The Tameness of Quantum Physics

Thomas W. Grimm

Utrecht University

Based on: 2112.08383 and

works with Mike Douglas, Lorenz Schlechter
Benjamin Bakker, Christian Schnell, Jacob Tsimerman

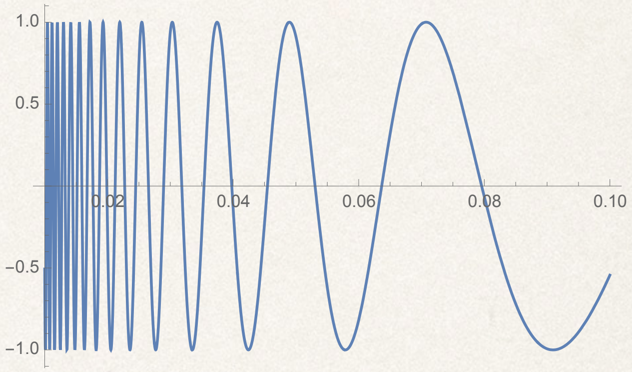
Western Hemisphere Colloquium on Geometry and Physics - February 2023

Mathematics can be wild

- Analysis: topologies and maps can be involved

  → complicated sets: Cantor sets, …

  → complicated functions:



$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases} \qquad f(x) = \sin(1/x)$$

  - common feature: no proper graphical representation

- Logic: Gödel’s first incompleteness theorem

  → there are statements that are undecidable

Physics is more tame, isn’t it?

Mathematics can be wild

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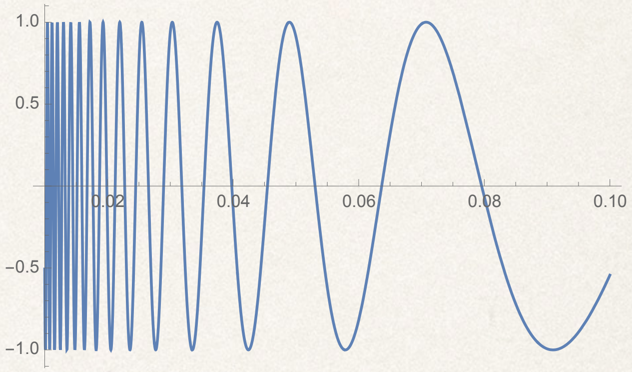

  - common feature: no proper graphical representation

- Logic: Gödel's first incompleteness theorem

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What is a good Tameness Principle?

Finiteness as a tameness principle?

- Longstanding question: Is number of distinct effective theories from string theory below fixed cut-off finite? e.g. [Douglas '03] [Acharya,Douglas '06]
  - much recent activity: finiteness of spectra, ranks of gauge groups

    [Adams,DeWolfe,Taylor] [Kim,Shiu,Vafa] [Kim,Tarazi,Vafa] [Cvetic,Dierigl,Lin,Zang] [Dierigl,Heckman] [Font,Fraiman,Grana,Nunez,DeFreitas] [Hamada,Vafa] [Taylor etal],[Kim,Shiu,Vafa],[Lee,Weigand],[Tarazi,Vafa] [Hamada,Montero,Vafa,Valenzuela]

- In certain string theory models finiteness is now part of general mathematical theorem [Bakker,TG,Schnell,Tsimerman]

Finiteness criterion seems to be a yes/no-criterion:
Can we turn finiteness into a structural criterion?

Finiteness as a tameness principle?

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> Tameness principle: demand that theories are formulated within 'Tame geometry' or 'o-minimal geometry'

(needed in the proof of [Bakker,TG,Schnell,Tsimerman])

What is tameness?

- Tameness implements a generalized finiteness principle
  → restricts sets and functions: tame sets + tame functions

- Avoid wild sets and functions:
  → no sets with infinite disconnected components: integers, lattices…
  → no functions with infinitely many isolated minima, maxima, zeros as in $f(x) = \sin(1/x)$

- Comes from logic: o-minimal structures
  can avoid logical undecidability [Tarski] (Gödel's theorems are over integers)

- Grothendieck's dream to develop math. framework for geometry:
  → tame topology [Esquisse d'un programme]

Tameness - a generalized finiteness principle

- Tameness from theory of o-minimal structures (model theory, logic)

  intro book [van den Dries]

  Recent lectures: 2022 Fields institute program (6 months)

- structure $\mathcal{S}$: collect subsets of $\mathbb{R}^n$, $n = 1, 2, \ldots$
  - closed under finite unions, intersections, products, and complements
  - closed under linear projections
  - sets defined by all real polynomials included (algebraic sets)

- tame/o-minimal structure $\mathcal{S}$: only subsets of $\mathbb{R}$ that are in $\mathcal{S}$ are finite unions of points and intervals

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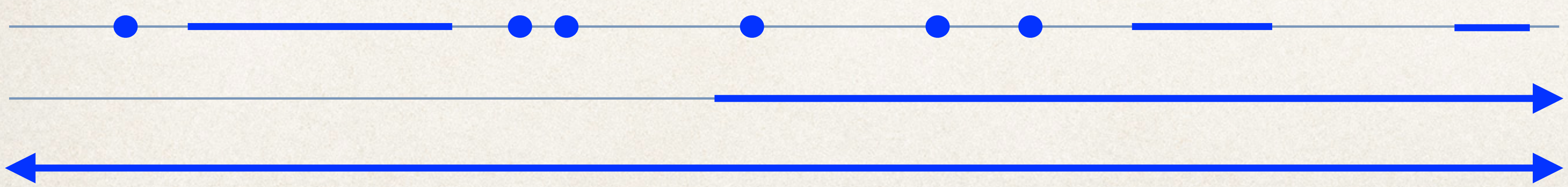

