The Lattice-Continuum Correspondence in Quantum Mechanics

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Continuum and lattice quantum field theories

Continuum QFT: an assignment of operators to (sub)manifolds + consistent rules for computing expectation values

Lattice QFT: a tensor product of finite-dimensional Hilbert spaces associated to components of a finite graph + a Hamiltonian

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Why continuum?

- Marvelous mathematical structure and insights
- Useful for describing our own world

Why lattice?

- Amenable to numerics
- Experiments (condensed matter, AMO, optics)
- Nonperturbative definition

The lattice-continuum correspondence

- Some notions are more natural in one framework than in the other, e.g. chiral theories in the continuum, or confinement on the lattice

- **Puzzles:** Is there a lattice formulation of every continuum QFT? How can the entire continuum structure emerge from a lattice?

The lattice-continuum correspondence

- Some notions are more natural in one framework than in the other, e.g. chiral theories in the continuum, or confinement on the lattice
- **Puzzles:** Is there a lattice formulation of every continuum QFT? How can the entire continuum structure emerge from a lattice?
- **This talk** will answer this for QFT in (0+1)D $\equiv$ QM
- Lattice QM: finite target space, e.g. $\mathbb{Z}_K$
  Continuum QM: target is an n-manifold for $n > 0$, e.g. $\mathbb{R}^n$ or S^n
  Here the focus will be on $n = 1$ (one-dimensional targets)

Why ask these questions?

- To unify divergent viewpoints on QFT
- To pave the road to new rigorous definitions and proofs:
  - Proof of Abelian bosonization in (1+1)D [1912.01022]
  - Derivation of OPE coefficients in the Ising CFT [1912.13462]
  - A definition of continuum QM that does not rely on functional analysis [this talk!]
  - A nonperturbative formulation of many higher-dimensional QFTs of interest, with gravity as the ultimate prize [in progress]

Rough outline

1. **Smoothing** and **Gaussianization**: elementary procedures that restrict a lattice theory to a subtheory with continuum properties. Most easily presented in the canonical/Hamiltonian formalism

2. Same as above, but in the path integral formalism. Here we also define **temporal smoothing**, a mutilation of the path integral that allows its evaluation and leads to various familiar concepts

3. Fermions and supersymmetry (if time allows). Lattice origins of the SUSY harmonic oscillator; the simplest nonlagrangian SUSY theory; and a no-go theorem (Witten index $= 0$ in any SUSY lattice theory)

Smoothing: a natural example

- Lattice QM with Hilbert space $\mathcal{H} = \mathrm{span}\left\{|\mathrm{e}^{\mathrm{i}\phi}\rangle\right\}$, $\phi \equiv \frac{2\pi}{K} n \equiv n\, \mathrm{d}\phi$
- Algebra generated by **clock** and **shift** operators [Schwinger 1960]

$$Z|\mathrm{e}^{\mathrm{i}\phi}\rangle = \mathrm{e}^{\mathrm{i}\phi}|\mathrm{e}^{\mathrm{i}\phi}\rangle, \quad X|\mathrm{e}^{\mathrm{i}\phi}\rangle = |\mathrm{e}^{\mathrm{i}(\phi - \mathrm{d}\phi)}\rangle$$

- Free clock model, $H = \frac{2 - X - X^\dagger}{2(\mathrm{d}\phi)^2}$, diagonal in the momentum basis

$$|p\rangle \equiv \frac{1}{\sqrt{K}} \sum_{\phi=\mathrm{d}\phi}^{2\pi} \mathrm{e}^{\mathrm{i}p\phi}|\mathrm{e}^{\mathrm{i}\phi}\rangle, \quad -\frac{K}{2} \leq p < \frac{K}{2}$$

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- The **smooth subspace** contains only low momentum states:

$$\mathcal{H}_\mathrm{S} \equiv \mathrm{span}\left\{|p\rangle\right\}, \quad -p_\mathrm{S} \le p < p_\mathrm{S}, \quad 1 \ll p_\mathrm{S} \ll K$$

- **Smoothing** $\equiv$ projecting to a unital algebra that preserves $\mathcal{H}_\mathrm{S}$

$$X^n \mapsto (X^n)_\mathrm{S} = X^n, \quad Z^p \mapsto (Z^p)_\mathrm{S}; \quad (Z^{p_1+p_2})_\mathrm{S} \neq (Z^{p_1})_\mathrm{S}(Z^{p_2})_\mathrm{S}$$

Gaussianization

- The subspace $\mathcal{H}_\mathrm{S}$ is also spanned by smooth position states

$$|\mathrm{e}^{\mathrm{i}\varphi}\rangle \equiv \frac{1}{\sqrt{2p_\mathrm{S}}} \sum_{p=-p_\mathrm{S}}^{p_\mathrm{S}-1} \mathrm{e}^{-\mathrm{i}p\varphi}|p\rangle, \quad \varphi \equiv \frac{2\pi}{2p_\mathrm{S}} n \equiv n\mathrm{d}\varphi, \quad 1 \le n \le 2p_\mathrm{S}$$

