

Volumes and Random Matrices

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- ▶ Today we will be discussing the volume of the moduli space \mathcal{M}_g of Riemann surfaces of genus g , and also the volume of the corresponding moduli space \mathfrak{M}_g of super Riemann surfaces.
- ▶ We will also consider Riemann surfaces with punctures and/or boundaries.
- ▶ We will discuss how these volumes are related to random matrix ensembles.
- ▶ This is actually an old story with a very contemporary twist.
- ▶ Main references (apart from classic ones on torsion, Weil-Petersson volumes, and super Riemann surfaces): P. Saad, S. Shenker, D. Stanford, arXiv:1903.11115 (SSS), D. Stanford and EW, arXiv:1907.03363 (SW). Also see P. Norbury, arXiv:1712.03662, Y. Huang, R. Penner, and A. M. Zeitlin, arXiv:1907.09978. A written version of today's lecture will be on the arXiv tomorrow.

I will start with ordinary Riemann surfaces and the corresponding classical moduli space \mathcal{M}_g , and postpone super Riemann surfaces to the end of the lecture.

- ▶ What is meant by the volume of \mathcal{M}_g ?
- ▶ One answer is that \mathcal{M}_g has a natural (Weil-Petersson) symplectic structure. It parametrizes a family of flat $PSL(2, \mathbb{R})$ connections A over a genus g surface Σ – modulo the action of the mapping class group of Σ . This leads to a natural definition of a symplectic structure:

$$\omega = \frac{1}{4\pi} \int_{\Sigma} \text{Tr} \delta A \wedge \delta A.$$

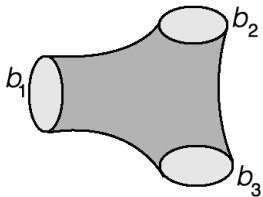
The volume is then

$$V_g = \int_{\mathcal{M}_g} \text{Pf}(\omega) = \int_{\mathcal{M}_g} e^{\omega}.$$

- ▶ From this point of view, a Riemann surface is not just a complex manifold of dimension 1.
- ▶ It is the quotient of the upper half plane $H \cong SL(2, \mathbb{R})/U(1)$ by a discrete group, and accordingly it carries a hyperbolic metric, which is a Riemannian metric of constant curvature $R = -2$.

Volumes can also be related to intersection theory of tautological (Mumford-Morita-Miller) classes on \mathcal{M}_g . Mirzakhani proved a sort of converse of this statement: from a knowledge of the volumes (for Riemann surfaces possibly with geodesic boundary, as discussed momentarily) one can deduce the tautological intersection theory. (These facts do not generalize directly to \mathfrak{M}_g , as there is not a natural intersection theory on a supermanifold.) For brevity I will not explain any detail about this part of the story.

Volumes for surfaces with boundary are introduced as follows. Let Σ be a hyperbolic Riemann surface of genus g with n boundaries. We require the boundaries to be geodesics of prescribed lengths b_1, b_2, \dots, b_n .



Let $\mathcal{M}_{g, \vec{b}}$ be the moduli space of such objects.

$\mathcal{M}_{g,\vec{b}}$ has a symplectic form and volume that can be defined precisely as before

$$\omega = \frac{1}{4\pi} \int_{\Sigma} \text{Tr} \delta A \wedge \delta A$$

$$V_{g,\vec{b}} = \int_{\mathcal{M}_{g,\vec{b}}} \text{Pf}(\omega) = \int_{\mathcal{M}_{g,\vec{b}}} e^{\omega}.$$

Mirzakhani showed in her thesis that $V_{g,\vec{b}}$ is a polynomial in b_1, b_2, \dots, b_n , and that the canonical intersection numbers are the coefficients of the top degree terms in this polynomial.

- ▶ The relation of volumes to intersection numbers gives one way to compute them but it is hard to use this to get explicit formulas.
- ▶ This relationship shows that volumes are related to random matrix ensembles.
- ▶ My 1990 conjecture about intersection numbers was motivated by work (of physicists Douglas and Shenker; Gross and Migdal; Brezin and Kazakov) relating random matrix ensembles to two-dimensional gravity.
- ▶ Kontsevich's proof was based on a connection of the intersection numbers to a different type of random matrix ensemble that he discovered.
- ▶ However the role of the random matrices in all these considerations was rather obscure, at least to me.
- ▶ What I will explain today gives a much more direct link to random matrices.

- ▶ In her thesis, Mirzakhani discovered a new direct way to compute the bosonic volumes $V_{g, \vec{b}}$.
- ▶ I will explain how Saad, Shenker, and Stanford, following Eynard and Orantin, reinterpreted her results in terms of a random matrix ensemble.
- ▶ In this approach, the role of the random matrix ensemble is much more transparent than in previous work, in my opinion.
- ▶ Then I will explain how Stanford and I developed a superanalog of this and obtained Mirzakhani-style formulas for the super-volumes $\widehat{V}_{g, \vec{b}}$.

