theory with a 0-form symmetry G.

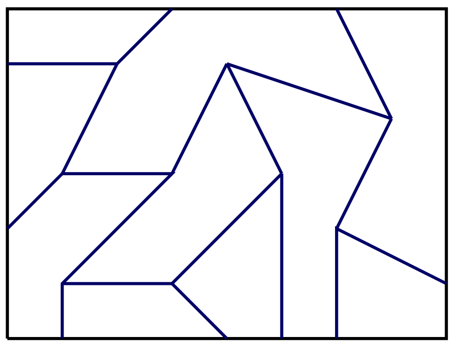
The symmetries of this theory are generated by topological lines $D_1^{(g)}$, which fuse according to the group multiplication in G

$$D_1^{(g)}\,,\ g \in G, \qquad D_1^{(g)} \otimes D_1^{(h)} = D_1^{(gh)}$$

This defines a fusion category: $\mathcal{C} = \mathrm{Vec}(G)$.

Gauging G: $\mathcal{T}/G$ is defined by:

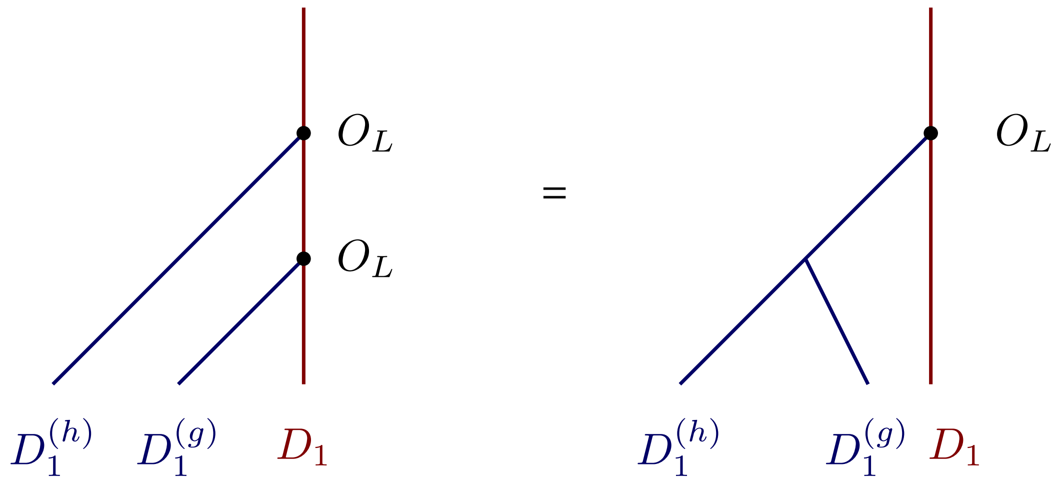
- Couple to G-background: insert a fine mesh of topological lines $D_1^{(g)}$:



- Sum over such backgrounds

What are the symmetries of $\mathcal{T}/G$? In the gauged theory, we only allow lines D_1 which are gauge-invariant, i.e. configurations of lines that can move freely through the G-mesh.

- $D_1^{(g)}$ lines can end from the left on D_1 subject to



- Similarly: couple to G-backgrounds from the right.
- Compatibility of left and right coupling

$\Rightarrow$ The $D_1^{(g)}$ lines are invisible to the lines D_1 in $\mathcal{T}/G$.

Mathematically, in order to gauge we need to pick an algebra A in $\mathcal{C}$ and solve the above bimodule conditions. For $A = \oplus_{g\in G} D_1^{(g)}$ in $\mathrm{Vec}(G)$:

$$\mathrm{Bimod}_{\mathrm{Vec}(G)}(A) = \mathrm{Rep}(G) = \text{representations of } G$$

This approach generalizes to higher-categories.

Example: Dual Symmetry in 2d

Let $\mathcal{T}$ be a 2d theory with a 0-form symmetry G, generated by topological lines $D_1^{(g)}$, which fuse according to the group multiplication in G

$$D_1^{(g)}\,,\; g \in G, \qquad D_1^{(g)} \otimes D_1^{(h)} = D_1^{(gh)}$$

This defines a fusion category: $\mathcal{C} = \text{Vec}(G)$.

Gauging G means introducing a dynamical G gauge field:

- There is a dynamical G gauge field a and Wilson lines in G-representations $\mathbf{R}$

$$D_1^{(\mathbf{R})} = \text{Tr}_{\mathbf{R}}\, e^{\int a}$$

- These Wilson lines fuse according to the representations of G, $\text{Rep}(G)$:

$$D_1^{(\mathbf{R}_1)} \otimes D_1^{(\mathbf{R}_2)} = \bigoplus_{\mathbf{R}_3} N_{\mathbf{R}_3}^{\mathbf{R}_1\mathbf{R}_2} D_1^{(\mathbf{R}_3)}$$

For abelian groups, e.g.

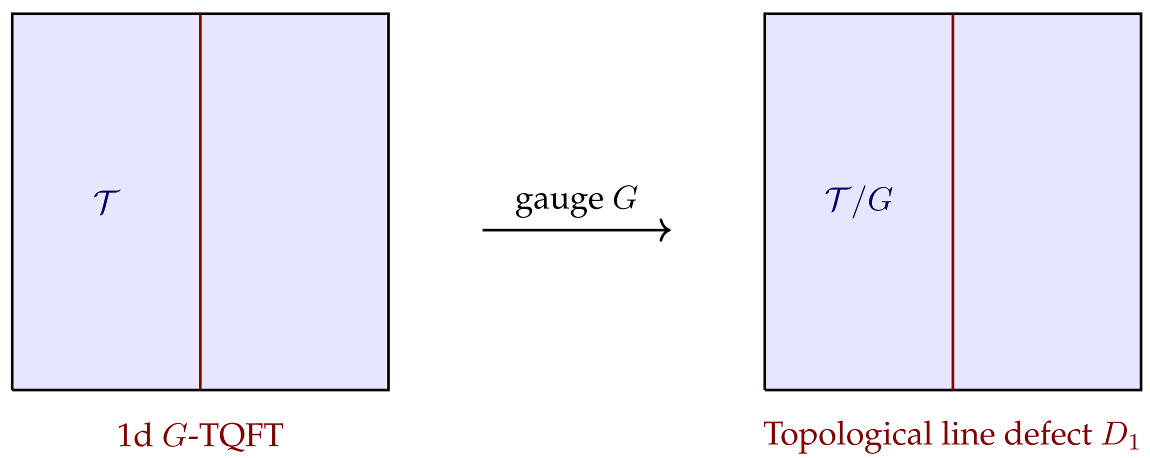
$$\text{Rep}(\mathbb{Z}_N) \equiv \text{Hom}(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$$

Theta-Defects

Theta-defects are a complementary perspective, which naturally generalizes to higher dims.