finitely many points/intervals

infinitely long intervals

whole real line

Tameness - a generalized finiteness principle

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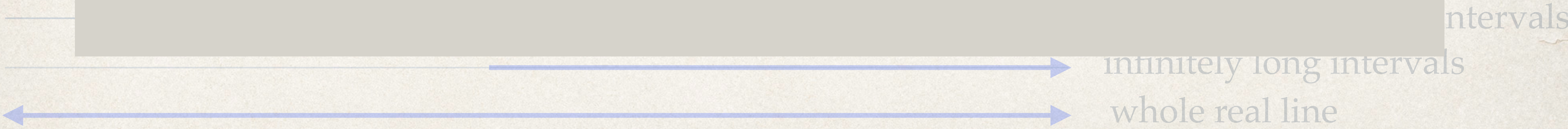
- structure $\mathcal{S}$: collect subsets of $\mathbb{R}^n$, $n = 1, 2, \ldots$

sets in o-minimal structure $\mathcal{S}$: tame sets

functions with graph being a tame set: tame functions

$\rightarrow$ tame manifold, tame bundles… a tame geometry

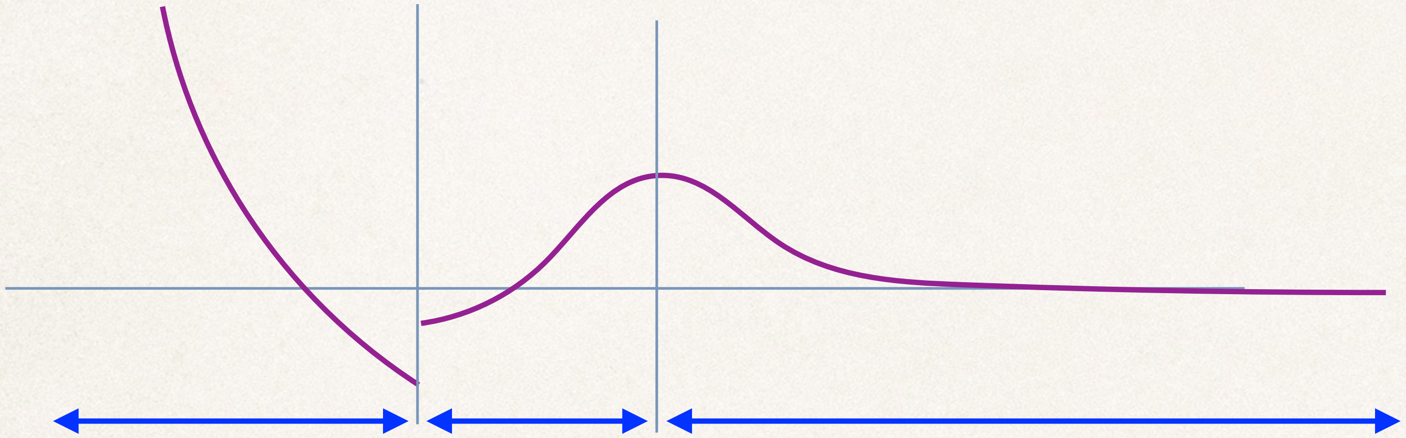
- ta



Examples and Non-Examples

- Consider function: $f : \mathbb{R} \to \mathbb{R}$

tame function

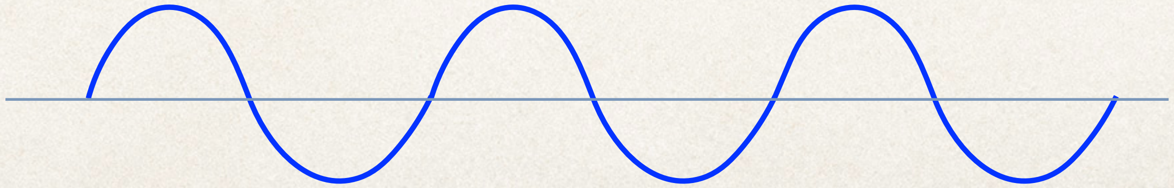


split $\mathbb{R}$ into finite number of intervals: f is either constant, or monotonic and continuous in each open interval

→ finitely many minima and maxima, tame tail to infinity

- Periodic functions $f(x+n) = f(x)$ are never tame (when not constant)

$\sin(x), x \in \mathbb{R}$



Examples of o-minimal structures

- Note: There are many known o-minimal structures.
  - examples are obtained by stating which functions are allowed to generate the sets → non-trivial

- Simplest structure: $\mathbb{R}_{\text{alg}}$ (used e.g. in real algebraic geometry)
  - sets in $\mathbb{R}^n$ are zero-sets of finitely many real polynomials:

$$P_k(x_1, ..., x_n) = 0 \quad \cap \quad \hat{P}_l(x_1, ..., x_n) > 0 \qquad \begin{matrix} k = 1, .., m \\ l = 1, ..., \hat{m} \end{matrix}$$

needed for projection property

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- General structure: add more functions $f_i : \mathbb{R}^m \to \mathbb{R}$ to generate sets

$$P_k(x_1, ..., x_m, f_1(x), ..., f_n(x)) = 0$$

→ complete sets compatible with structure axioms (i.e. add sets obtained by projection, unions,…)