- ▶ Let S^1 be a circle. An analog of \mathcal{M}_g as a symplectic manifold is $\text{diff}S^1/\text{PSL}(2, \mathbb{R})$ or $\text{diff}S^1/\text{U}(1)$, viewed as homogeneous symplectic manifolds.
- ▶ In fact, $\text{diff}S^1/\text{PSL}(2, \mathbb{R})$ is sometimes called “universal Teichmüller space” (for example see F. G. Gardiner and W. J. Harvey, arXiv:math/0012168).
- ▶ $\text{diff}S^1/\text{PSL}(2, \mathbb{R})$ and $\text{diff}S^1/\text{U}(1)$ have natural symplectic forms ω because they are coadjoint orbits of $\text{diff}S^1$.
- ▶ Writing $\mathcal{X} = \text{diff}S^1/\text{PSL}(2, \mathbb{R})$ or $\mathcal{X} = \text{diff}S^1/\text{U}(1)$, we cannot make sense of the infinite-dimensional “volume”

$$V_{\mathcal{X}} = \int_{\mathcal{X}} e^{\omega}.$$

- ▶ It is believed that there is no reasonable definition of this volume.

- ▶ The infinite dimensional integral $\int_{\mathcal{X}} e^{\omega}$ is too divergent even for physicists.
- ▶ But we can do the following: Consider a subgroup $U(1) \cong S^1 \subset \text{diff } S^1$, consisting of rigid rotations of S^1 . In other words, for some parametrization of S^1 by an angle θ , $U(1)$ acts by $\theta \rightarrow \theta + \text{constant}$.
- ▶ Then there is a moment map H for this action of $U(1)$; in other words, if V is the vector field on \mathcal{X} that generates $U(1)$ and i_V is contraction with V , then

$$dH = -i_V \omega.$$

- ▶ Then introducing a real constant β , the integral

$$Z(\beta) = \int_{\mathcal{X}} \exp(H/\beta + \omega)$$

does make sense, as understood by physicists.

We are in an infinite-dimensional version of a situation that was studied by Duistermaat and Heckman, and then reinterpreted by Atiyah and Bott in terms of equivariant cohomology. Let \mathcal{Y} be a symplectic manifold with symplectic form ω and action of $U(1)$. Let p_1, \dots, p_s be the fixed points of the $U(1)$ action. For simplicity I assume that there are finitely many. Let H be the moment map for the $U(1)$ action. The Duistermaat-Heckman/Atiyah-Bott (D-H/A-B) formula gives

$$\int_{\mathcal{Y}} \exp(H/\beta + \omega) = \sum_i \frac{\exp(H(p_i)/\beta)}{\prod_{\alpha} (e_{i,\alpha}/2\pi\beta)},$$

where the $e_{i,\alpha}$ are integers that represent the eigenvalues of the $U(1)$ action on the tangent space to \mathcal{Y} at p_i .

In the present example, there is only one fixed point in the $U(1)$ action on $\text{diff}S^1/\text{PSL}(2, \mathbb{R})$ or $\text{diff}S^1/U(1)$. The product over eigenvalues at this fixed point becomes formally $\prod_{n=2}^{\infty} n/2\pi\beta$ which is treated with (for example) ζ -function regularization. The result is

$$Z(\beta) = \frac{C}{4\pi^{3/2}\beta^{3/2}} \exp(\pi^2/\beta),$$

where the constant C depends on the regularization and so is considered inessential, but the rest is “universal.” (This problem was first studied by A. Kitaev followed by Maldacena and Stanford; the explanation I have sketched is in D. Stanford and EW, arXiv:1703.04612. There are many other derivations of this formula in the physics literature.)

I have described this somewhat abstractly. To use the D-H/A-B formula, we did not need to know what is the moment map H (only its value at the fixed point). But in fact it is a function of interest. To pick the $U(1)$ subgroup of $\text{diff } S^1$ that was used in this “localization,” we had to pick an angular parameter θ on the circle; an element of $\text{diff } S^1$ maps this to another parameter t , and H is the integral of the Schwarzian derivative $\{t, \theta\}$.

It is convenient to take an inverse Laplace transform of the formula for $Z(\beta)$ and write

$$Z(\beta) = \int_0^\infty dE \rho(E) \exp(-\beta E)$$

with

$$\rho(E) = \frac{C'}{4\pi} \sinh(2\pi\sqrt{E}).$$

(There are similar formulas for the other case $\text{diff } S^1/U(1)$.)