Consider a 2d theory $\mathcal{T}$, finite 0-form symmetry G: $\mathcal{C}_{\mathcal{T}} = \text{Vec}(G)$.

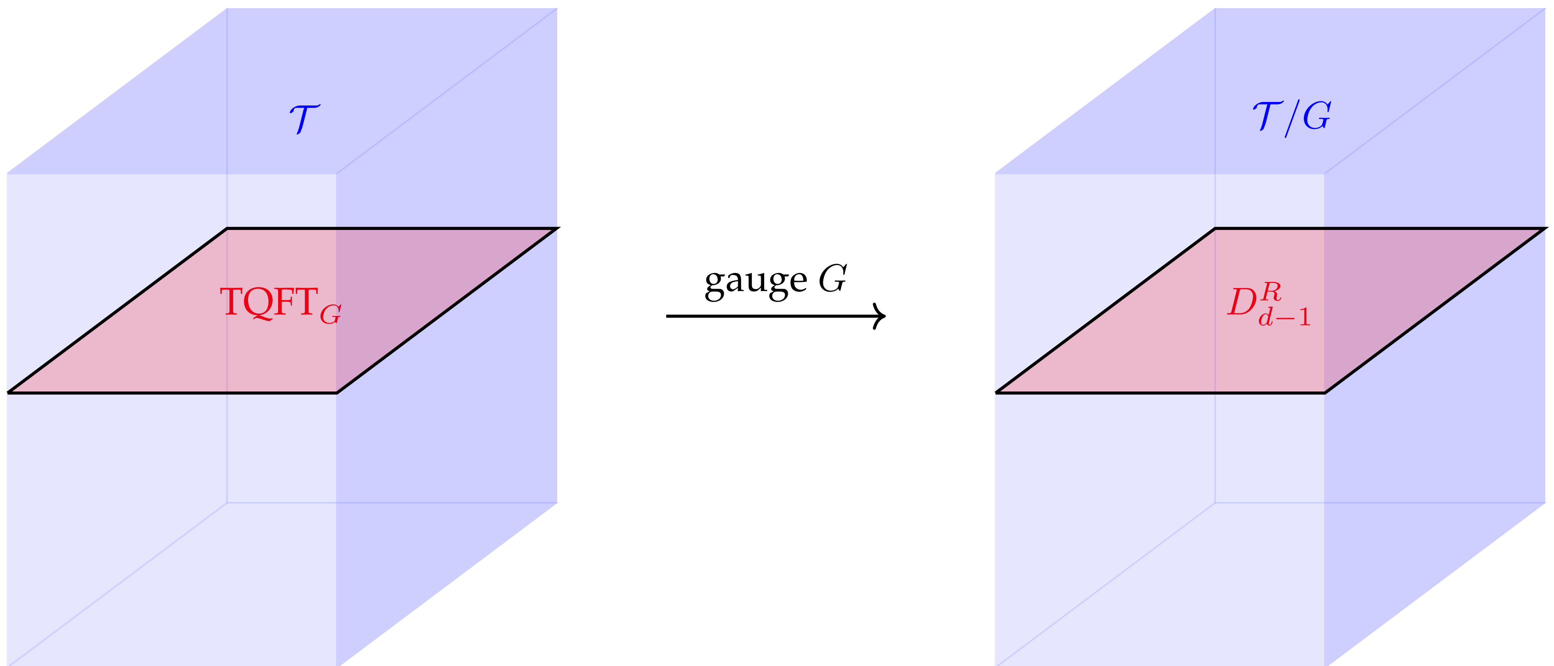
Stacking a 1d TQFT with G-symmetry, and gauging the diagonal G results in topological lines $D_1^{(\mathbf{R})}$, $\mathbf{R}$ rep of G in the gauged theory:



1d G-TQFTs:
Characterized by the number of vacua and G action on them, i.e. a G-representation. They form a subset $\text{Rep}(G)$ of the symmetry of $\mathcal{T}/G$.

Theta-Defects: higher dimensions

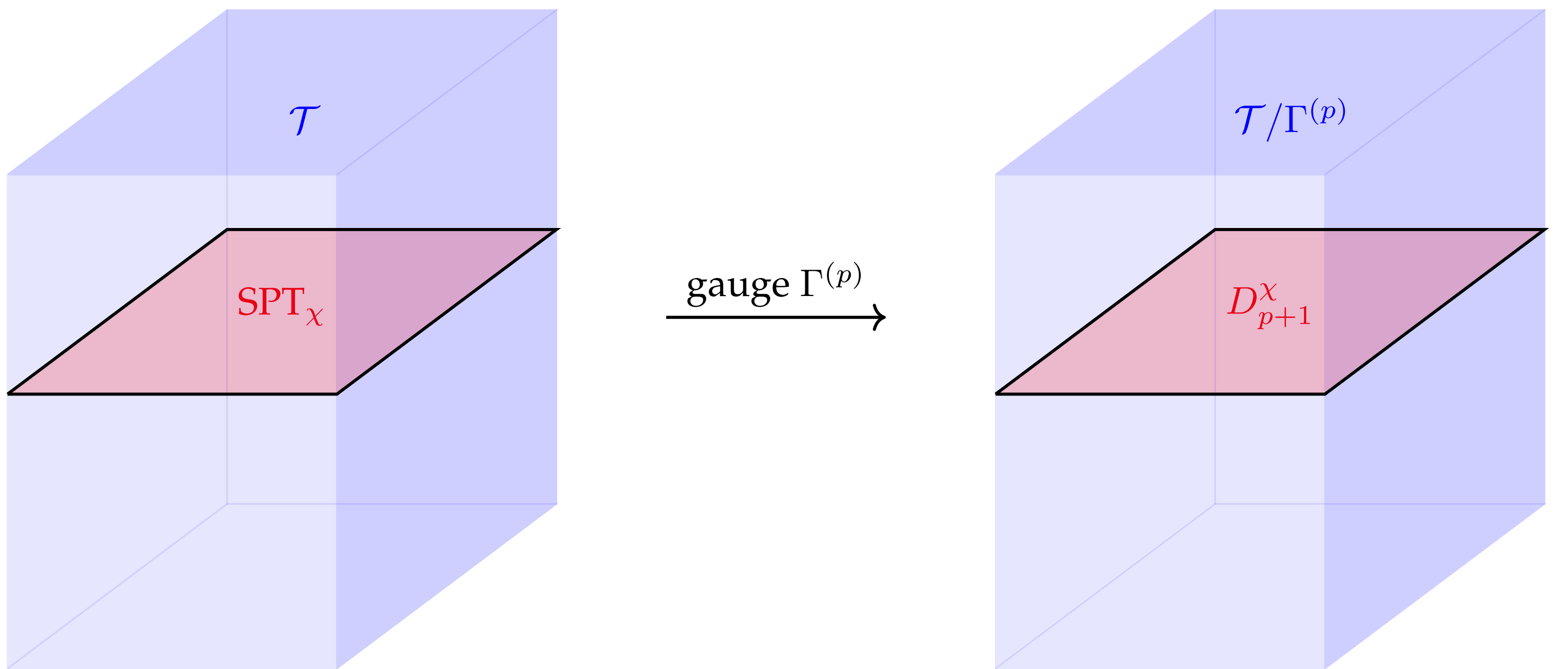
[Bhardwaj, SSN, Wu][Bhardwaj, SSN, Tiwari]



Stack a fully extended $(d-1)$dim G-TQFT and gauge diagonal G
$(d-1)$-category $(d-1)\text{Rep}(G)$:

- Objects: $(d-1)$-dim G-TQFTs
- 1-morphisms: $(d-2)$-dim G-interfaces between TQFTS
- 2-morphisms: $(d-3)$-dim G-junctions between interfaces
- etc.