Examples of o-minimal structures

- Note: There are many known o-minimal structures.
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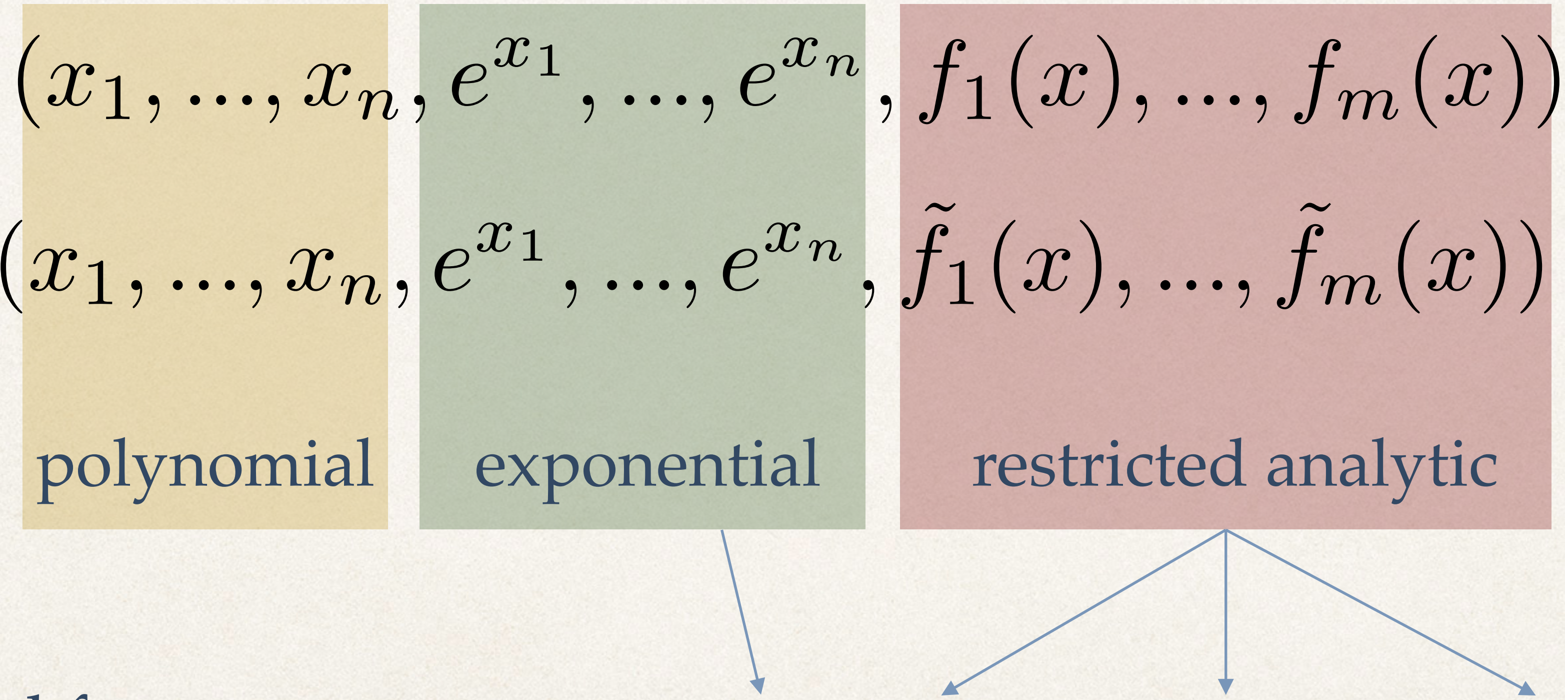
- Some important examples:
  - $\mathbb{R}_{\text{alg}}$ extended by exponential function: $\mathbb{R}_{\text{exp}}$ [Wilkie '96]
  - $\mathbb{R}_{\text{alg}}$ extended by restricted real analytic functions: $\mathbb{R}_{\text{an}}$ [Denef, van den Dries '88]
  - exp and restricted real analytic functions: $\mathbb{R}_{\text{an,exp}}$ [van den Dries, Macintyre,Marker '94]
  - several more recent examples: e.g.
    (1) add solutions to certain first-order differential equations
    (2) structure including $\Gamma(x)|_{(0,\infty)}$ and $\zeta(x)|_{(1,\infty)}$ [Rolin,Servi,Speissegger '22]

Examples of o-minimal structures

- Note: There are many known o-minimal structures.
  - examples are obtained by stating which functions are allowed to generate the sets → non-trivial
- Sets in $\mathbb{R}_{\text{an,exp}}$ given by finitely many equalities and inequalities:



$$P_k(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}, f_1(x), \ldots, f_m(x)) = 0$$

$$\tilde{P}_l(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}, \tilde{f}_1(x), \ldots, \tilde{f}_m(x)) > 0$$

needed for tameness of complex exponential:

$$e^z = e^r(\cos(\phi) + i\sin(\phi)) \quad 0 \leq \phi \leq c$$

A currently active field of mathematics

- Historically routed in logic, there has been much recent activity in the field of tame geometry relating to different parts of mathematics

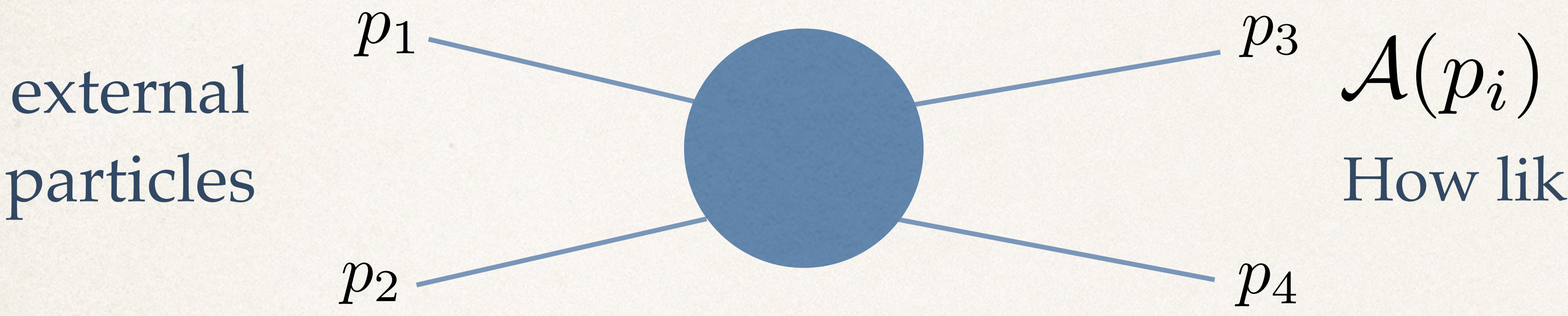
- Tameness used in many recent proofs of deep mathematics conjectures:
  - Klingler's Ax-Schanuel conjecture for Hodge structures [Bakker,Tsimerman '17]
    several subsequent generalizations, e.g. to mixed Hodge structures
  - Griffiths' conjecture [Bakker,Brunebarbe,Tsimerman '18]
  - André-Oort conjecture [Pila,Shankar,Tsimerman '21]
  - Geometric André-Grothendieck Period Conjecture [Bakker,Tsimerman '22]

