3d mirror symmetry and its discontents

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August 10, 2020

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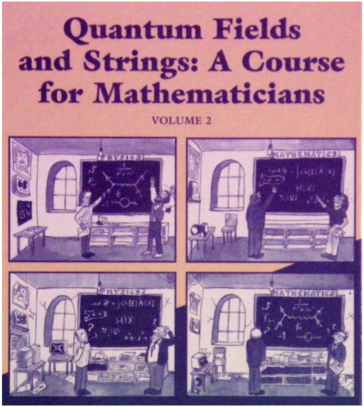
Quantum field theory is a subject which has caused a great deal of confusion for mathematicians (myself included) over the years:



I started my career thinking that I was studying geometric representation theory, and somehow ended up in the deep dark forest of quantum field theory by accident.

In particular, I was interested in understanding the representation theory of non-commutative algebras by understanding their relationship to their semi-classical limits.

Paradigmatic example: the universal enveloping algebra of a semi-simple Lie algebra $U(\mathfrak{g})$ has semi-classical limit $(\mathfrak{g}^*, \{\}_{KKS})$.

- This is the universal deformation of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, which is a **symplectic singularity**.
- Can also represent representation theory of $U(\mathfrak{g})$ in the geometry of the Springer **symplectic resolution** $T^*G/B \to \mathcal{N}$.

Several other pairs of algebras and symplectic singularities have related properties, and are much less thoroughly investigated.

These sort of objects appear naturally in a 3d theory with $\mathcal{N} = 4$ supersymmetry.

1. For each topological twist in this theory, the local operators (the Hilbert space of S^2) carry a Poisson bracket: this is a secondary product, and comes from integrating over the S^2 of possible directions two points can collide from.
2. Turning on an Ω-background (working equivariantly for a $U(1)$ rotating $\mathbb{R}^3$) gives a non-commutative deformation of this space of local operators.

All the examples I was interested in come from 3d field theories!

There's not just one choice of topological twist, though. Any 3d $\mathcal{N} = 4$ has two particularly nice twists we'll call Q_A and Q_B.

These are related to the factors in the isomorphism of R-symmetry groups

$$SU(2) \times SU(2) \cong \mathrm{Spin}(4).$$

These are the "same" as the A and B-twists of 2d $\mathcal{N} = (2, 2)$ theories under one way of placing it on the boundary of a 3-d theory, corresponding to the inclusion of R-symmetry:

$$\mathrm{Spin}(2) \times \mathrm{Spin}(2) \cong U(1) \times U(1) \hookrightarrow SU(2) \times SU(2) \cong \mathrm{Spin}(4).$$

Conjecture (2d Mirror symmetry)

Kähler manifolds come in pairs such that the associated sigma models are "equivalent" in a way that switches the A and B twists.

Homological mirror symmetry simply says that the categories of boundary conditions compatible with the two twists are interchanged.

There is much, much more to say about the $d = 2$ case, but I want to think instead about the $d = 3$. This suggests an intriguing possibility:

Conjecture (3d mirror symmetry)

$3d, \mathcal{N} = 4$ *theories come in pairs which are "equivalent" in a way that switches the A and B twists.*

There's a hierarchy of dualities here which are all compatible under boundary conditions, and which all have interesting mathematical manifestations:

4d mirror symmetry (S-duality, E-M duality)	geometric Langlands
3d mirror symmetry (S-duality)	S(ymplectic) duality
2d mirror symmetry (T-duality)	homological mirror symmetry (Kontsevich)

Question for mathematicians:

How does this duality of theories manifest? How are mathematically comprehensible objects on both sides related?

Lot of interesting examples and hints of relations, but the full picture isn't in focus yet. We're a lot closer than we were though, thanks to recent work by many authors (Braverman, Bullimore, Costello, Creuzig, Finkelberg, Dimofte, Gaiotto, Garner, Geracie, Hilburn, Nakajima,...)

The ring of local operators for these twisted theories carries:

1. a commutative algebra structure (from colliding operators)
2. a Poisson bracket compatible with this product (from integrating over the S^2 of choices of how to collide 2 points in $\mathbb{R}^3$).
3. a grading such that product has degree 0 and $\{-.-\}$ has degree $-2(= 1 - 3)$.

