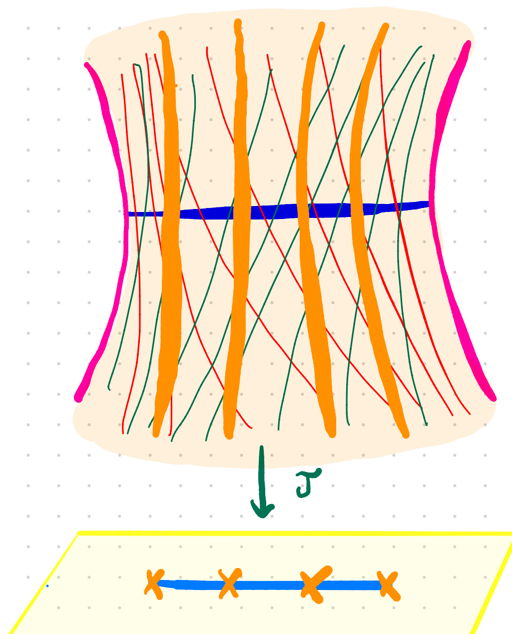
Branes in symplectic groupoids



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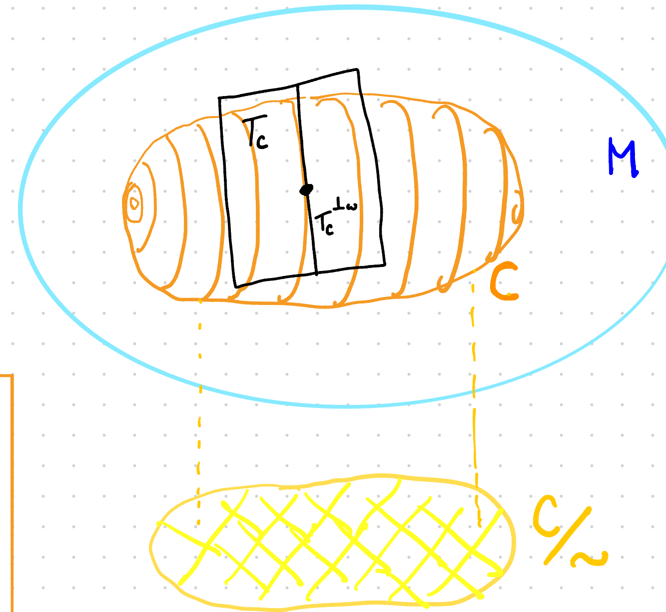
I Coisotropic A-branes

(M, ω) symplectic

$((C, (\mathcal{U}, \nabla)))$ Brane in M



- C coisotropic i.e. $T_C^{\perp\omega} \subset T_C$
- $\mathcal{U} \to C$ Hermitian line bundle

  ∇ unitary connection

such that ① F^∇ descends to $T_C / T_C^{\perp\omega}$

② $(\omega^{-1} F^\nabla)^2 = -1$ on $T_C / T_C^{\perp\omega}$

$\Bigg\}$ $F^\nabla + i\omega$ is holomorphic symplectic on $C/\sim$

$\left(\dim_{\mathbb{R}} (C/\sim) = 4k \right)$

Examples of A-branes

1. $\mathcal{L}$ Lagrangian

$T_{\mathcal{L}}^{\perp\omega} = T_{\mathcal{L}}$

$F^{\nabla} = 0$, i.e.

$(\mathcal{U}, \nabla)$ flat bundle on $\mathcal{L}$

2. $\mathcal{Z}$ Spacefilling

$T_{\mathcal{Z}}^{\perp\omega} \simeq \{0\}$

$I = \omega^{-1} F^{\nabla}$ complex structure on M

$\Omega = F^{\nabla} + i\omega$ holomorphic symplectic form

Kapustin - Orlov :

must be coisotropic

multiplication by i

$$\begin{tikzcd} ? \arrow[r, hook] \arrow[d, leftrightarrow] & \left(T^4 = T^2 \times T^2,\ \omega = -dx_1 \wedge dy_1 + dx_2 \wedge dy_2\right) \arrow[d, leftrightarrow, "\text{Mirror}"] \\ \Gamma_{m_i} \arrow[r, hook, "\text{brane}"'] & E \times E \end{tikzcd}$$

$E = \mathbb{C}/\mathbb{Z}[i]$

$$\begin{array}{c} (U, \nabla) \\ \downarrow \\ T^4 \end{array}$$

$F^\nabla = dx_1 \wedge dy_2 + dx_2 \wedge dy_1$ defines space-filling brane on T^4

$$F^\nabla + i\omega = -i\,(dx_1 + i\,dx_2) \wedge (dy_1 + i\,dy_2)$$

complex symplectic str. on T^4.