  → very active field connecting logic, number theory, and geometry

Tameness in perturbative Quantum Field Theories (QFTs)

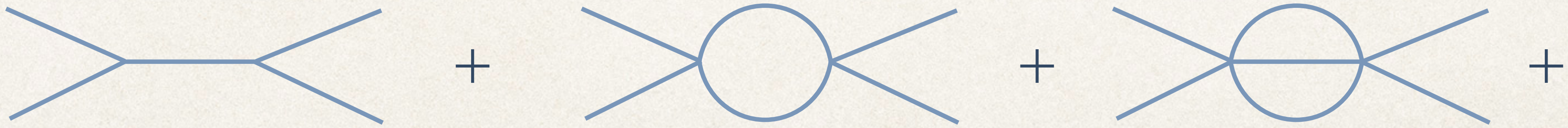
Perturbative QFTs

- Scattering amplitudes



- Physics: defined using path integrals - "sum over all possible processes"

- Perturbative expansion: small coupling expansion $\lambda \ll 1$

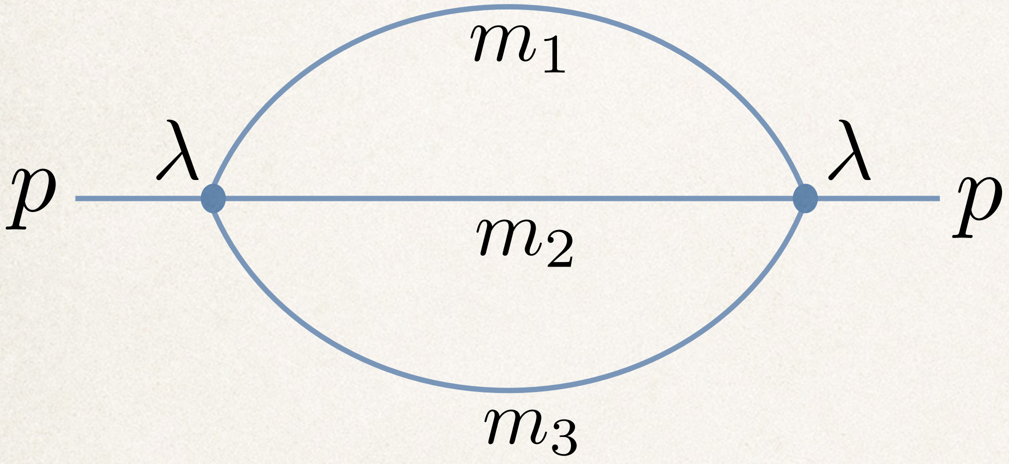


$\rightarrow$ summing till fixed loop number: finite number of Feynman integrals

Perturbative QFTs

- Theorem: For any renormalizable QFT with finitely many particles and interactions all finite-loop amplitudes are tame functions of the masses, external momenta, and coupling constants.

[Douglas,TG,Schlechter]



$$\mathcal{A}_2(m_1, m_2, m_3, \lambda, p)$$

tame in all parameters

hidden finiteness property in all QFT amplitudes

Remarks: - theorem is non-trivial: interesting implications for Feynman amplitudes (symmetry ⟷ relations) [in progress]

Why is this true?

- amplitudes are composed of finitely many Feynman integrals

- Basic idea: Feynman integrals are tame by relating them to period integrals of some auxiliary compact geometry Y_{graph}

  review book by [Weinzierl] + many original works

- Use: all steps only involve tame maps, period integrals are tame maps in o-minimal structure $\mathbb{R}_{\text{an,exp}}$

  seminal paper [Bakker,Klingler,Tsimerman '18]

  [Bakker,Mullane '22] related integration results [Comte,Lion,Rolin]

- Note: renormalizable theories have only finitely many counterterms → tameness preserved by finite composition

Tameness of full QFTs?

Observables in QFT

- interested in physical observables in local QFTs

particle described by the field ϕ

$\rightarrow$ dynamics encoded by Lagrangian, e.g. $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$

parameters of the model

- compute correlation functions:

(up to normalization)

$$\langle \mathcal{O}_1(y_1)...\mathcal{O}_k(y_k)\rangle_{\lambda,m} = \int D\phi\, \mathcal{O}_1(y_1)...\mathcal{O}_k(y_k)\, e^{-\int_\Sigma d^d y \mathcal{L}(\lambda,m)}$$

local operator at some space-time point $y_1 \in \Sigma$ (e.g. polynomial in ϕ)

path integral over all field configurations

exponential weight by parameter-dep. Lagrangian

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parameters of the model

- compute correlation functions: (up to normalization)

$$\langle \mathcal{O}_1(y_1)...\mathcal{O}_k(y_k)\rangle_\lambda$$

$\rightarrow$ complicated function on product of space-time $\Sigma \times ... \times \Sigma$ and parameter space $\mathcal{P}$

Tameness at non-perturbative level

- check tameness of partition functions $Z(\lambda) = \langle 1 \rangle_\lambda$ of solvable theories:
  - 0d theory: sine-Gordon $Z = \int_{-\pi}^{\pi} d\phi\, e^{4\lambda \sin^2(\phi)} = 2\pi e^{2\lambda} I_0(2\lambda)$