  These are smearings of original states $|\mathrm{e}^{\mathrm{i}\phi}\rangle$ over an angle $\mathrm{d}\varphi \gg \mathrm{d}\phi$

- Define the **Gaussian subspace** $\mathcal{H}_\mathrm{G} \subset \mathcal{H}_\mathrm{S}$ as

$$\mathcal{H}_\mathrm{G} \equiv \mathrm{span}\left\{|\mathrm{e}^{\mathrm{i}\varphi}\rangle\right\}, \quad -\varphi_\mathrm{G} \le \varphi < \varphi_\mathrm{G}$$

- **Gaussanization** is a corresponding projection of operators to a unital subalgebra that preserves $\mathcal{H}_\mathrm{G}$,

$$(X^n)_\mathrm{S} = X^n \mapsto (X^n)_\mathrm{G}, \quad (Z^p)_\mathrm{S} \mapsto (Z^p)_\mathrm{G}$$

  For a generic pair of operators, $(\mathcal{O}_1\mathcal{O}_2)_\mathrm{G} \neq (\mathcal{O}_1)_\mathrm{G}(\mathcal{O}_2)_\mathrm{G}$

Remarks

- Smoothing restricts to wavefunctions that vary slowly:

$$\psi(\phi + \mathrm{d}\phi) = \psi(\phi) + O(p_{\mathrm{S}}/K)$$

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- Smooth/Gaussian states and operators defined this way behave just like continuum objects in the QM of a particle on S^1 or $\mathbb{R}$

- The hierarchy $\mathcal{H}_{\mathrm{G}} \subset \mathcal{H}_{\mathrm{S}} \subset \mathcal{H}$ is somewhat analogous to the **rigged Hilbert space** used by Gel'fand as a rigorous foundation of QM,

$$\mathrm{Dom}(\Delta) \subset L^2(\mathbb{R}) \subset \mathrm{Dom}^{\times}(\Delta)$$

When and how to use these reductions?

- If there exists an energy eigenspace invariant under these projections, we can talk about a **flow** to an effective continuum QM
- In the free clock model, all energy eigenstates are invariant under smoothing, and none under Gaussianization ("the Hamiltonian of a free particle on $\mathbb{R}$ has no normalizable eigenstates")
- If an invariant eigenspace exists, it is natural to work with

$$P \equiv \frac{X - X^\dagger}{2\mathrm{i}\,\mathrm{d}\phi}, \quad Q \equiv \frac{Z - Z^\dagger}{2\mathrm{i}}$$

Acting on Gaussian states, these are "canonical" operators

$$P_{\mathrm{G}}|\mathrm{e}^{\mathrm{i}\varphi}\rangle \approx -\mathrm{i}\hat{\partial}_\varphi|\mathrm{e}^{\mathrm{i}\varphi}\rangle, \quad Q_{\mathrm{G}}|\mathrm{e}^{\mathrm{i}\varphi}\rangle \approx \varphi|\mathrm{e}^{\mathrm{i}\varphi}\rangle, \quad [Q, P]_{\mathrm{G}}|\mathrm{e}^{\mathrm{i}\varphi}\rangle \approx \mathrm{i}|\mathrm{e}^{\mathrm{i}\varphi}\rangle$$

NB: it is crucial to multiply first and then Gaussianize!

Smoothing via path integrals

$$\mathfrak{Z} \equiv \mathrm{Tr}\,\mathrm{e}^{-\beta H} = \sum_{\{\varphi_\tau\}} \prod_{\tau=\mathrm{d}\tau}^{\beta} \langle \varphi_{\tau+\mathrm{d}\tau} | \mathrm{e}^{-\mathrm{d}\tau H} | \varphi_\tau \rangle$$

$$\mathrm{d}\tau \equiv \frac{\beta}{N_0}, \quad \varphi_{\beta+\mathrm{d}\tau} \equiv \varphi_{\mathrm{d}\tau}$$