Problem: $\mathrm{Fuk}(T^4, \omega)$ consists of **Lagrangians** only

Solution: (L, ∇) lies in **completion** $\mathrm{Tw}^{\pi}\, \mathrm{Fuk}(T^4)$

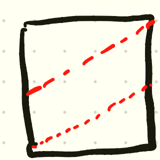
(L, ∇) defines transform $\mathrm{Fuk}(T^2) \longrightarrow \mathrm{Fuk}(T^2)$

2 $(\pi_1^{-1}(\mathcal{L}_1),\ \pi_1^{*}\mathcal{U}_1 \otimes \mathcal{U}) \xmapsto{(\pi_2)_*}$ 3 flat sections over Bohr-Sommerfeld fibers

$$\begin{tikzcd} & (T^4, \omega) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (T^2, dx_1 \wedge dy_1) & & (T^2, dx_2 \wedge dy_2) \end{tikzcd}$$

1 $(\mathcal{L}_1, \mathcal{U}_1) \mapsto (\pi_1^{-1}(\mathcal{L}_1),\ \pi_1^{*}\mathcal{U}_1 \otimes \mathcal{U})$

Lagrangian brane



Thm: (-, J. Lackman) This defines an endofunctor of $Fuk(T^2)$ <u>not</u> in the image of fully-faithful (Abouzaid - Smith)

$$Fuk\left((T^2)^- \times T^2\right) \longrightarrow End\left(Fuk(T^2)\right)$$

but which defines an object

Idempotent completion

$$(\mathcal{U}, \nabla) \in \underbrace{Tw}_{\text{twisted complexes: sums, shifts, cones.}}{}^{\pi} Fuk(T^4)$$

which is mirror to Γ_{m_i} as desired.

II Application to Kähler geometry

Symplectic manifold: $M = T^*N, \quad \omega_{can} = d\theta_{can}$

Lagrangian: $\mathcal{L} = (N, (\mathbb{C}_M, d)) \hookrightarrow M$ zero section.

Spacefilling: Fix I complex structure on N

$$Z_0 = (M, (\underbrace{\mathcal{U}_0}_{\mathbb{C}_M}, \underbrace{\nabla_0}_{d + i I^*\theta_{can}})) \qquad F^{\nabla_0} = d I^*\theta_{can} = I^*\omega_{can}$$

$(M, \quad \Omega_0 = F^{\nabla_0} + i\omega_{can}) \cong$ holomorphic symplectic cotangent bundle $\cong T^*_{1,0}N$

Kähler prequantization

Let ω Kähler on (N, I) and $(\mathcal{U}, \nabla)$ a prequantization $F^{\nabla} = \omega$

$$Z_1 = \left(\mathcal{U}_1 = \mathcal{U}_0 \otimes \pi^* \mathcal{U} \,,\; \nabla_1 = \nabla_0 \otimes 1 + 1 \otimes \pi^* \nabla \right)$$

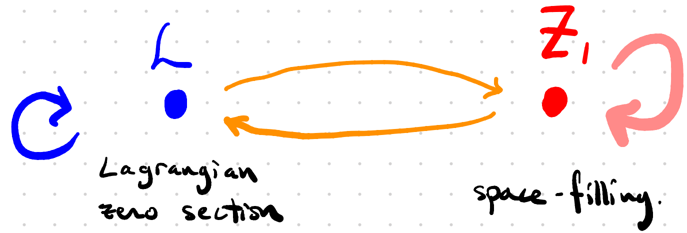
holomorphic symplectic str: $\left(I_1 = I + \omega_{can}^{-1} \pi^* \omega \,,\; \Omega_1 = I^* \omega_{can} + \pi^* \omega + i\, \omega_{can} \right)$

Z_1 = twisted cotangent bundle

Gukov-Witten approach to Geometric Quantization

Pair of branes

realize symplectic as a submfld $\mathcal{L}$ of complexification



$$\underset{\text{hol. symp}}{\Omega}\Big|_{\mathcal{L}} = \omega_{real}$$

$$\mathrm{Hom}(\mathcal{L},\mathcal{L}) \circlearrowright \mathrm{Hom}(\mathcal{L},\mathcal{Z}_1) \circlearrowleft \mathrm{Hom}(\mathcal{Z}_1,\mathcal{Z}_1)$$

Ex.:

$M = T^*S^2$

$\mathcal{L}$ = zero section

$\mathcal{Z}_1$ = twisted cotangent

$$\mathbb{C} \circlearrowright H^0(\mathbb{P}^1, \mathcal{O}(1)) \circlearrowleft U\mathfrak{sl}_2\mathbb{C} \big/ (J^2 - 1/4)$$

Geometric Quantization of (S^2, ω)

Holomorphic symplectic quantization

Summary idea:

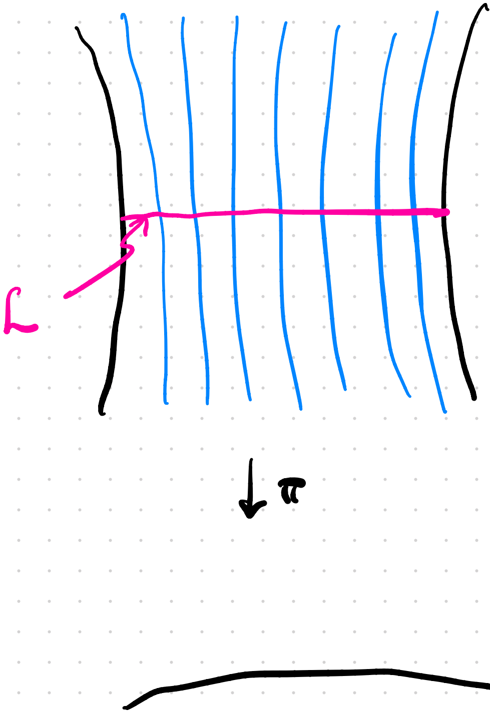
- Kähler structure on N encoded by a pair $\mathcal{L}, \mathcal{Z}$ of A-branes in (T^*N, ω_{can})
- Geometric quantiz. of Kähler structure is $\mathrm{Hom}(\mathcal{L}, \mathcal{Z})$.

How to define $\mathrm{Hom}(\mathcal{L}, Z)$ Lagrangian $\rightarrow$ Space filling

Proposed Method :

1. choose a holomorphic Lag. polarization of (Z, Ω) transverse to $\mathcal{L}$
2. lift $(\mathcal{U}, \nabla)$ to a holomorphic prequantization
3. take holomorphic flat sections along Bohr-Sommerfeld leaves intersecting $\mathcal{L}$

   $\Rightarrow$ sheaf on Polarization param. space.
4. take global holom. sections



N

(Z_1, Ω) twisted cotangent bundle

fibers of $\pi : Z_1 \longrightarrow N$ are holom.

π - flat sections of $(U_1, \nabla_1) \cong \mathcal{O}(U) \to N$

Lagrangian $\mathcal{L} \Rightarrow \mathcal{O} \to N$

$$\mathrm{Hom}(\mathcal{L}, Z) = H^0(N, U)$$

assembling branes to form an algebra:

$$Z_i = \left(T^*N,\ \left(\mathcal{U}_0 \otimes \pi^* \mathcal{U}^{\otimes i},\ \nabla_0 \otimes 1 + 1 \otimes \nabla^{\otimes i} \right) \right)$$

$$\begin{tikzcd}
 & T^*M \times_M T^*M \arrow[dl] \arrow[dr, "+"] & \\
T^*M \times T^*M & & T^*M
\end{tikzcd}$$

$$Z_i * Z_j = Z_{i+j} \qquad\qquad \mathcal{L} * \mathcal{L} = \mathcal{L}$$

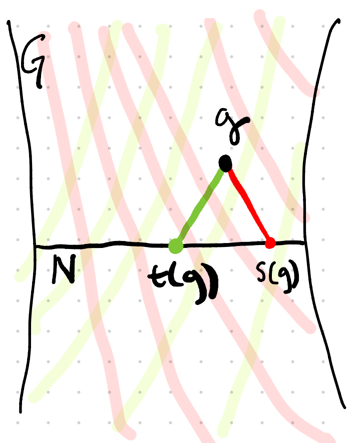
$$\boxed{\mathrm{Hom}(\mathcal{L}, Z_i) \otimes \mathrm{Hom}(\mathcal{L}, Z_j) \to \mathrm{Hom}(\mathcal{L}, Z_{i+j})}$$

making $A = \bigoplus_{k \geq 0} H^0(N, \mathcal{U}^{\otimes k})$ into $\mathbb{Z}$-graded algebra