I would like to explain why this formula was considered problematical and how Saad, Shenker, and Stanford (SSS) interpreted it. But this will require explaining a little more physics. General relativity is difficult to understand as a quantum theory. Searching for understanding, physicists have looked for a simpler model in a lower dimension. Two dimensions is a good place to look. An obvious idea might be to start with the Einstein-Hilbert action in two dimensions, $I = \int_{\Sigma} d^2x \sqrt{g} R$, with R the Ricci scalar of a Riemannian metric g . This does not work well, as in two-dimensions this action is a topological invariant, according to the Gauss-Bonnet theorem. Instead it turns out to be better to add a scalar (real-valued) field ϕ . What turns out to be for many purposes a simple and illuminating model of two-dimensional gravity is “Jackiw-Teitelboim (JT) gravity,” with action

$$I = \frac{1}{\kappa} \int_{\Sigma} d^2x \sqrt{g} \phi (R + 2).$$

The form of the action

$$I = \frac{1}{\kappa} \int_{\Sigma} d^2x \sqrt{g} \phi(R + 2)$$

implies that a classical solution will have $R + 2 = 0$, so in other words it is a hyperbolic Riemann surface. The Feynman path integral for compact Σ without boundary (or with geodesic boundary of prescribed length) is very simple. The path integral

$$Z_{\Sigma} = \frac{1}{\text{vol}} \int \mathcal{D}\phi \mathcal{D}g \exp \left(-\frac{1}{\kappa} \int d^2x \sqrt{g} \phi(R + 2) \right)$$

is studied by integrating first over ϕ (after rotating the integration contour $\phi \rightarrow i\phi$) and gives a delta function setting $R + 2 = 0$. The prefactor $1/\text{vol}$ is a schematic way to indicate that we have to divide by the diffeomorphism group. So the integral “localizes” on the moduli space of two-manifolds with hyperbolic structure.

If Σ is orientable and of genus g , the moduli space of two-manifolds with hyperbolic structure is the usual moduli space \mathcal{M}_g of Riemann surfaces of genus g , and one can show that the integral over \mathcal{M}_g gives its usual volume:

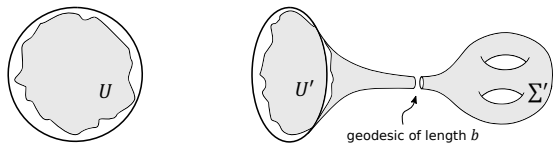
$$Z_\Sigma = C^{\chi(\Sigma)} \int_{\mathcal{M}_g} \exp(\omega).$$

(Here $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Σ , and C is a constant, independent of g , that depends on the regularization used in defining the Feynman path integral. It can be absorbed in the normalization of the Weyl-Petersson form ω , since the dimension of \mathcal{M}_g is a fixed multiple of χ , namely -3χ .)

If Σ is unorientable, there is a more complicated and very interesting story, for which unfortunately there is not time today.

So far, I've assumed that Σ is a compact surface (without boundary or with geodesic boundary of prescribed length). What really led to progress in the last few years was applying JT gravity to, roughly speaking, the whole upper half-plane H – the universal Teichmüller space. But it turned out that literally taking all of H is not the right thing to do. This would be rather like trying to calculate the naive integral $\int_{\text{diff } S^1 / \text{PSL}(2, \mathbb{R})} \exp(\omega)$. It turns out that a better thing to do is to consider not all of H but a very large region $U \subset H$.

Such a large region $U \subset H$ is sketched on the left of this figure:



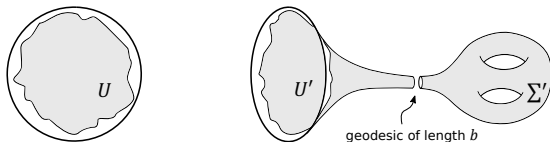
(What is on the right will be discussed later.) One considers JT gravity on a two-manifold that topologically is a disc. We will not, however, use the “geodesic” boundary conditions that I mentioned previously. Instead we will specify the induced metric of the boundary, which I will call h , and the boundary values of ϕ .

On a manifold with boundary, the Einstein-Hilbert action needs a (Gibbons-Hawking-York) boundary correction, which also appears here:

$$I = \frac{1}{\kappa} \int_{\Sigma} d^2x \sqrt{g} \phi (R + 2) + \frac{1}{\kappa} \int_{\partial\Sigma} dx \sqrt{h} \phi (K - 1),$$

where K is the extrinsic curvature of the boundary $\partial\Sigma$, and h is the induced metric of the boundary. A classical solution is going to have $R + 2 = 0$ so (as Σ is topologically a disc) it will be a region in H . The boundary condition is such that the induced metric h of the boundary is specified (and taken to have extremely large circumference). $\phi|_{\partial\Sigma}$ is also specified to be a large constant. In a certain scaling limit as the length of the boundary goes to infinity and $\phi|_{\partial\Sigma}$ also becomes large, with fixed ratio β , the Feynman integral turns into our friend

$$\int_{\text{diff } S^1 / \text{PSL}(2, \mathbb{R})} \exp(H/\beta + \omega).$$



How this happens needs some explanation. $\text{diff } S^1/\text{PSL}(2, \mathbb{R})$ comes in when one compares a natural parameter on ∂H (unique up to the action of $\text{PSL}(2, \mathbb{R})$) to the arclength parameter of $\partial \Sigma$. The key step that relates JT gravity to the integral that we discussed over $\text{diff } S^1/\text{PSL}(2, \mathbb{R})$ is that $\int_{\partial \Sigma} (K - 1)$, in the limit that the perimeter of ∂U and the constant value of ϕ are both large, with fixed ratio, becomes a multiple of the moment map H . Hence the Feynman integral of JT gravity on the disc Σ becomes our friend

$$\int_{\text{diff } S^1/\text{PSL}(2, \mathbb{R})} \exp(H/\beta + \omega).$$

After steps I have explained, this becomes

$$Z(\beta) = \int_0^\infty dE \rho(E) \exp(-\beta E), \quad \rho(E) = C \sinh(2\pi\sqrt{E}).$$