This is the same information as an affine algebraic variety $\mathfrak{M}_{A/B}/\mathbb{C}$ with Poisson bivector Π, and a $\mathbb{C}^*$ action with $t \cdot \Pi = t^{-2}\Pi$.

Physics seems to suggest that this should have an underlying hyperkähler metric.

We should get **two** of these for each theory, for the A and B twists. You'll often hear these called the Coulomb and Higgs branches.

The best understood theories are gauge theories, constructed from a compact connected Lie group G and a complex representation V.

For those who like physics terminology, we couple a hypermultiplet valued in V to a vectormultiplet for the group G.

1. The Higgs branch is the usual hyperkähler quotient of T^*V by G.
2. The Coulomb branch is much more mysterious. Can't be calculated "classically," and nature of "quantum corrections" is hard to describe precisely.

   However, Braverman-Finkelberg-Nakajima have given a precise description of this ring, which is a bit complicated and geometric, but can be represented algebraically.

These varieties swap roles under 3d mirror symmetry.

Examples:

(V,G)	Higgs	Coulomb
$(\mathbb{C}^{n+1}, U(1))$	$T^*\mathbb{P}^n$	$\mathbb{C}^2/\mathbb{Z}_{n+1}$
$(\mathbb{C}^{n+1}, U(1)^n)$	$\mathbb{C}^2/\mathbb{Z}_{n+1}$	$T^*\mathbb{P}^n$
$(\mathrm{Mat}_{n\times n} \times \mathbb{C}^n, GL_n)$	$\mathrm{Sym}^n(\mathbb{C}^2)$	$\mathrm{Sym}^n(\mathbb{C}^2)$
$(\mathrm{Mat}_{n\times n-1} \times \mathrm{Mat}_{n-1\times n-2} \times \cdots, GL_{n-1} \times GL_{n-2} \times \cdots)$	$\mathrm{Nil}_{n\times n}$	$\mathrm{Nil}_{n\times n}$
$(0, G)$	pt/G	Toda phase space
quiver gauge theory	Nakajima quiver variety	affine Grassmannian slice

We've already found some dual pairs: (1) and (2) are dual to each other, and (3) and (4) are self-dual.

Particularly interesting special cases:

1. G is commutative ($\mathfrak{M}_B$ is a hypertoric variety)
2. (V, G) corresponds to a linear or cyclic quiver for a dominant weight ($\mathfrak{M}_B$ is the corresponding Nakajima quiver variety)

In both of these cases, the dual of a gauge theory is again a gauge theory.

Theorem

In the cases above:

1. $\mathfrak{M}_A$ *is the Gale dual hypertoric variety,*
2. $\mathfrak{M}_A$ *is the rank-level dual quiver variety.*

I stumbled into this story from the perspective of non-commutative algebras and their representations. This requires turning on a (twisted) Ω-background to obtain noncommutative algebras U_A, U_B.

Theorem (W. (Soergel, BLPW, RSVV, ...))

There is a derived equivalence (Koszul duality) between pieces of the module categories of U_A and U_B called category $\mathcal{O}$.

The proof of this theorem depends on the BFN description of the Coulomb branch, and the geometry of the category $\mathcal{C}_A$ of line operators compatible with the A-twist.

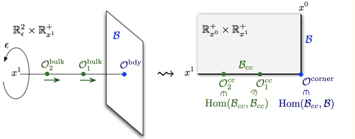
However, it's a bit unclear what it means from a physics perspective.

Quantization results in some well-known algebras with quantum integrable systems (i.e. commutative subalgebras):

$$
\begin{array}{rcl}
T^*\mathbb{P}^n & \to & \mathrm{Diff}(\mathbb{P}^n) \supset U(\mathfrak{t}) \\
\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell) & \to & \text{rational Cherednik algebra} \\
 & & \quad \supset \text{Dunkl-Opdam operators} \\
\mathrm{Nil}_{n\times n} & \to & U(\mathfrak{sl}_n) \\
 & & \quad \supset \text{Gelfand-Tsetlin subalgebra} \\
\text{affine Grassmannian slice} & \to & \text{shifted Yangian} \\
\text{Nakajima quiver variety} & \to & \text{quantum Hamiltonian reduction}
\end{array}
$$

What does this mean physically?

We can do reduction on a cigar transverse to our line: this reduces us to a 2 dimensional theory.