    $\longrightarrow$ modified Bessel function is tame (construct geometry → period)
  - 1d theory: harmonic oscillator (finite temperature partition function)

    $$Z(\beta, m) = \frac{1}{2\sinh \beta/(2m)}$$

    $\longrightarrow$ tame in β, m
  - 2d free Yang-Mills: $SU(2)$ example $Z_{SU(2)} = e^{\frac{A\lambda}{16}}(\theta_3(e^{-\frac{A\lambda}{16}}) - 1)$

    $\longrightarrow$ tame in λ, A, theta tame on fundamental domain [Peterzil,Starchenko]
  - 2d theories: (2,2) GLSMs appearing in Type II CY compactifications

    $Z_{S^2} = e^{-K} = \bar{\Pi}\eta\Pi$ tame due to relation to periods

Challenges in mathematics

- Consider in 0d: $S = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad \rightarrow \quad Z = \sqrt{\frac{3}{\lambda}} e^{\frac{3m^4}{4\lambda}}\, m\, K_{\frac{1}{4}}\left(\frac{3m^4}{4\lambda}\right)$ tame?

→ we expect Z to be tame, but tameness of $K_\alpha(x)$ has not been proved

→ tameness of exponential periods introduced by [Konzewitsch,Zagier] ?

Challenges in mathematics

- More generally: QFTs on points recently e.g. [Gasparotto,Rapakoulias,Weinzierl]

correlation functions in 0d are ordinary integrals

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda = \int d\phi_1 ... d\phi_k \, \mathcal{O}_1 ... \mathcal{O}_n \, e^{-S^{(0)}(\phi,\lambda)}$$

tame?

Conjecture [van den Dries][Kaiser]: Given a real-valued tame function $f(\lambda, \phi)$ (in some o-minimal structure $\mathcal{S}$) the integral

$$g(\lambda) = \int d\phi_1 ... d\phi_k f(\phi, \lambda)$$

is also a tame function (in some o-minimal structure $\widetilde{\mathcal{S}}$).

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$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda = \int d\phi_1 ... d\phi_k \, \mathcal{O}_1 ... \mathcal{O}_n \, e^{-S^{(0)}(\phi,\lambda)}$$

tame?

Note: This is a theorem for $\mathcal{S} = \mathbb{R}_{\mathrm{an}}$ yielding to $\widetilde{\mathcal{S}} = \mathbb{R}_{\mathrm{an,exp}}$.

[Comte,Lion,Rolin]

However, for non-perturbative results, we need exponential to be in $\mathcal{S}$.

Challenges in mathematics

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correlation functions in 0d are ordinary integrals

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda = \int d\phi_1 ... d\phi_k \, \mathcal{O}_1 ... \mathcal{O}_n \, e^{-S^{(0)}(\phi,\lambda)}$$

tame?

$\Rightarrow$ math. conjecture implies:

physical observables $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda$ are tame functions of parameters λ labelling the theory if

(1) $\mathcal{O}_1, \mathcal{O}_2, ...$ are tame functions of λ, ϕ

(2) $S^{(0)}(\phi, \lambda)$ is tame function of λ, ϕ

Are observables of every QFT tame?

- No! e.g. consider discrete symmetry group G

$$Z(g \cdot \lambda) = Z(\lambda)$$ → never tame if dim(G) is infinite

→ tameness requires that all such symmetries are gauged or eventually broken in full Z

→ One of best understood conjectures about Quantum Gravity: 'No global symmetries in QG'

[Banks,Dixon][Banks,Seiberg]

- black hole arguments
- confirmed in all String Theory settings
- proved within AdS/CFT for most global symmetries

[Harlow,Ooguri]

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- Non-tameness of Lagrangian: easy to get non-tame Lagrangian by picking non-tame potential $V(x)$

Simple: $$V(\theta) = A\cos(\theta) + B\cos(\alpha\,\theta) \qquad \alpha \text{ irrational}$$

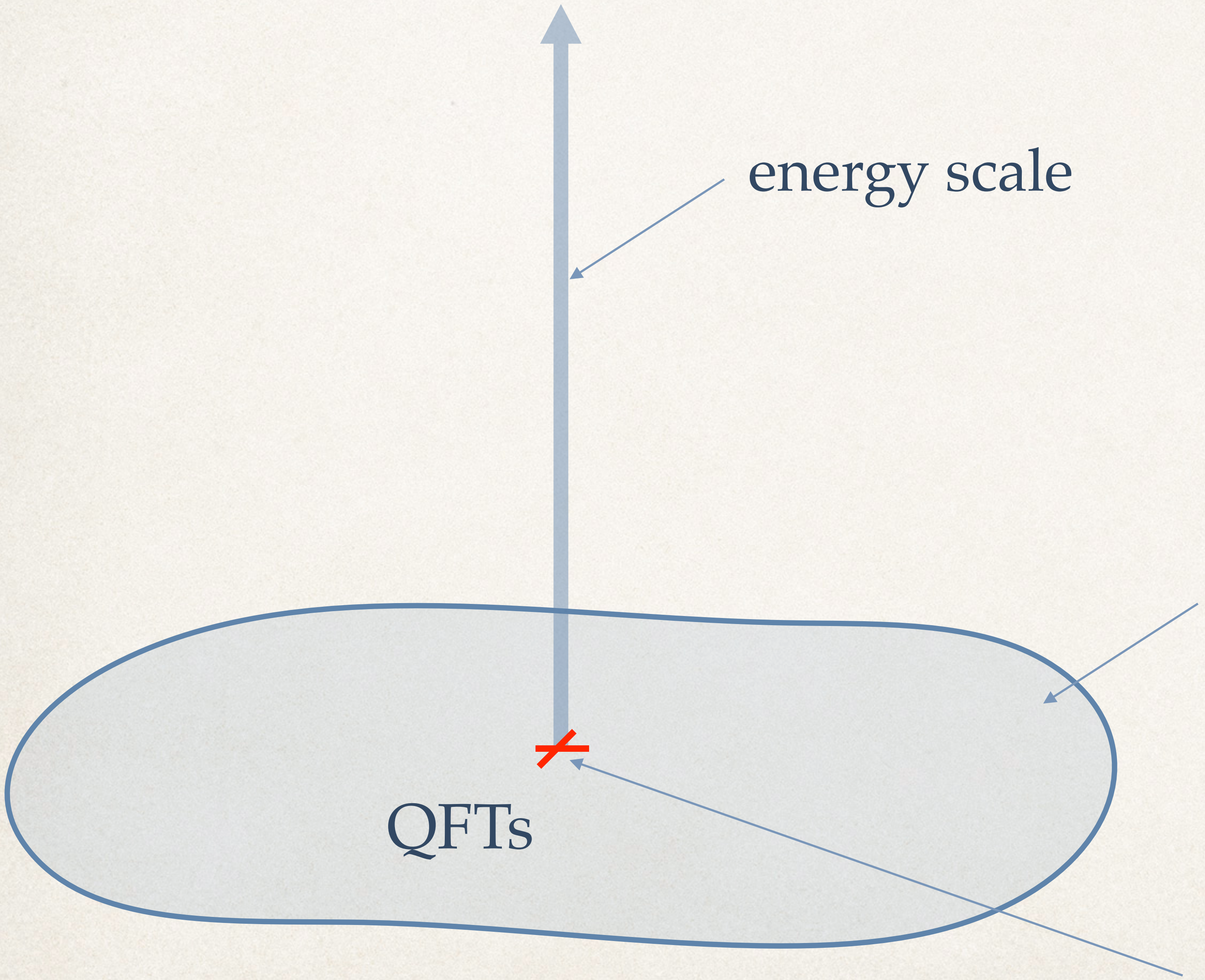
Fancy: $$W_\xi = Y P_\xi(X_1, \ldots, X_k)^2 + \sum_a Z_a(\sin 2\pi i X_a)^2$$ [Tachikawa]