- ▶ This is a deeply problematic answer for the Feynman integral on the disc.
- ▶ To understand this, one should be familiar with holographic duality between gravity in the bulk of spacetime and an ordinary quantum system on the boundary.
- ▶ If the bulk were 4-dimensional, the boundary would be 3-dimensional and the “ordinary quantum system” on the boundary would be a quantum field theory – perhaps not a very familiar concept
- ▶ But here the bulk is 2-dimensional and so the boundary is just 1-dimensional.
- ▶ An ordinary quantum system in 1 dimension is just described by giving a Hilbert space \mathcal{J} and a Hamiltonian operator \mathcal{H} acting on \mathcal{J} .
- ▶ The basic recipe of holographic duality predicts that
$$Z(\beta) = \text{Tr}_{\mathcal{J}} \exp(-\beta\mathcal{H}).$$

In a moment, we will check that that prediction is false, but before doing so, I want to explain that this is actually not entirely a surprise:

- ▶ Analogous calculations (going back to Hawking, Gibbons, and others in the 1970's) have always given the same problem
- ▶ The problem is the essential mystery about quantum black holes
- ▶ The calculations were always done in models (like four-dimensional General Relativity) that were too complicated for a complete calculation, and there was always a possibility that a more complete calculation would make the issue go away
- ▶ Holographic duality and a variety of other developments that I am omitting made it possible to ask the question in a model – JT gravity – that is so simple that one can do a complete calculation, demonstrating the problem.

To see that the prediction of the duality is false:

- ▶ If we do have a Hilbert space \mathcal{J} and a Hamiltonian \mathcal{H} acting on \mathcal{J} such that the operator $e^{-\beta\mathcal{H}}$ has a trace, then \mathcal{H} must have a discrete spectrum with eigenvalues E_1, E_2, \dots (which moreover must tend to infinity fast enough) and

$$\text{Tr} \exp(-\beta\mathcal{H}) = \sum_i e^{-\beta E_i} = \int_0^\infty dE \sum_i \delta(E - E_i) e^{-\beta E}.$$

- ▶ However, the integral over $\text{diff } S^1/\text{PSL}(2, \mathbb{R})$ gave

$$Z(\beta) = \int_0^\infty dE \cdot C \sinh(2\pi\sqrt{E}) e^{-\beta E}.$$

- ▶ The function $C \sinh(2\pi\sqrt{E})$ is not a sum of delta functions, so the prediction of the duality is false.

However, the interpretation via JT gravity gives us a key insight that we did not have when we were just abstractly integrating over $\text{diff } \mathcal{S}^1 / \text{PSL}(2, \mathbb{R})$:

- ▶ The constant C is exponentially large near the classical limit ($\kappa \rightarrow 0$). We interpret it as e^S where S is the classical black hole entropy, of order $1/\kappa$ or $1/\hbar$ and thus large.
- ▶ When C is exponentially large, the function $C \sinh(2\pi\sqrt{E})$, which we now write as $e^S \sinh(2\pi\sqrt{E})$, can be well-approximated as a sum of delta functions.
- ▶ One must look very closely to see the difference.

- ▶ The novel idea of Saad, Shenker, and Stanford (SSS) was to interpret $e^S \sinh(2\pi\sqrt{E})$ as not the density of states of a particular Hamiltonian, but as the average density of states of an ensemble of Hamiltonians – a random matrix.
- ▶ In terms of the physics involved, this interpretation was sort of heretical and highly stimulating, but I do not think I will be able to convey this well.

What made them go in this direction?

- ▶ One clue was given by the work of A. Kitaev which had pointed in the direction of things I am telling you about. His work had involved a random ensemble (more complicated than the one used by SSS), but unfortunately there isn't time today to describe this ensemble.
- ▶ Another clue was the prior history of relations between two-dimensional gravity and random matrix theory.
- ▶ Finally a clue related more directly to today's lecture had to do with volumes of moduli spaces of Riemann surfaces.
- ▶ Mirzakhani, as I said at the beginning, had found a new way to compute these volumes and Eynard and Orantin (EO) (arXiv:0705.3600) had interpreted her work in terms of a random matrix ensemble.
- ▶ And the eigenvalue density of the EO ensemble was precisely $e^S \sinh(2\pi\sqrt{E})!$ (with a different interpretation of the constant S and a different normalization of the energy E).

The sort of random matrix ensemble that we are interested in is the following.