Groupoids

(T^*N, ω_{can}) is an example of a symplectic groupoid

Lie groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} N$ with a multiplicative symp. form ω



$$\begin{tikzcd}
 & G \times_M G \arrow[dl] \arrow[dr, "m"] & \\
G \times G & & G
\end{tikzcd}$$

$$\pi_1^*\omega + \pi_2^*\omega = m^*\omega$$

$$s_*\omega^{-1} = -\sigma$$

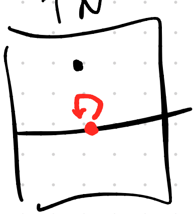
$$t_*\omega^{-1} = \sigma$$

(N, σ) Poisson

$G = T^*N$ is a simple case of a bundle of (abelian) groups, $s = t = \pi_N$

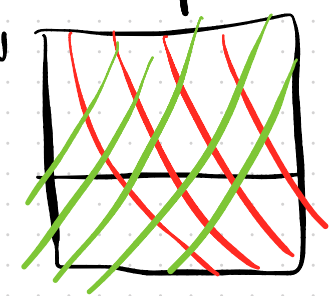
Deforming T^*N using an R-matrix (Weinstein, Xu)

- N toric Kähler $\Rightarrow$ $\mathbb{T}_{\mathbb{C}} = (\mathbb{C}^*)^n$ $\qquad \mathbb{T}_{\mathbb{C}} \times N \xrightarrow{\rho} N$ action
- $R \in \Lambda^2 \mathfrak{t}_{\mathbb{C}}$ defines $\rho_* R = \sigma$ invariant Poisson



- $\mathbb{T}_{\mathbb{C}}$ action lifts to Hamiltonian action on $(T^*N, \Omega_0 = I^*\omega_{can} + i\omega_{can})$

$$J : T^*N \longrightarrow \mathfrak{t}^*_{\mathbb{C}} \qquad \text{moment map.}$$



Weinstein Xu

- New source: $\quad s(g) = e^{-\frac{1}{2} R J(g))} \pi_N(g)$

  New target: $\quad t(g) = e^{\frac{1}{2} R(J(g))} \pi_N(g)$

  multiplication: $\quad m(g,h) = e^{-\frac{1}{2} R J(h)} \cdot g + e^{\frac{1}{2} R J(g)} \cdot h$

- line bundle $(\mathcal{U}_0, \nabla_0)$ is multiplicative using 1-cocycle

$$\lambda_R(g,h) = e^{i\pi R(J(g), J(h))}$$

- Same works for all $\{\mathcal{Z}_i\}_{i \in \mathbb{Z}}$.

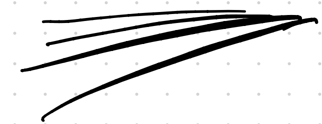
To compute $\mathbb{Z}$-graded algebra $\quad (\mathbb{C}P^2, \omega_{FS})$

- holom. Lagrangian polarization

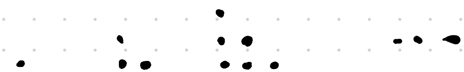
for Z_k provided by complex moment map J

$(\mathbb{C}P^2, \sigma)$

non-toric



BS fibres intersecting $\mathcal{L}$



$$\left\{ Z_0 \quad Z_1 \quad Z_2, \dots \right.$$

$$\mathrm{Hom}(\mathcal{L}, Z_k) \cong H^0(N, U^{\otimes k})$$

- Convolution

$$f_{k,\lambda} * f_{\ell,\eta} = e^{i\pi C(\lambda,\eta)}\left(\left(e^{-\frac{1}{2}C(\eta)} * f_{\ell,\eta}\right) \otimes \left(e^{\frac{1}{2}C(\lambda)} * f_{k,\lambda}\right)\right)$$

weights

$$= e^{-i\pi C(\lambda,\eta)} f_{k,\lambda} \otimes f_{\ell,\eta}$$

$\Rightarrow$ Noncomm. algebraic toric variety.

Future Steps:

Any Gen. cx mfld $(M, \mathbb{J})$ has underlying Poisson str.

and so symplectic groupoid $(G, \omega) \rightrightarrows (M, \mathbb{J})$

and the anchor is a g. holo map.

$$(G, \omega) \xrightarrow{(s,t)} (\bar{M} \times M, \bar{\mathbb{J}} \times \mathbb{J})$$

Given a pair of branes C_1, C_2 in $(M, \mathbb{J})$

define a Coisotropic A-brane $B(C_1, C_2) = (s,t)^{-1}(\bar{C_1} \times C_2)$.

$$\mathrm{Hom}(C_1, C_2) := \mathrm{Hom}_{(G,\omega)}(\mathrm{Id}_M, B(C_1, C_2))$$