Existence of supersymmetric vacua is undecidable!

- tameness depends on the UV origin of the theory

QFTs compatible with quantum gravity

A tameness conjecture



energy scale

QFTs

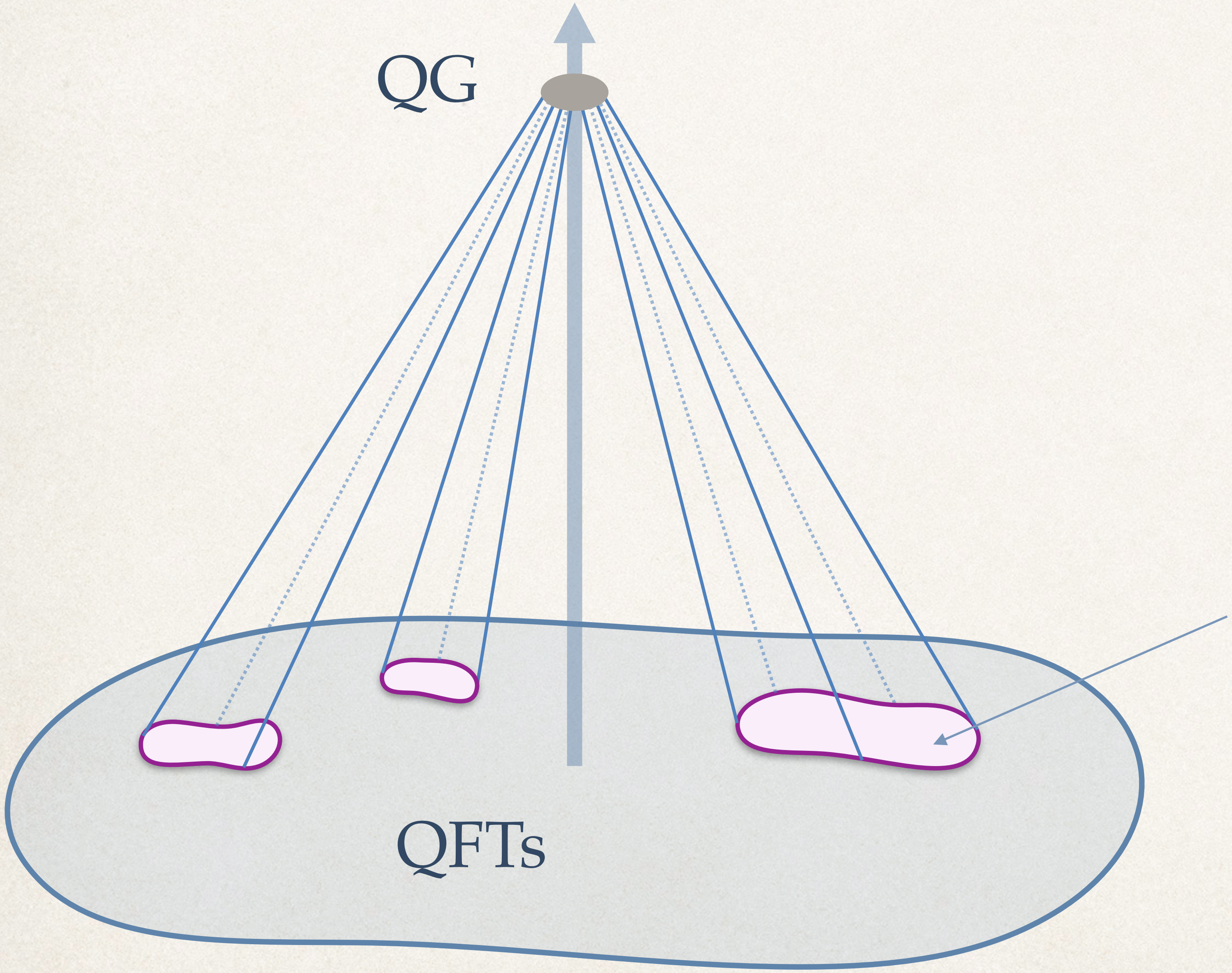
set of *d*-dimensional QFTs

e.g. defined by a Lagrangian

$$\mathcal{L}^{(d)}(\phi, \lambda)$$

cut-off scale of QFTs

A tameness conjecture



set $\mathcal{T}$ of d-dimensional QFTs

parameterized by tame Lagrangian varying over tame field and parameter space

A tameness conjecture

Conjecture [TG '21]:

All effective theories valid below a fixed finite energy cut-off scale that can be consistently coupled to quantum gravity are labelled by a tame parameter space and must have scalar field spaces and Lagrangians that are tame in an o-minimal structure.