- ▶ M will be an $N \times N$ hermitian matrix for some N ; we are really interested in N very large or $N \rightarrow \infty$.
- ▶ Picking some suitable function $T(M)$, we consider the integral

$$Z(T; N) = \frac{1}{\text{vol}(\text{U}(N))} \int dM \exp(-N \text{Tr} T(M)).$$

- ▶ This integral or rather its logarithm has an asymptotic expansion for large N :

$$\log Z(T; N) \sim N^2 F_0(T) + F_1(T) + \frac{1}{N^2} F_2(T) + \dots$$

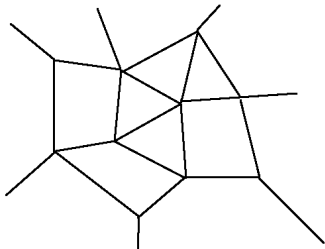
$$= \sum_{g=0}^{\infty} N^{2-2g} F_g(T).$$

- ▶ The expansion

$$\begin{aligned}\log Z(T; N) &\sim N^2 F_0(T) + F_1(T) + \frac{1}{N^2} F_2(T) + \dots \\ &= \sum_{g=0}^{\infty} N^{2-2g} F_g(T).\end{aligned}$$

is constructed by standard Feynman diagram methods ('t Hooft, 1974).

- ▶ In that context $F_g(T)$ is the sum of connected Feynman diagrams of genus g .
- ▶ Here the “genus” is the genus of a two-manifold on which a given Feynman diagram can be naturally drawn.



However:

- ▶ Instead of a Feynman diagram expansion, we can just try to evaluate the integral.
- ▶ We diagonalize M , writing $M = U\Lambda U^{-1}$,
 $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$.
- ▶ The measure is

$$dM = dU \prod_i d\lambda_i \prod_{j < k} (\lambda_j - \lambda_k)^2.$$

- ▶ Here dU is Haar measure on the group $U(N)$ and the integral over U just cancels the factor $1/(\text{vol}U(N))$ in the integral we are trying to do.
- ▶ We reduce to

$$Z(T; N) = \int d\lambda_1 d\lambda_2 \cdots d\lambda_N \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_k \exp(-NT(\lambda_k)).$$

- ▶ The integrand

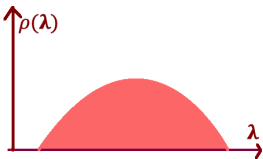
$$\prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_k \exp(-NT(\lambda_k))$$

has a sharp maximum as a function of the λ 's,

- ▶ Remember there are many of them. If the density of λ 's is $N\rho(\lambda)$ for some function ρ (constrained so $\int d\lambda\rho(\lambda) = 1$) then the integrand is

$$\exp\left(N^2\left(-\int d\lambda\rho(\lambda)T(\lambda) + \int d\lambda d\lambda'\rho(\lambda)\rho(\lambda')\log|\lambda - \lambda'|\right)\right).$$

- ▶ For a nice function T , the exponent has a unique maximum at some function $\rho(\lambda)$, which might look like this:



If $F_0(T)$ is the value of the exponent at its maximum, then the leading approximation to the integral is

$$Z \sim \exp(N^2 F_0(T))$$

or

$$\log Z \sim N^2 F_0(T) + \dots$$

How can we compute the corrections to this leading behavior?

Something nice happens, but I won't have time to explain it. One should define the “spectral curve” in the $y - \lambda$ plane:

$$y^2 = -\rho^2(\lambda).$$

Once one knows $\rho(\lambda)$, one can forget about doing integrals and one can forget the original function T . The whole expansion

$$\log Z(T; N) \sim N^2 F_0(T) + F_1(T) + \frac{1}{N^2} F_2(T) + \dots = \sum_{g=0}^{\infty} N^{2-2g} F_g(T)$$

(and everything else about this ensemble that we might want to know) can be worked out just using a knowledge of the spectral curve. A very useful version of this process is the “topological recursion” of Eynard and Orantin.

Now let us go back to volumes of moduli spaces:

- ▶ As I explained, SSS interpreted the function $e^S \sinh(2\pi\sqrt{E})$ as the density of eigenvalues $N\rho(E)$ for a random matrix from the type of ensemble that I described.
- ▶ In principle, the procedure is to start with a function T and compute the corresponding density of energy levels $N\rho_T(E)$, where I make the dependence on T explicit.
- ▶ Then we take $N \rightarrow \infty$ while adjusting the function T so that $N\rho_T(E)$ converges to the desired $e^S \sinh(2\pi\sqrt{E})$. (This is called double-scaling.)
- ▶ But we can skip all that work because everything we want to compute only depends on the spectral curve and we know the spectral curve is going to be

$$y^2 = -\sinh^2(2\pi\sqrt{E}).$$

In short, all we have to do is to start with the spectral curve

$$y^2 = -\sinh^2(2\pi\sqrt{E})$$

and apply topological recursion to compute the expansion

$$\log Z(T; N) = \sum_{g=0}^{\infty} e^{S(2-2g)} F_g(T)$$

(and other quantities of interest that are introduced momentarily) where after double-scaling, the expansion parameter is e^{-S} rather than $1/N$.