By choosing different limits of the parameters of the parameters, we can get the 2d sigma model to $\tilde{\mathfrak{M}}_A$ or $\tilde{\mathfrak{M}}_B$.

Proposal (Bullimore-Dimofte-Gaiotto-Hilburn)

- These category $\mathcal{O}$'s can be interpreted as boundaries to a particular twist of our 2d theory (an A-model in the presence of real FI and mass parameters).
- The Koszul duality is the fact that this category remains unchanged as we deform one twist to another.

The most profitable approach of recent years has been to consider categories $\mathcal{C}_A, \mathcal{C}_B$ of line operators compatible with each twist. These both have interesting geometric descriptions in the gauge theory case.

Framework (DGGH)

$$\mathcal{C}_A \cong \mathrm{QCoh}(\mathrm{Map}(S^1, V/G)_{\mathrm{dR}}) \cong D\big(V((t))/G((t))\big)$$
$$\mathcal{C}_B \cong \mathrm{QCoh}(\mathrm{Map}(S^1_{\mathrm{dR}}, V/G)) \twoheadrightarrow \mathrm{QCoh}(\mathrm{Map}(T^*(V/G))$$

where X_{dR} is the deRham stack of X (think of the deRham complex as functions on a dg-manifold).

- The category $\mathrm{QCoh}(X_{\mathrm{dR}})$ is the D-modules $D(X)$.
- $\mathrm{Map}(S^1_{\mathrm{dR}}, V/G)$ is is space of G-connections with a flat section of the associated V bundle on S^1.

The local operators appear here as endomorphisms of the trivial line operators $\mathbb{1}_A$, $\mathbb{1}_B$. This is actually how BFN construct the local operators of the A-twist:

Theorem (Braverman-Finkelberg-Nakajima)

$$\mathrm{Ext}^\bullet(\mathbb{1}_A, \mathbb{1}_A) = H_*^{BM}\left(\frac{V[[t]]}{G[[t]]} \times_{\frac{V((t))}{G((t))}} \frac{V[[t]]}{G[[t]]}\right).$$

However, there is more information in these categories than just the trivial line; the stacks appearing before are not affine.

So what else is in this category?

- Compatible with the B-twist, the natural objects are "Wilson operators": the coherent sheaves associated to G-representations.
- Compatible with the A-twist, the natural objects are "vortex operators": these correspond to pushforward D-modules

$$U/H \to V[[t]]/G[[t]]$$

with $H \subset G[[t]]$ and $U \subset V[[t]]$; in QFT terms, H is breaking of the gauge symmetry along the line, and U the singularities allowed in the fields.

Question

In cases where two gauge theories are dual, how do these line operators match?

In the case of G abelian, there's a nice match between these operators.

When G is non-abelian, the story is more complicated.

Theorem (Assel-Gomis)

In a cyclic quiver gauge theory, the Wilson operators have a vortex description as more complicated pushforward D-modules, but usually not just U/H.

On the other hand, there are coherent sheaves with a simple vortex description which are provably not Wilson operators (the Procesi bundle on a Hilbert scheme).

Thus, thinking of a Higgs branch as a Coulomb branch as well gives us access to a new collection of coherent sheaves which were hard to find before.

One of the mathematically recognizable objects we find are resolutions of singularities. These are relevant for physics, because they arise when we turn on real mass or FI parameters.

1. For $\mathfrak{M}_B$, we can try to construct resolutions based on GIT theory for the action of G.
2. For $\mathfrak{M}_A$, we can extend our gauge group G by adding a flavor torus F, to obtain a larger group $\tilde{G}$. The Coulomb branch of (V, G) is a quotient of that for $(V, \tilde{G})$ by ${}^L F$, and we can apply GIT theory here.

Coherent sheaves on these resolutions are quotients of the full category of line operators:

1. GIT quotients $\tilde{\mathfrak{M}}_B$ are constructed by removing unstable locus from $T^*(V/G)$; thus $\mathrm{QCoh}(\tilde{\mathfrak{M}}_B)$ is a quotient of $\mathcal{C}_B$ by the subcategory of objects supported on the unstable locus.

The variety $\tilde{\mathfrak{M}}_A$ depends on a **flavor** $\varphi\colon U(1) \to F \subset \mathrm{Aut}_G(V)$. Multiplication by $\varphi(t)$ gives an automorphism of $V((t))/G((t))$, so we can apply this to any D-module on this category.