- Conjecture implies several finiteness conjectures proposed in the past
  e.g. [Douglas][Acharya,Douglas][Vafa][Hamada,Montero,Vafa,Valenzuela]
- Part of the Swampland Program searching for universal constraints from Quantum Gravity
- bold conjecture based mostly on evidence from string theory, but fits nicely with other conjectures about the nature of Quantum Gravity

Evidence from String Theory:
A Tameness Theorem

String theory and higher dimensions

Study effective four-dimensional theories arising from String Theory

⇒ String Theory formulated consistently in 10 space-time dimensions

or 12 space-time dimensions (F-theory)

Product Ansatz for the higher-dimensional space-time manifold:

$$\mathbb{S} \times Y$$

our 4-dimensional
space-time

8-dimensional
compact manifold

→ Four-dimensional physics depends on choice of Y

Fifth force problem: deformations of Y correspond to massless fields
→ fifths force → immediate contradiction with experiment

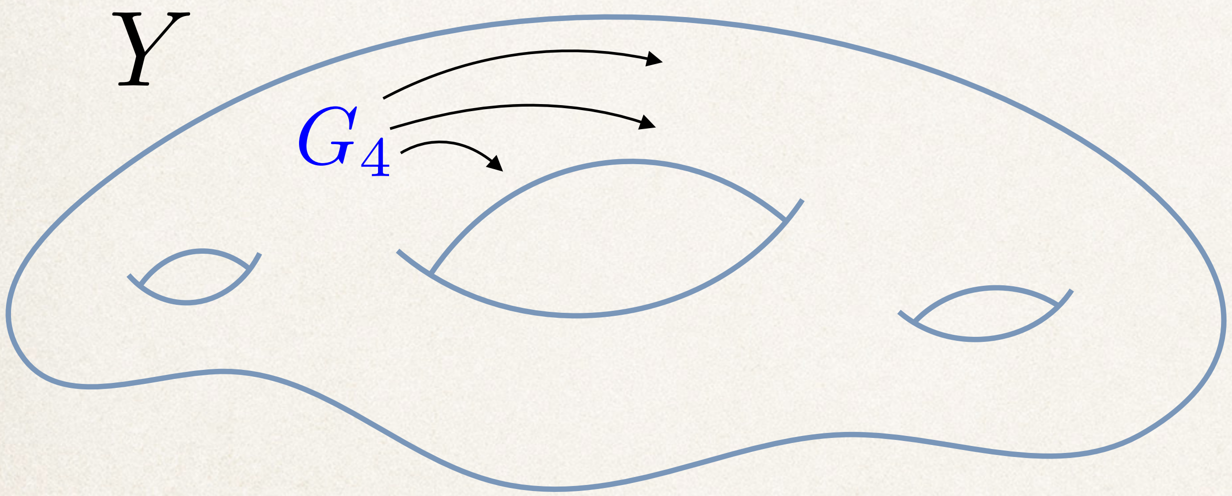
Solutions with background fields

Solution: Flux Compactifications review: [Graña] [Kachru,Douglas]

…[Becker,Becker '96],[Gukov,Vafa,Witten '99],[Giddings,Kachru, Polchiski '03],[TG,Louis '04]…

rough idea: introduce generalization of electromagnetic field, called G_4 on eight-dimensional manifold Y

differential 4-form 'flux'



equations of motion (Maxwell eq):
$G_4 \in H^4(Y,\mathbb{R})$

quantization:

$$\boxed{G_4 \in H^4(Y,\mathbb{Z})}$$

Aim: choice of G_4 solves fifth-force problem,
but parametrizing set of discrete parameters on lattice
→ disaster for tameness?

Best understood solutions

- Solution to 12-dimensional theory (F-theory) of the form:

  solving Einstein's equations and other equations of motion

  - 12d manifold: $\mathbb{S} \times Y$ ← Calabi-Yau fourfold:
    Kähler, vanishing first Chern class

  - 4-form flux: $G_4 \in H^4(Y, \mathbb{Z})$

$$\boxed{\int_Y G_4 \wedge G_4 = \ell}$$

cancellation of charge on compact Y

$$\boxed{*G_4 = G_4 \qquad G_4 \wedge J = 0}$$

self-dual flux

(true in cohomology)

Hodge star operator on Y

Kähler form on Y

Best understood solutions

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  - 4-form flux: $G_4 \in H^4(Y, \mathbb{Z})$ $\boxed{\int_Y G_4 \wedge G_4 = \ell}$ cancellation of charge on compact Y

$$\boxed{*G_4 = G_4 \qquad G_4 \wedge J = 0}$$

self-dual flux

(true in cohomology)

$\Rightarrow$ should be read as a condition on the choice of complex structure and Kähler structure on Y $\Rightarrow$ solve fifth force problem

Finiteness conjectures

- Concrete conjecture: The number of solutions in the described setting is finite. Finitely many choices for G_4.