Now we can compute volumes:

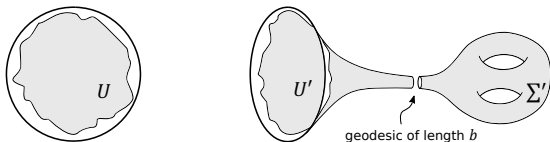
- ▶ To compute the volumes of moduli spaces, we compute the average of $\text{Tr} \exp(-\beta\mathcal{H})$ in this random matrix ensemble (where $\mathcal{H} = M$).
- ▶ That can be done explicitly, applying topological recursion to the spectral curve

$$y^2 = -\sinh^2(2\pi\sqrt{E}).$$

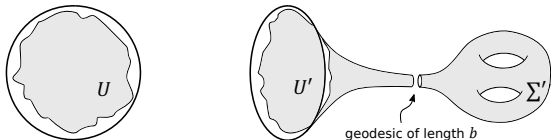
- ▶ The result is an expansion in powers of e^{-2S} .

To interpret the result in terms of volumes:

- ▶ The Feynman diagram expansion of $\langle \text{Tr} \exp(-\beta H) \rangle$ in this ensemble involves Feynman diagrams drawn on an oriented two-manifold with one boundary component as in the picture



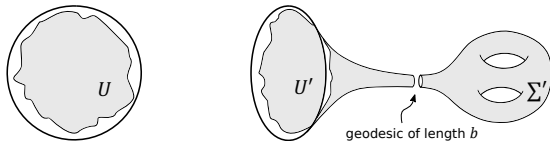
- ▶ When we make a Feynman diagram expansion, the trace $\text{Tr} \exp(-\beta \mathcal{H})$ turns into a boundary.
- ▶ The picture on the left actually corresponds to the leading answer, the JT integral on $\text{diff} S^1 / \text{SL}(2, \mathbb{R})$, which we've interpreted as $\int_0^\infty dE \rho(E) e^{-\beta E}$. We are now interested in the higher topologies shown on the right. They contribute the higher order terms in the expansion in e^{-2S} . (The genus g term is of relative order $\exp(-2gS)$.)



- ▶
- ▶ On the right is a Riemann surface $\Sigma = U' \cup \Sigma'$. Here U' represents $\text{diff} S^1 / U(1)$.
- ▶ The JT integral on this homogeneous space is similar to the integral over $\text{diff} S^1 / \text{PSL}(2, \mathbb{R})$ that we discussed before, but it depends on a parameter b (which appears in the choice of a coadjoint orbit corresponding to $\text{diff} S^1 / U(1)$).
- ▶ Let us just write $\Theta(b; \beta)$ for the JT integral on this orbit.
- ▶ The other half of Σ is a Riemann surface Σ' of genus $g \geq 1$ with a geodesic boundary of length b . Let $\mathcal{M}_{g,b}$ be its moduli space and $V(g, b)$ the corresponding volume.
- ▶ Then JT gravity in this geometry gives

$$\int_0^\infty db b \Theta(b; \beta) V(g, b).$$

On the other hand, a particular term in the expansion of the matrix integral in powers of e^{-S} is supposed to equal JT gravity on the surface $\Sigma = U' \cup \Sigma'$



Comparing the result one gets that way to

$$\int_0^\infty db \Theta(b; \beta) V(g, b).$$

where $\Theta(b; \beta)$ is known by DH/AB localization, one gets explicit results for $V(g, b)$.

- ▶ This procedure gives the right answer for $V(g, b)$ (and it can be extended to give the right answer for moduli spaces of surfaces with different numbers of geodesic boundaries) because Eynard and Orantin showed that topological recursion applied to the spectral curve

$$y^2 = -\sinh^2(2\pi\sqrt{E})$$

recovers a recursion relation discovered by Maryam Mirzakhani by means of which she had computed the volumes.

- ▶ Matching with Mirzakhani's recursion relation was how Eynard and Orantin determined which spectral curve to use. Having determined the spectral curve, their main insight was that Mirzakhani's recursion relation is equivalent to topological recursion for that spectral curve.
- ▶ Another way to find the right spectral curve is to use the relation of volumes to intersection numbers and the general relation of intersection numbers to matrix ensembles and spectral curves.

What Saad, Shenker, and Stanford obtained was a physical interpretation of the procedure of Mirzakhani and Eynard/Orantin. This was very interesting for physicists, but if you only care about volumes, you might not be sure why it is important. One answer is that possibly we've gained a better understanding of the relation between $\text{diff}S^1/\text{SL}(2, \mathbb{R})$ and $\mathcal{M}_{g,n}$. Also, we possibly now have a more direct understanding of the relation of intersection theory to random matrices. Yet another possible answer is given by my work with Stanford. We ran the whole story for super Riemann surfaces.