Generalized gauging



Gauging a p-form symmetry of a theory $\mathcal{T}$:
Stack a $(p+1)$-dim TQFT SPT_χ, protected by $\Gamma^{(p)}$, associated to $\Gamma^{(p)}$ character χ. Gauging the diagonal $\Gamma^{(p)}$, results in TQFT becoming a topological defect in the gauged theory, which generates a $(d-p-2)$-form symmetry.

Example: 3d gauge theories

Consider the 3d pure gauge theory with gauge group

$$\mathrm{PSO}(4N)$$

This is obtained e.g. from $\mathrm{Spin}(4N)$ by gauging the center symmetry $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\frac{\mathrm{Spin}(4N)}{Z} = \mathrm{PSO}(4N)$$

The theory $\mathrm{PSO}(4N)$ has magnetic 0-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$, and in addition there is an outer automorphism coming from the action on the Dynkin diagram. The combined 0-form symmetry is generated by

$$D_2^{(g)} : \qquad g \in G = D_8 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$$

Its symmetry category is G-graded 2-vector spaces

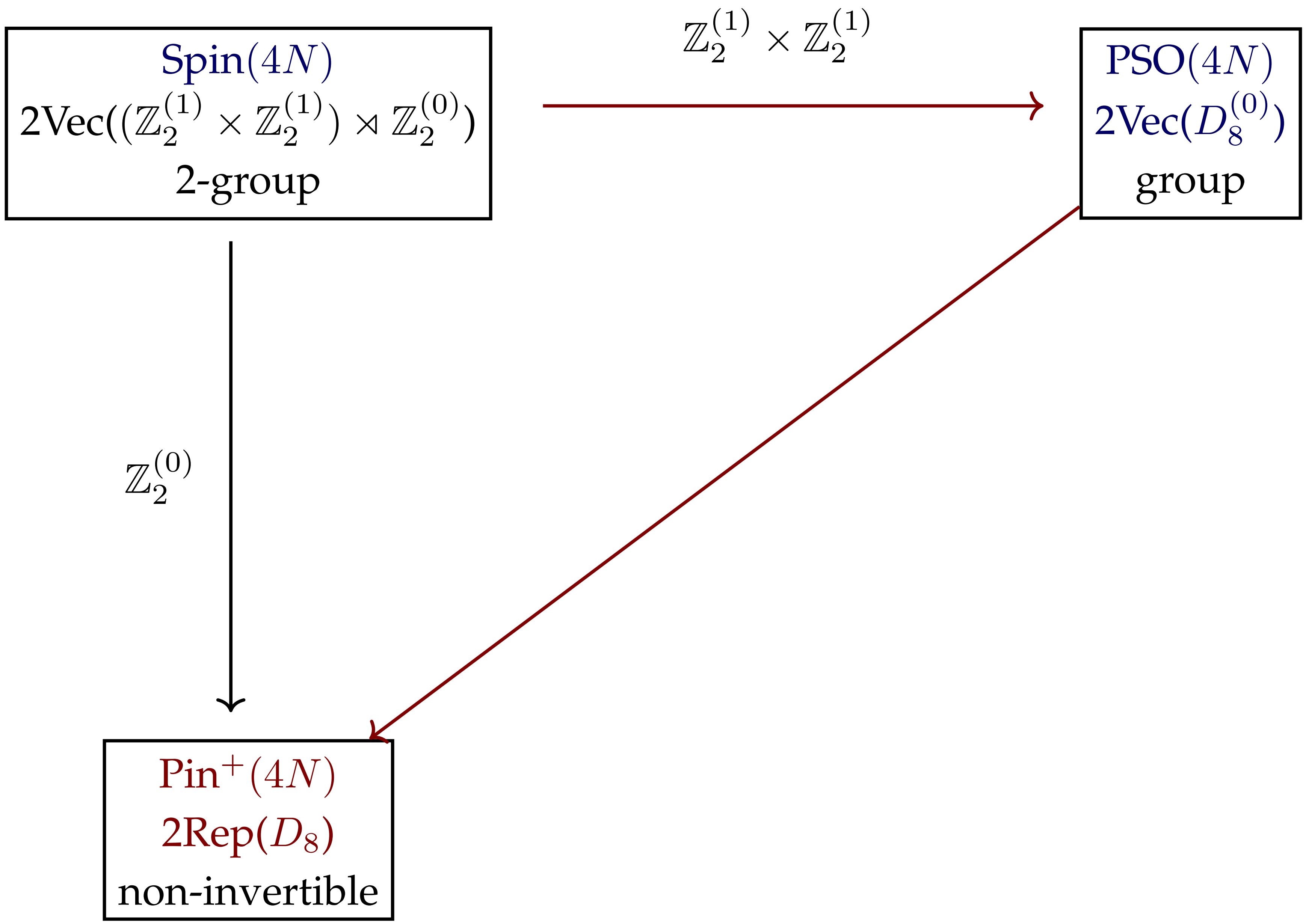
$$2\mathrm{Vec}(D_8)$$

Gauging the full 0-form symmetry results in a theory with symmetry category

$$2\mathrm{Rep}(D_8)$$

This is in fact $\mathrm{Pin}^+(4N)$, and the symmetries are non-invertible.

Categorical Symmetry Webs: 3d $\mathfrak{so}(4N)$



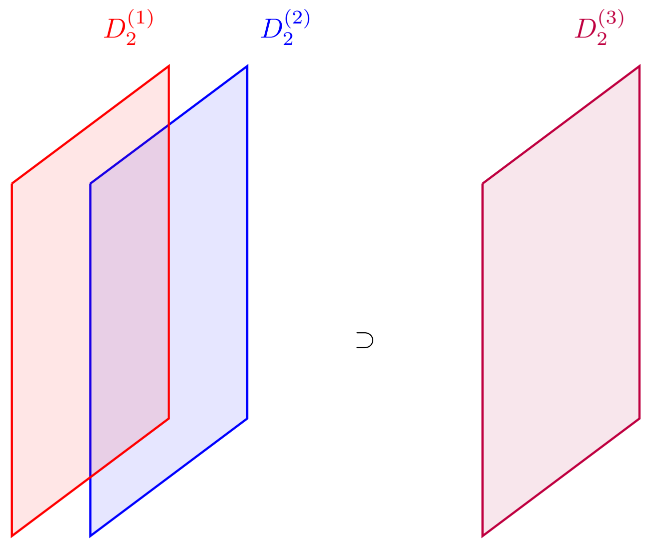
What is $2\text{Rep}(G)$, precisely?