  [Douglas '03] [Acharya,Douglas '06]

- Answer: Yes (on fixed topology for Y). [Bakker,TG,Schnell,Tsimerman '21]

  → Gen. finiteness theorem by [Cattani,Deligne,Kaplan '95] on Hodge classes

- Hodge theory formulation (focus on primitive part)

$$H^4(Y,\mathbb{C}) = H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4}$$ (Hodge decomposition)

Hodge classes: $G_4 \in H^4(Y,\mathbb{Z}) \cap H^{2,2}$ [CDK] → loci are complex algebraic

Self-dual classes: $G_4 \in H^4(Y,\mathbb{Z}) \cap (H^{4,0} \oplus H^{2,2} \oplus H^{0,4})$

→ loci can be real!

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- Tameness results:
  - [Bakker,Klingler,Tsimerman '18] show that period map is definable in $\mathbb{R}_{\mathrm{an,exp}}$ new proof of the theorem of [Cattani,Deligne,Kaplan] using tame geometry

- Finiteness / Tameness theorem: Locus of integral self-dual classes with bounded self-intersection is $\mathbb{R}_{\mathrm{an,exp}}$-definable closed real-analytic subspace of Hodge bundle and restriction of projection to the base to this locus has finite fibers.

  [Bakker,TG,Schnell,Tsimerman]

Tameness of CFTs

Structures from physical theories

- Idea: associate a structure to any set of physical theories

- Starting point for QFTs:
  - set of QFTs $\mathcal{T}$, e.g. specified Lagrangians $\mathcal{L}^{(d)}(\phi, \lambda)$
  - set $\mathcal{S}$ of Euclidean spacetimes (Σ, g)

$\rightarrow$ both sets should be definable in some structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$

Example:

$\mathcal{T}$: polynomial Lagrangians with real valued parameters

$\mathcal{S}$: spacetimes $\mathbb{R}^d, T^d, S^d$ with standard metric

$$\longrightarrow \quad \mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}} = \mathbb{R}_{\text{alg}}$$

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- Extend structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ by physical observables:
  add correlation/partition functions:

$$f_\alpha(y,\lambda) = \langle \mathcal{O}_1(y_1)...\mathcal{O}_1(y_n)\rangle_\lambda \longrightarrow$$

new structure

$\mathbb{R}_{\mathcal{T},\mathcal{S}}$

Structures from physical theories

- Idea: associate a structure to any set of physical theories

- Starting point for QFTs:
  - set of QFTs $\mathcal{T}$, e.g. specified Lagrangians $\mathcal{L}^{(d)}(\phi, \lambda)$
  - set $\mathcal{S}$ of Euclidean spacetimes (Σ, g)

  $\rightarrow$ both sets should be definable in some structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$

- Extend structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ by physical observables:
  add correlation/partition functions:

  Example: harmonic oscillator in quantum mechanics (Euclidean)

  $$\mathbb{R}_{\mathcal{T},\mathcal{S}} = \mathbb{R}_{\text{exp}}$$

Structures from physical theories

- First-order logic: $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ should be rich enough to formulate statements about the physical observables

- Tameness questions:

(1): If $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$ is o-minimal, is $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ o-minimal?

  - Are physical observables tame?

(2): What are simple conditions on theories such that $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$ is o-minimal?

  - Tameness of the set of physical theories?

For (2): Tameness conjecture → effective theories compatible with QG and valid below fixed cut-off scale

Structures from physical theories

- First-order logic: $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ should be rich enough to formulate statements about the physical observables

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(2): What are simple conditions on theories such that $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$ is o-minimal?

  - Tameness of the set of physical theories?

We consider: $\mathbb{R}_{\mathrm{QFT}d},\ \mathbb{R}_{\mathrm{EFT}d},\ \mathbb{R}_{\mathrm{CFT}d},\ \mathbb{R}_{\mathrm{QG}},\ \ldots$

Conformal field theories

- CFTs require no UV completion with quantum gravity
- CFTs are axiomatically well-defined theory set containing $\mathcal{T}$
  assume: CFT is unitary and local
- In [Douglas,TG,Schlechter II] we argue that CFT observables are tame and discuss various conditions which we believe ensure that $\mathcal{T}$ is tame set

Conjecture 1 (Tame observables):
All observables of a tame set $\mathcal{T}$ of CFTs are tame functions.

Alternative: Structure $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ for such theories is o-minimal.

$$\mathbb{R}_{\mathrm{CFT}d}$$

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Conjecture 1 (Tame observables):
All observables of a tame set $\mathcal{T}$ of CFTs are tame functions.

evidence from considering expansion into conformal partial waves

implications: conditions on gaps for operators
finite radius of convergence of conformal perturbation

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Conjecture 2 (Tame theory spaces):

Theory space $\mathcal{T}$ of CFTs in $d=2$ is tame set if

- central charge is bounded by $\hat{c}$
- lowest operator dimensions bounded from below by Δ_{min}

implies conjectures by [Douglas,Acharya][Kontsevich,Soibelman]

Conformal field theories

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  assume: CFT is unitary and local
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Conjecture 2 (Tame theory spaces):

Theory space $\mathcal{T}$ of CFTs in $d>2$ is tame set if

- appropriate measure of degrees of freedom is bounded by $\hat{c}$
- theories differing by discrete gaugings are identified

- many challenging cases: e.g. 3d Chern-Simons matter theories
  → show that there are no infinite discrete families

Conclusions

- Suggested that tameness of set of well-defined physical theories and their observables as a general principle

- Showed tameness of perturbative QFT amplitudes and certain non-perturbative settings

- Evidence for tameness from effective theories arising in String Theory
  → tameness theorem for self-dual integral classes, ‘flux vacua’

- Near future: tameness of space of Conformal Field Theories and their correlation functions

Conclusions

Often made statement:

All fields of mathematics are relevant in physics (especially string theory) apart from mathematical logic.

Fascinating new perspective:

Sets of well-defined QFTs and physical observables → First order structure (model) with tameness property 'o-minimal structure'

Much left to be explored at this new interface of physics-mathematics:
implications of tameness (computational + understanding QFTs)
relation to other QG conjectures,…,
connection with decidability

Thanks!