Theorem (BFN)

The category $\mathrm{QCoh}(\tilde{\mathfrak{M}}_A)$ *is the quotient of* $\mathcal{C}_A$ *by the operators* X *such that*

$$\mathrm{Ext}^\bullet(\varphi(t^{-k})\mathbb{1}, X) = 0 \quad \text{for} \quad k \gg 0.$$

We can always reconstruct the varieties $\tilde{\mathfrak{M}}_A/\tilde{\mathfrak{M}}_B$ from these categories of sheaves, by looking at the powers of a line bundle to get a projective coordinate ring.

Conjecture (Kawamata: "K-equivalence = D-equivalence")

Two crepant (in particular, symplectic) resolutions $\tilde{X}_i \to X$ of the same variety have $D^b(\mathrm{QCoh}(\tilde{X}_1)) \cong D^b(\mathrm{QCoh}(\tilde{X}_2))$.

Theorem (Kaledin)

Two symplectic resolutions of the same variety $\tilde{X}_i \to X$ (subject to some mild hypotheses) have $D^b(\mathrm{QCoh}(\tilde{X}_1)) \cong D^b(\mathrm{QCoh}(\tilde{X}_2))$.

Kaledin's proof proceeds by constructing (via some characteristic p dark magic) a non-commutative algebra A such that

$$D^b(\mathrm{QCoh}(\tilde{X}_1)) \cong D^b(A\text{-mod}) \cong D^b(\mathrm{QCoh}(\tilde{X}_2)).$$

The algebra A is a **noncommutative crepant resolution of singularities.** The image of A under these equivalences is a vector bundle called a **tilting generator**.

Let $C_p^\lambda \subset U(1)$ be the p-torsion points, acting on $V((t))$ by loop rotation twisted by a flavor λ:

$$s \cdot v(t) = \lambda(s) \cdot v(st) \;\text{ for }\; s \in e^{\frac{2\pi i}{p}\mathbb{Z}}.$$

Note that the fixed points of C_p^λ on $V((t))$ are exactly $t^{-\lambda}V((t^p))$, and the obvious isomorphism $\mathrm{Fr}\colon V((t)) \cong t^{-\lambda}V((t^p))$ intertwines the action of $G((t))$ with that of $G((t^p))$.

We can define a vortex line operator $\mathcal{M}_\lambda$ by the pushforward by the map

$$X_\lambda = \left(\frac{G((t)) \times V[[t]]}{G[[t]]}\right)^{C_p} \to t^{-\lambda}V((t^p)) \cong V((t)).$$

Every component of X_λ is of the form $(G((t^p)) \times U)/P$ for

- P a parahoric in $G((t^p))$ and
- $U \subset t^{-\lambda}V((t^p))$ a P-invariant subspace.

Let me not torture you with the actual combinatorics of describing these, but this perspective shows that $\mathcal{M}_\lambda$ has a combinatorial construction.

Thus, in the language of Dimofte-Garner-Geracie-Hilburn, this is the vortex line operator attached to the Lagrangian conormal to $\mathrm{Fr}^{-1} U \subset V((t))$ with the action of the group $\mathrm{Fr}^{-1} P \subset G((t))$.

Note that we chose λ (the quantization parameter) and φ (the choice of symplectic resolution) independently.

Theorem

The algebra $A_\lambda = \mathrm{End}_{\mathcal{C}_A}(\mathcal{M}_\lambda)$ *defines a non-commutative crepant resolution of singularities for* $\mathfrak{M}_A$ *(independent of* φ*).*

Kaledin's equivalence is given by projecting the subcategory $\langle \mathcal{M}_\lambda \rangle$ to $D^b(\mathrm{QCoh}(\tilde{\mathfrak{M}}_A))$.

Applications:

- Explicit presentations of endomorphisms: for quiver varieties, get “KLR algebras on cylinders.”
- Explicit presentations of wall-crossing functors and construction of a Schober, can verify Bezrukavnikov-Okounkov conjecture “by hand.”
- Explicit stability conditions coming from assigning slopes to simple A-modules.
- Same line operators define tilting generators on K-theoretic Coulomb branch where Kaledin’s trick doesn’t work to build a global tilting generator.
- This was key in work of Gammage-McBreen-W. on mirror symmetry for multiplicative hypertoric varieties.

Thanks for listening.