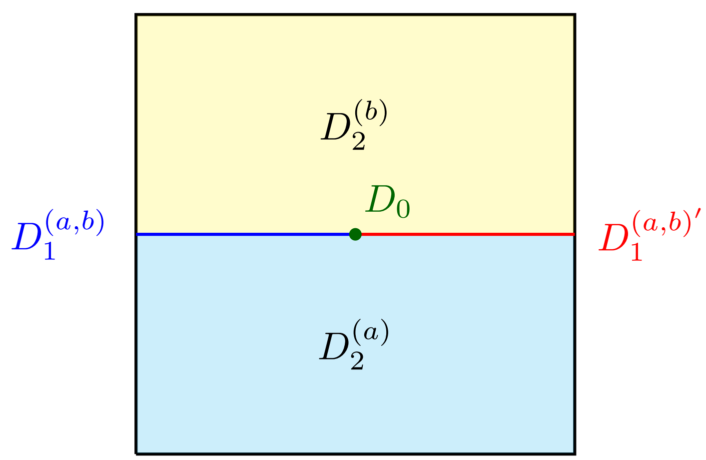
Fusion 2-categories

Objects: D_2 topological surfaces; e.g. $\Gamma^{(d-3)}$ form symmetry

1-Morphisms: D_1 topological lines; e.g. $\Gamma^{(d-2)}$ form symmetry

2-Morphisms: D_0 topological point operators

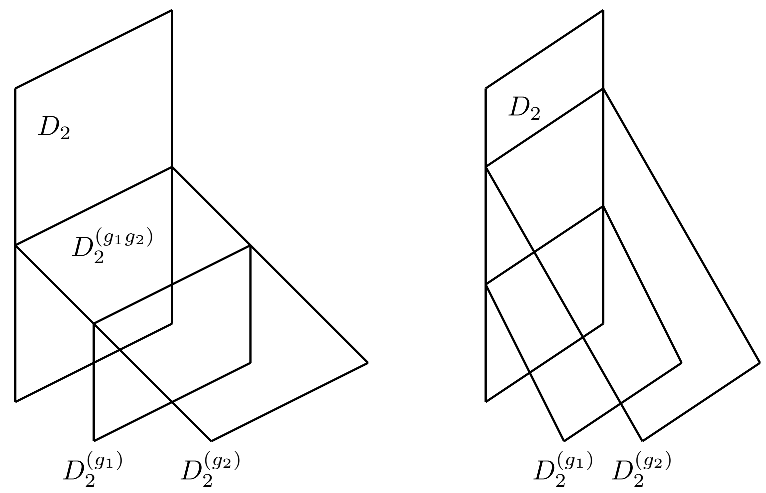




Gauging 2-categories

Start with a theory with a 2-fusion category symmetry $\mathcal{C}$, with a G-symmetry generated by $D_2^{(g)}$, $g \in G$. To gauge G there are again two approaches:

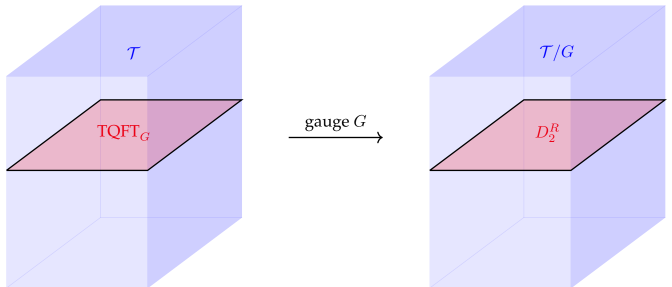
- Coupling the theory to a G-background, and solving for consistent endings of G-surfaces.
  $\Rightarrow$ Bimodules for algebras in 2-categories. [Bartsch, Bullimore, Ferrari, Pearson][Bhardwaj, Bottini, SSN, Tiwari]



- Stacking G-TQFTs and gauging diagonal G. [Bhwardwaj, SSN, Wu]

Generalized Gauging

Our philosophy is to attach 2d G-TQFTs:



A 2d TQFT (finite, modulo Euler number counter-term) is characterized by an integer $n \in \mathbb{N}$, the number of vacua:

$$T_n = \oplus_{i=1}^{n} \text{trivial}_i$$

The space of local operators has a basis id_i, $i = 1, \cdots, n$ with

$$\text{id}_i \text{id}_j = \delta_{ij} \text{id}_i$$

Defects are domain walls between vacua $\mathcal{I}_{ij}$ which compose as

$$\mathcal{I}_{ij} \circ \mathcal{I}_{kl} = \delta_{jk} \mathcal{I}_{i,l}$$

These form a fusion 2-category

$$\text{2Vec} : \qquad T_n \otimes T_m = T_{nm}$$

2d G-TQFTs to 2d Defects

What are 2d G-TQFTs?

- Single Vaccum: 2d G-SPTs, which are classified by

$$\alpha \in H^2(G, U(1))$$

- Multiple Vacua: we can have spontaneous symmetry breaking to $H < G$.
  A minimal set of G-TQFTs is determined by

$$T_{(H,\alpha)}: \qquad H < G\,, \qquad \alpha_H \in H^2(H, U(1))$$

  and any G-TQFT can be written as

$$\bigoplus_{(H,\alpha)} T_{(H,\alpha)}$$

  and has $\sum_{H<G} |G/H|$ many vacua.

In the gauged theory: TQFTs become 2d topological defects $D_2^{(H,\alpha)}$ with fusion:

$$D_2^{(H,\alpha_H)} \otimes D_2^{(K,\alpha_K)} = \bigoplus_{HaK \in H\backslash G/K} D_2^{(H\cap {}^a K, \alpha_H + \alpha_K)}$$

Example: $G = \mathbb{Z}_2$

$G = \mathbb{Z}_2$ then $H = 1$ or $\mathbb{Z}_2$ and α trivial.

Objects:

- $D_2^{(H=1)} \equiv D_2^{(-)}$: TQFT with two vacua $|\pm\rangle$, which is a non-trivial defect.
- $D_2^{(H=\mathbb{Z}_2)} \equiv D_2^{(\mathrm{id})}$: TQFT with 1 vacuum $|0\rangle$, trivial defect (identity).