- ▶ First of all, what is a super Riemann surface?
- ▶ There are various approaches, but for today's purposes, we get a super Riemann surface by just replacing $SL(2, \mathbb{R})$, which is the group of linear transformations of \mathbb{R}^2 that preserve the symplectic form $du dv$, with $OSp(1|2)$, which is the supergroup of linear transformations of $\mathbb{R}^{2|1}$ that preserves the symplectic form $du dv - d\theta^2$.
- ▶ $OSp(1|2)$ is a Lie supergroup of dimension $3|2$. Its Lie algebra carries a nondegenerate bilinear form that I will denote as Tr .
- ▶ The superanalog of the upper half plane H is $\hat{H} = OSp(1|2)/U(1)$. Thus \hat{H} is a smooth supermanifold of real dimension $2|2$; it carries a complex structure in which it has complex dimension $1|1$.

A digression for those familiar with other approaches to super Riemann surfaces: The definition I've given is related to a standard definition as follows:

- ▶ First, \widehat{H} carries a canonical “completely unintegrable distribution” making it a super Riemann surface. (There is no natural splitting of $\mathfrak{osp}(1|2)$ as the direct sum of even and odd parts, but the choice of a point in \widehat{H} determines such a splitting, and the odd part defines a subbundle of the tangent bundle to \widehat{H} which is the unintegrable distribution.)
- ▶ Now if we are given a flat $\mathrm{OSp}(1|2)$ connection (of appropriate topological type) on an ordinary 2-manifold Σ , then its monodromies define a homomorphism $\rho : \pi_1(\Sigma) \rightarrow \mathrm{OSp}(1|2)$. Set $\Gamma = \rho(\pi_1(\Sigma))$. Then $\widehat{\Sigma} = \widehat{H}/\Gamma$ is a smooth supermanifold of real dimension $2|2$ that inherits a complex structure and unintegrable distribution from \widehat{H} . It is a super Riemann surface.
- ▶ If some of these matters are unfamiliar, they are not really needed for today.

- ▶ With the “hyperbolic” definition that I have given of \mathfrak{M}_g , we can imitate the definition of a symplectic form and a volume that I gave in the ordinary case.
- ▶ The symplectic form of \mathfrak{M}_g is

$$\hat{\omega} = \frac{1}{4\pi} \int_{\Sigma} \text{Tr } \delta A \wedge \delta A.$$

- ▶ The volume is

$$\hat{V}_g = \int_{\mathfrak{M}_g} \sqrt{\text{Ber}(\hat{\omega})}.$$

(Ber is the Berezinian, the superanalog of the determinant.)

- ▶ One can also define moduli spaces $\mathfrak{M}_{g, \vec{b}}$ of super Riemann surfaces with geodesic boundaries of specified lengths $\vec{b} = (b_1, b_2, \dots, b_n)$ and corresponding volumes $\hat{V}_{g, \vec{b}}$.

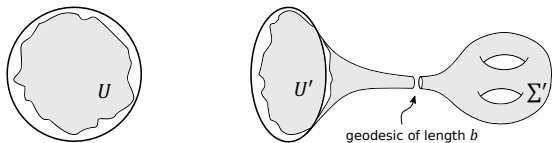
- ▶ It is possible to describe the super volumes \widehat{V}_g in purely bosonic terms.
- ▶ The “reduced space” of \mathfrak{M}_g is the moduli space \mathcal{M}'_g that parametrizes an ordinary Riemann surface Σ with a spin structure, which we can think of as a square root $K^{1/2}$ of the canonical bundle $K \rightarrow \Sigma$.
- ▶ The normal bundle to \mathcal{M}'_g is the vector bundle $U \rightarrow \mathcal{M}'_g$ whose fiber is $H^1(\Sigma, K^{-1/2})$. Viewing U as a real vector bundle (of twice the complex dimension), we denote its Euler class as $\chi(U)$.
- ▶ The symplectic form $\widehat{\omega}$ of \mathfrak{M}_g restricts along \mathcal{M}'_g to the ordinary symplectic form ω of \mathcal{M}'_g (which is a finite cover of \mathcal{M}_g).

By general arguments about symplectic supermanifolds, one can show that

$$\widehat{V}_g = \int_{\mathcal{M}'_g} \chi(U) e^\omega.$$

Thus what I will say about the supervolumes can be interpreted as a purely classical statement about \mathcal{M}'_g .

The superanalog of JT gravity is JT supergravity, which computes volumes of the moduli spaces $\mathfrak{M}_{g,\vec{b}}$ of super Riemann surfaces, in general with geodesic boundaries of prescribed lengths. As before, it is important to consider the special case of a Riemann surface which is the super upper half plane \widehat{H} , or more precisely a very large piece of it, as in the left hand side of the familiar picture



The boundary of that large piece (or simply the boundary of \widehat{H}) is what I will call $S^{1|1}$, the superanalog of a circle.