- Conventionally, insert a(n over)complete set of states at each time τ
- **Smooth path integrals:** insert the undercomplete set $\{|\mathrm{e}^{\mathrm{i}\varphi}\rangle\}$
- This is justified ($\mathfrak{Z}_\mathrm{S} \approx \mathfrak{Z}$) if $\beta \gg \frac{2\log K}{p_\mathrm{S}^2}$ for the free clock model
- At $\beta \sim (\log K)/p_\mathrm{S}^2$ there is a **roughening transition** [Parisi 1979]
- If $p_\mathrm{S}^2 \mathrm{d}\tau \ll 1$, the smooth partition function has the familiar form

$$\mathfrak{Z}_\mathrm{S} \approx \frac{(\mathrm{d}\varphi)^{N_0}}{(2\pi\mathrm{d}\tau)^{N_0/2}} \sum_{\{\varphi_\tau\}} \mathrm{e}^{-\frac{1}{2}\sum_{\tau=\mathrm{d}\tau}^{\beta} \frac{(\Delta_\tau\varphi)^2}{\mathrm{d}\tau}} \equiv \int [\mathrm{d}\varphi]\, \mathrm{e}^{-\frac{1}{2}\int_0^\beta \mathrm{d}\tau (\partial_\tau \varphi)^2}$$

Evaluating smooth path integrals

- Usual idea: Fourier-transform φ_τ and integrate modes one by one
- $\varphi_\tau \equiv \varphi_\tau + 2\pi$ means Fourier modes are not independent variables!
- The product over Matsubara frequencies is intractable, anyway

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- The product over Matsubara frequencies is intractable, anyway
- Solutions: keep only small fluctuations around saddle points, and then simply **discard** their high frequency modes

$$\varphi_\tau \equiv \varphi_\tau^{\mathrm{cl}} + \delta\varphi_\tau, \quad -\varphi_{\mathrm{G}} \leq \delta\varphi_\tau < \varphi_{\mathrm{G}}, \quad \varphi_{\mathrm{G}} \ll \tfrac{1}{\sqrt{N_0}}$$

$$\delta\varphi_\tau = \textstyle\sum_{n=-\frac{1}{2}N_0}^{\frac{1}{2}N_0-1} \delta\varphi_n \, \mathrm{e}^{\mathrm{i}\omega_n\tau} \mapsto \delta\varphi(\tau) \equiv \textstyle\sum_{n=-n_{\mathrm{S}}}^{n_{\mathrm{S}}-1} \delta\varphi_n \, \mathrm{e}^{\mathrm{i}\omega_n\tau}$$

- This **temporal smoothing** is not an approximation and has no canonical counterpart. It is a miracle of physics that there exists a procedure (**renormalization**) that takes the resulting quantity $\widetilde{\mathfrak{Z}}_{\mathrm{S}} = \prod_{n=1}^{n_{\mathrm{S}}-1} \frac{1}{\omega_n^2}$ and outputs part of the sought answer $\mathfrak{Z}_{\mathrm{S}}$

Two consequences of temporal smoothing

- Many familiar concepts are only defined after temporal smoothing. For example, **dilatations**:

$$\delta\varphi_\tau \mapsto \lambda^\Delta \delta\varphi_{\lambda\tau} \quad \text{vs} \quad \delta\varphi(\tau) \mapsto \lambda^\Delta \delta\varphi(\lambda\tau)$$

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- Consider the path integral for a two-point function:

  $$G(\tau) = \mathfrak{Z}^{-1} \sum_{\{\varphi^{\mathrm{cl}}\}} \int [\mathrm{d}\delta\varphi]\, \delta\varphi_0 \delta\varphi_\tau \, \mathrm{e}^{-S[\varphi]}$$

  Temporal smoothing affects both the action and the operator insertions. The smoothed quantity $\widetilde{G}(\tau)$ differs from $G(\tau)$ both by an overall renormalization (reflected by **counterterms** in the action) and by an additive term (reflected by **contact terms**)

Fermions

- Fermion QM = a clock model with $K = 2$
- No canonical smoothing procedure, since K is not large
- The Berezin path integral allows the definition of temporal smoothing even in the absence of canonical smoothing
- Ideal playground for exploring counter/contact terms

Supersymmetry

- Elementary definition: a theory is SUSY if it has nilpotent symmetries
- Standard setup in QM: a clock model coupled to a fermion, with nilpotent symm. generators (supercharges) and Hamiltonian given by

$$\mathcal{Q} = B^\dagger f, \quad \mathcal{Q}^\dagger = B f^\dagger, \quad H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

- **Minimal SUSY model:** two fermions, $H = n_{\mathrm{B}} + n_{\mathrm{F}} - 2n_{\mathrm{F}}n_{\mathrm{B}}$
- **SUSY harmonic oscillator:** $B = \mathrm{i}P + W(Q)$ for $W(Q) = \omega Q$
- In all examples there is a doubly degenerate ground state! This is a general feature of all SUSY models (a version of "fermion doubling")

Conclusions

- The construction shown here leads to a "finitary" definition of continuum QM that appears as powerful as the conventional ones
- Closely related ideas work in higher-dimensional QFTs. There it is necessary to separately smooth in both target and position spaces
- Provocative idea: maybe "It from Qubit" can be taken literally, and maybe we can get everything around us from a discrete setup...
  ... without handwaving!

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Thank you!