Fusion:

- Single vacuum: $|0\rangle \otimes |0\rangle$: $D_2^{(\mathrm{id})} \otimes D_2^{(\mathrm{id})} = D_2^{(\mathrm{id})}$
- One $\mathbb{Z}_2$-orbit: $\{|0\rangle \otimes |\pm\rangle\}$

$$D_2^{(\mathrm{id})} \otimes D_2^{(-)} = D_2^{(-)}$$

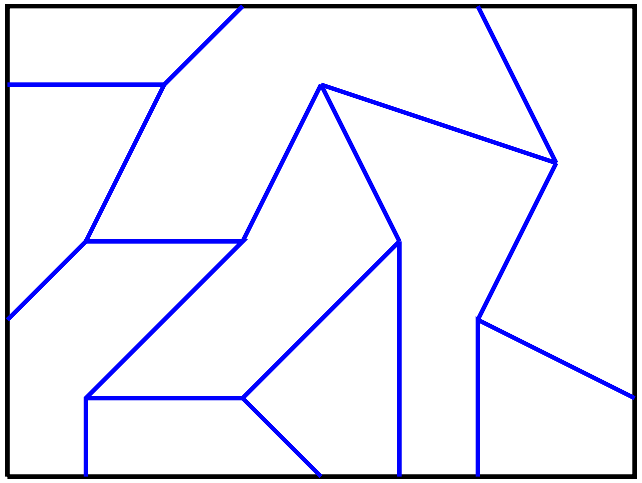
- Two $\mathbb{Z}_2$-orbits $|\pm\rangle \otimes |\pm\rangle$ and $|\pm\rangle \otimes |\mp\rangle$:

$$D_2^{(-)} \otimes D_2^{(-)} = 2D_2^{(-)}$$

This is known as the 2-fusion category: $2\mathrm{Rep}(\mathbb{Z}_2)$.

Relation to condensation defects:

The defects $D_2^{(-)}$ are precisely condensation defects, [Roumpedakis, Saifnashri, Shao] obtained by inserting a fine mesh of lines of the dual symmetry $D_1^{(-)}$

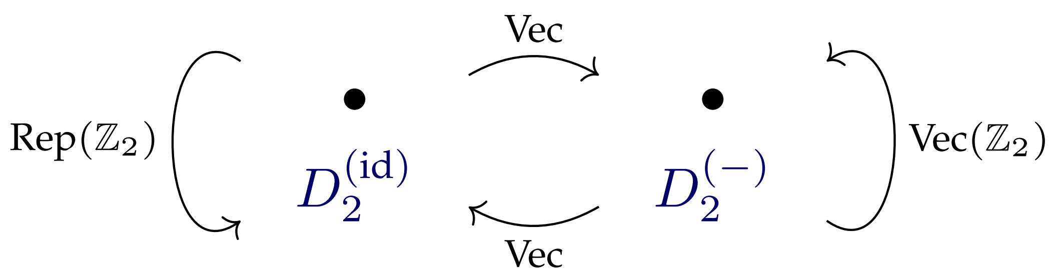


The topological defects obtained in this way obey non-invertible fusion, and are precisely the condensation defects of [Gaiotto, Johnson-Freyd][Rumpedakis, Seifnashri, Shao]:

$$D_2^{(H,\alpha_H)} \text{ correspond to 1-gauging } H < \widehat{G}^{(1)} \text{ on a surface.}$$

2Rep(G)

The generalized 0-form symmetry $\mathbb{Z}_2$ gauging in 3d, results in a theory $\mathcal{T}/\mathbb{Z}_2$ which has symmetry 2-category $2\mathrm{Rep}(\mathbb{Z}_2)$:

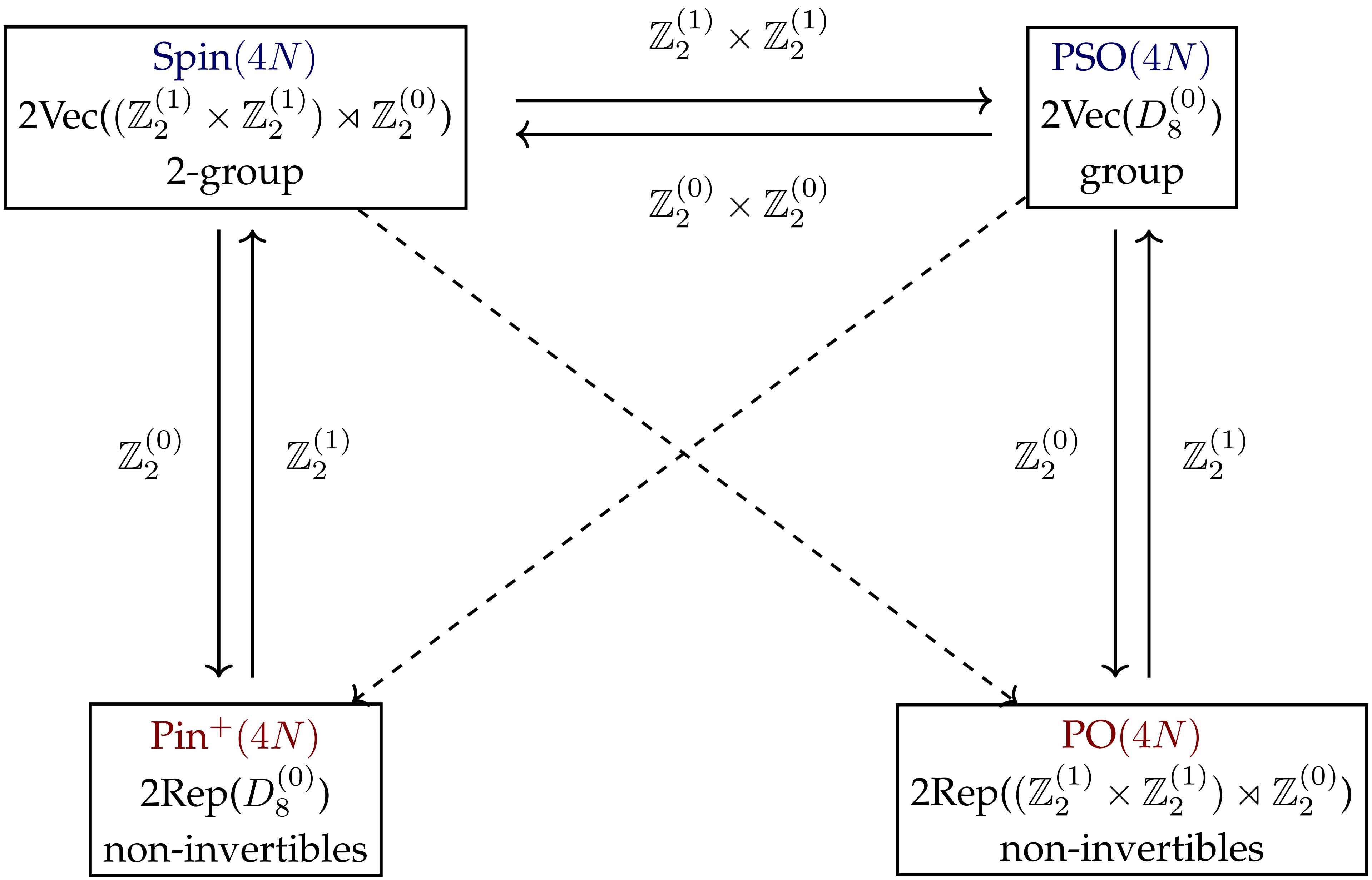


<u>General lesson:</u>

A 3d theory $\mathcal{T}$ with G 0-form symmetry has symmetry (sub-)category $2\mathrm{Vec}(G)$. The gauged theory $\mathcal{T}/G$ has symmetry (sub-)category $2\mathrm{Rep}(G)$.