The super JT path integral on a big piece of \widehat{H} is

$$Z = \int_{\text{Sdiff } S^1/\text{OSp}(1|2)} \exp(\mathbf{H}/\beta + \widehat{\omega}).$$

This is the closest universal super Teichmüller space analog of the supermoduli space volume. (Again \mathbf{H} is the moment map for a $U(1)$ subgroup.) In our first paper, Stanford and I computed this integral, again using D-H/A-B localization:

$$Z(\beta) = \int_0^\infty dE \exp(-\beta E) \widehat{\rho}(E), \quad \widehat{\rho}(E) = e^S \frac{\cosh(2\pi\sqrt{E})}{\sqrt{E}}.$$

Again, this is not $\text{Tr}_{\mathcal{J}} \exp(-\beta\mathcal{H})$ for any Hamiltonian \mathcal{H} acting on a Hilbert space \mathcal{J} , but now we know what to do: we have to consider an ensemble of random Hamiltonians.

We can rerun the previous story with a few changes:

- ▶ The formula for $Z(\beta)$ tells us the spectral curve:

$$y^2 = -\frac{1}{E} \cosh^2(2\pi\sqrt{E}).$$

- ▶ However the matrix ensemble is of a different type from before.
- ▶ One way to see that it must be different is to observe that the singularity near the endpoint of the spectrum is different from before: we have $\hat{\rho}(E) \sim 1/\sqrt{E}$, while for the type of ensemble considered before, the behavior near the endpoint is $\rho(E) \sim \sqrt{E}$.

In fact, because the dual quantum mechanical system is now supposed to be supersymmetric, we need to do random supersymmetric quantum mechanics, not just random quantum mechanics.

Supersymmetric quantum mechanics means that the Hilbert space \mathcal{H} is \mathbb{Z}_2 -graded by an operator

$$(-1)^F = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The Hamiltonian \mathcal{H} commutes with the \mathbb{Z}_2 -grading, but it is supposed to be the square of an operator Q that is odd, that is an operator that anticommutes with $(-1)^F$:

$$Q = \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix}, \quad \mathcal{H} = Q^2 = \begin{pmatrix} P^\dagger P & 0 \\ 0 & P P^\dagger \end{pmatrix}.$$

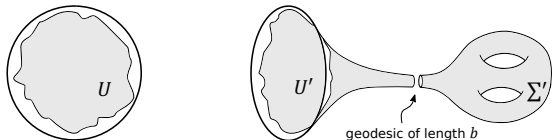
I've imposed that Q and $(-1)^F$ commute and that Q is self-adjoint. A random ensemble for Q is defined by the measure $\exp(-N \text{Tr} T(Q^2))$ for some function T . (This is one of the standard random matrix ensembles, constructed by Veerbarschot and Altland-Zirnbauer.)

Let μ be an eigenvalue of Q , and $f(\mu)d\mu$ the density of eigenvalues. One has $f(\mu) = f(-\mu)$. It is generic to have $f(0) \neq 0$. Since $\mathcal{H} = Q^2$, an eigenvalues of \mathcal{H} is $E = \mu^2$. Since $f(\mu)d\mu = f(E^{1/2})dE/2\sqrt{E}$, the density of eigenvalues behaves as $E^{-1/2}$ near $E = 0$. Thus this kind of ensemble is a good candidate for the present problem.

With this particular matrix ensemble, there is again a version of topological recursion. Applying this in the double-scaling limit with the spectral curve

$$y^2 = -\frac{1}{E} \cosh^2(2\pi\sqrt{E})$$

we get an expansion of $\langle \text{Tr}_{\mathcal{J}} \exp(-\beta\mathcal{H}) \rangle$ in powers of e^{-2S} . By the same logic as before, the terms in this expansion have an interpretation in terms of volumes of supermoduli spaces:



In this way, Stanford and I deduced a recursion relation that determines the volumes of the supermoduli spaces $\mathfrak{M}_{g, \vec{b}}$. Moreover, we were able to prove this formula by repeating what Mirzakhani had done for ordinary Riemann surfaces. By imitating Mirzakhani's derivation, we obtained a Mirzakhani-style recursion relation for the volumes of supermoduli spaces. And by imitating the arguments of Eynard and Orantin, we showed that the recursion relation that comes from the matrix ensemble agrees with the Mirzakhani-style recursion relation. Unfortunately, to explain all this would call for another occasion.

There are a few interesting refinements:

(1) The operator P can have a nonzero index (the difference in dimension between the odd and even subspaces of the Hilbert space \mathcal{J}). To compute volumes of $\mathfrak{M}_{g, \vec{b}}$, one takes the index to be zero. The same type of matrix ensemble, but with P assumed to have a nonzero index, appears to compute the volumes of moduli spaces of super Riemann surfaces with Ramond punctures, though we do not have a general proof of this.

(2) The spin structure of a Riemann surface or super Riemann surface can be even or odd. To compute volumes separately for each of the two cases, one has to also consider a somewhat different matrix ensemble, in which one still has $\mathcal{H} = Q^2$ (and the same spectral curve) but the Hilbert space is not assumed to be \mathbb{Z}_2 -graded.