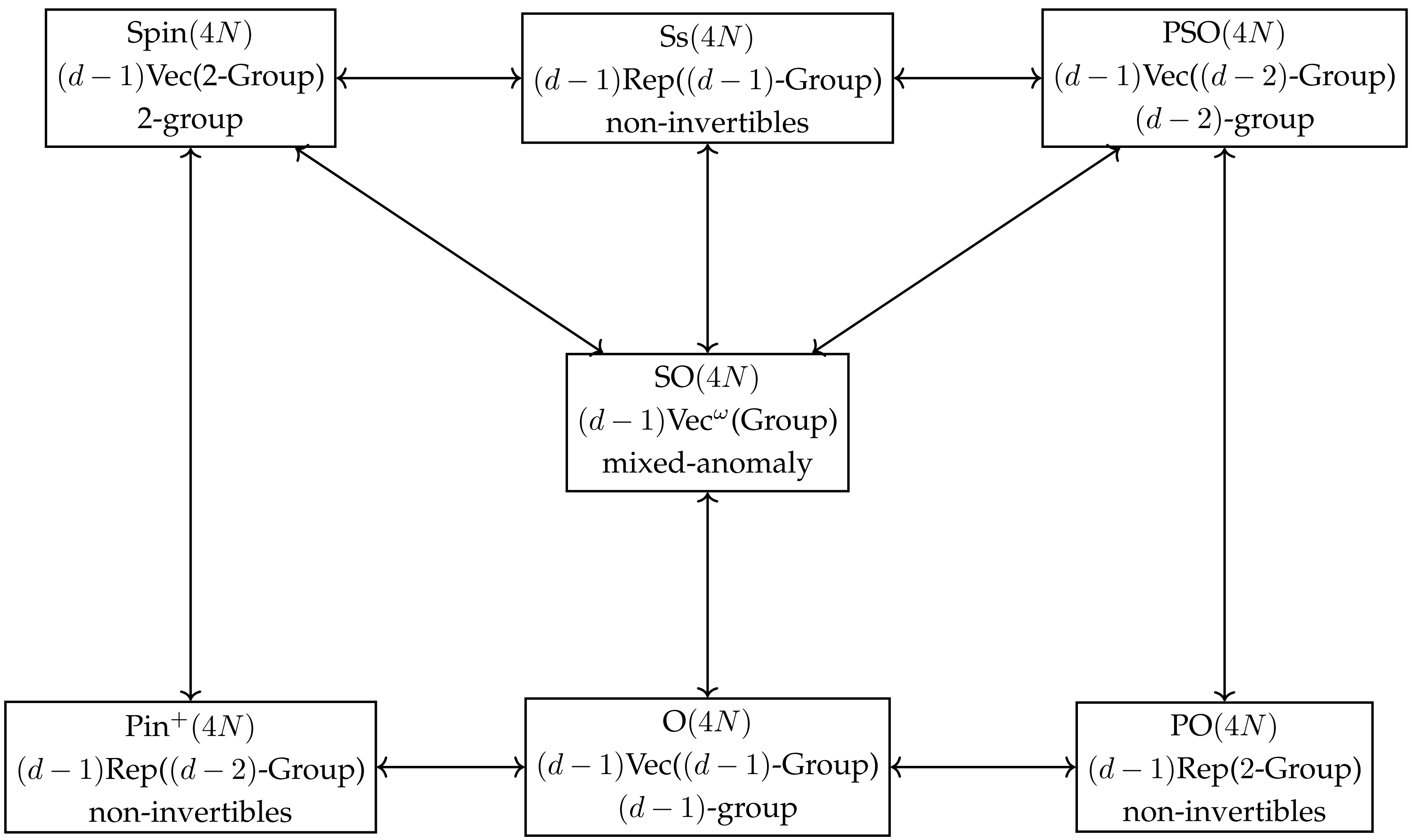
Categorical Symmetry Webs: 3d $\mathfrak{so}(4N)$

[Bhardwaj, Bottini, SSN, Tiwari]



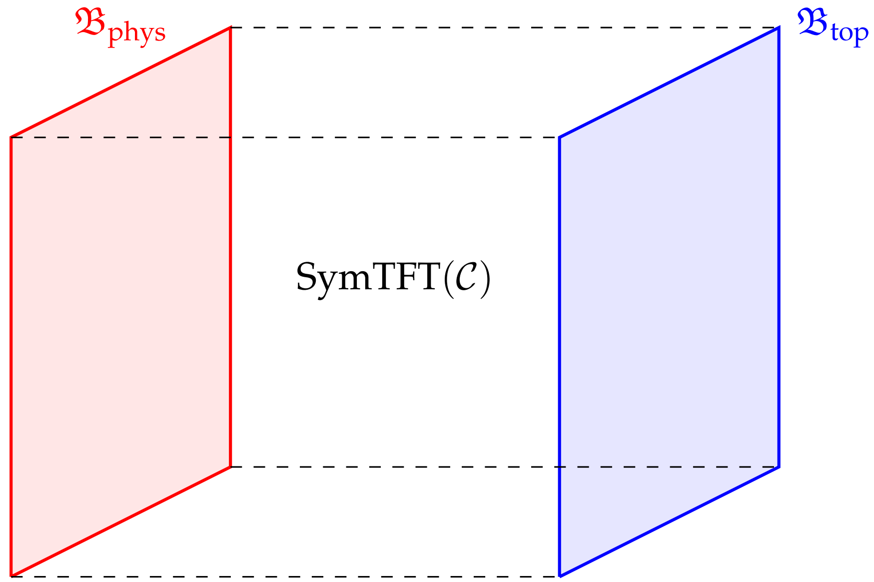
d-dim Categorical Symmetry Web

[Bhardwaj, Bottini, SSN, Tiwari]



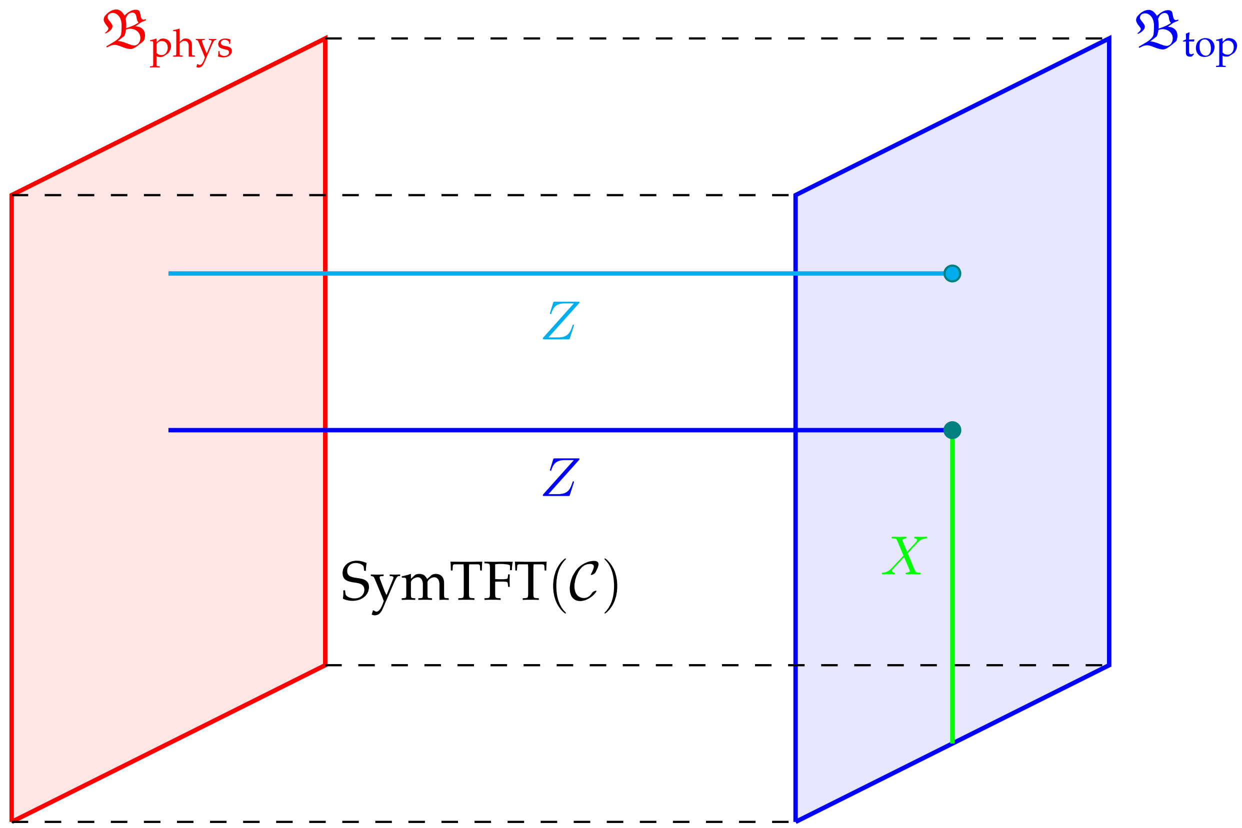
Symmetry TFT:
The "Everything Everywhere All at Once" of Symmetries

[Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] [Freed, Moore, Teleman]
Different choices of global forms (related by gauging) correspond to different b.c. on the so-called Symmetry TFT, which is a $(d+1)$dim TQFT that admits gapped boundary conditions:



B.c. result in different "global forms".
Example: Turaev-Viro $\text{TV}_{\mathcal{C}}$ for fusion symmetry $\mathcal{C}$.
More generally: Drinfeld center of the symmetry category. (2-categories [Bhardwaj, SSN, to appear])

Symmetries and Charged Operators from SymTFT



Defects that can end on the boundary resulting charged defects.
Defects that cannot end, result in symmetry generators $X \in \mathcal{C}$.

$\mathcal{C}$ can be any higher fusion category (not necessarily invertible). Some 2d and 4d examples in [Kaidi, Ohmori, Zheng, 2] and general structure will appear in [Bhardwaj, SSN]

Twisted Theta-Defects

- Theta-defects exist for any theory with G global symmetry: stack with a p-dimensional G-TQFT and gauge. These are ”universal” defects.
- There are Theta-defects, which are theory-dependent: e.g. if there is an obstruction to gauging on a defect, such as a ’t Hooft anomaly: Twisted Theta Defects.
- For 3d topological defects D_3 we can also consider G-TQFTs which do not necessarily admit gapped boundary conditions.

Example:

$G^{(0)} = \mathbb{Z}_{2M}$, $\Gamma^{(1)} = \mathbb{Z}_M$, with mixed anomaly.

$$\mathcal{A} = -\frac{2\pi}{M}\int A_1 \cup \frac{(B_2 \cup B_2)}{2}$$

The chiral symmetry generator $D_3^{(g)}$ transforms as [Kaidi, Ohmori, Zheng]

$$D_3^{(g)}(M_3) \to D_3^{(g)}(M_3)\exp\left(\int_{M_4} -\frac{2\pi i}{M}\frac{(B_2 \cup B_2)}{2}\right)$$

To gauge $\mathbb{Z}_M^{(1)}$, requires twisted theta-defect: stack with

$$\mathcal{A}^{M,1} = U(1)_M$$

which also does not admit gapped boundary conditions.
The Twisted Theta-defect is

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \otimes \mathcal{A}^{M,1}$$

with fusion

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2}\,\mathcal{N}_3^{(2)}$$

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \mathcal{C}_{\mathbb{Z}_M^{(1)}}(M_3) = \sum_{M_2 \in H_2(M_3,\mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3,\mathbb{Z}_M)|}$$

Connections to Geometry/Strings

Of course many of these theories that have categorical symmetries have realization in string theory.

Much of the defects/symmetries/etc is realizable in terms of topological operators in some Lagrangian for the SymTFT, and the interpretation in terms of branes [Apruzzi, Bah, Bonetti, SSN][Garcia-Extebarria].

Key open question:
determine the full categorical structure (including the SymTFT) directly from string theory/holography.

Example: Holographic dual of 4d $\mathcal{N} = 1$ SYM.

Holographic Dual Description of Non-Invertible Symmetries

[Apruzzi, Bah, Bonetti, SSN]

Consider the [Klebanov-Strassler] (KS) solution:

- N D3s at the conifold $C(T^{1,1})$ have a holographic dual in type IIB on $\mathrm{AdS}_5 \times T^{1,1}$, $\int F_5 = N$.
- $T^{1,1} \sim S^3 \times S^2$: wrap D5-branes on S^2, inducing $\int_{S^3} F_3 = M$
  $\Rightarrow$ breaks conformal invariance
- Dual to a cascade of Seiberg dualities, which for $N = kM$ end in pure $\mathfrak{su}(M)$ $\mathcal{N} = 1$ SYM:

$$ds^2 = \underbrace{\frac{r^2}{R^2} d\mathbf{x}^2 + \frac{R^2}{r^2} dr^2}_{M_5} + R^2 ds^2_{T^{1,1}} \,.$$

r= radial direction, RG-flow; $R(r) \sim \ln(\frac{r}{r_s})^{1/4}$, $r_s = r_0 e^{-N/gM^2 - 1/4}$.
The near horizon limit is $r \to r_0$.

The global form of gauge group is not fixed by this data alone.

Symmetries from Branes

[Apruzzi, Bah, Bonetti, SSN]

Proposal: in the near horizon limit, branes inserted in a holographic setup furnish symmetry generators. Close to the boundary $r \to \infty$:

$$T_{\mathrm{Dp}} \sim r^p, \qquad p > 0$$

such that the DBI part of the action decouples, and only topological couplings from the WZ term remain.

In the KS setup: D5-branes on $S^3 \times M_3 \subset T^{1,1} \times M_4$ have topological couplings in the near horizon limit $r \to r_0 \to \infty$

$$S_{\mathrm{D5}} = 2\pi \int_{M_3} \left(c_3 + \frac{M}{2} a_1 da_1 + a_1 db_1 \right)$$

The fields are c_3, from C_6 on S^3, b_1 from C_4 on S^3, $U(1)$ gauge field a_1 on the brane.

- b_1 Neumann: $SU(M)$: the second term is a trivial DW theory
- b_1 Dirichlet: $PSU(M)$: precisely the dressing with $U(1)_M$.

Branes realizing Non-Invertible Symmetries

[Apruzzi, Bah, Bonetti, SSN]

Non-invertible symmetries in 4d $\mathcal{N} = 1$ $PSU(M)$ SYM:

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2} \mathcal{N}_3^{(2)}$$

can be realized in terms of branes, which translates into (Myers effect)

$$D5(S^3) \otimes D5(S^3) = D7(T^{1,1})$$

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \mathcal{C}_{\mathbb{Z}_M^{(1)}}(M_3) = \sum_{M_2 \in H_2(M_3, \mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3, \mathbb{Z}_M)|}$$

has realization in terms of D5-branes is tachyon condensation [see also Bah, Leung, Waddleton]

$$D5(S^3) \otimes \overline{D5}(S^3) = \sum_{M_2} D3(M_2)$$

Action of Symmetries on Defects as Hanany-Witten Effect

[Apruzzi, Bah, Bonetti, SSN; PRL]**[Apruzzi, Gould, Bonetti, SSN – wip]**

How do symmetry generators act on charged operators (e.g. ′t Hooft and Wilson line operators)? Hanany-Witten effect:

- Charged line operators:
  D3s stretching along the radial direction and wrapped on $S^2 \times S^1$ give rise to ′t Hooft lines.

- Topological defects:
  D5s on $S^3 \times M_3$ generate the non-invertible codim 1 topological defects.

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X

Charge conservation implies that the total linking of the branes is conserved – in particular when we exchange the position of the D3 and D5: The linking is

$$\int_{\mathbb{R}^2_{x_1,x_2}\times S^3} F_5 = -\int_{\mathbb{R}_r\times S^2} F_3$$

which evaluates to

$$\int_{\mathbb{R}^2_{x_1,x_2}} db_1 = -\int_{\mathbb{R}_r} dc_0$$

On the D5:

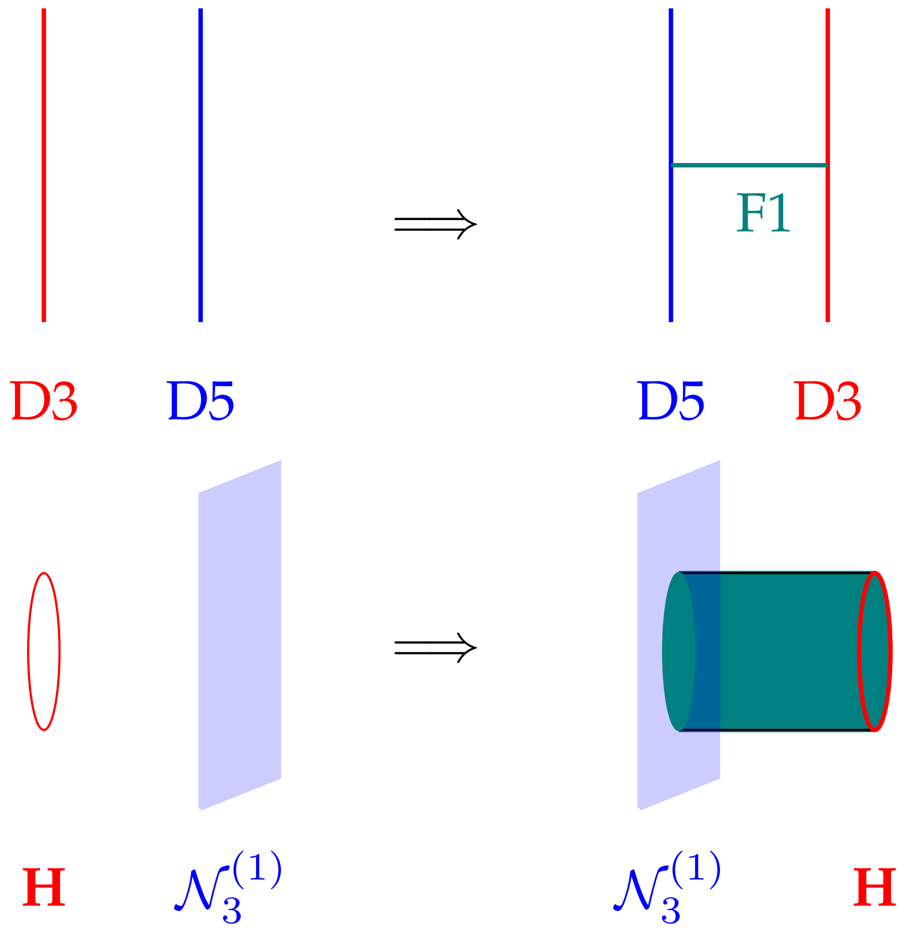
$$db_1 = -M da_1$$

As we pass the D3 through the D5:

$$db_1 = -M da_1 + \delta(p \in \mathbb{R}^2_{x_1,x_2})$$

which mean there is an additional object that intersects long x_3.

Preserving the linking requires the creation of an F1:



’t Hooft loop gets flux attachment when it crosses the non-invertible defect – similar to disorder operator in Kramers-Wannier duality.

Summary and Outlook

1. Higher fusion categories clearly have emerged as the framework to study symmetries in QFTs.
2. Non-Invertible symmetries are ubiquitous in QFTs. What is the associated ”representation theory”?
3. Symmetry TFT will play a key role in this. And also unifying the description of various ”global forms”.
4. String Theory: How is the Symmetry TFT realized in the brane-picture?
5. Action of branes on branes (a la Hanany-Witten) as action of non-invertible symmetries on charged objects. Can this be sharpened and mapped to a representation-theoretic statement?