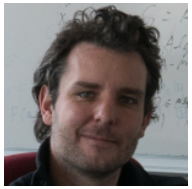
Quantum geometric Langlands as a fully extended TFT

David Jordan (University of Edinburgh) – and this lot:

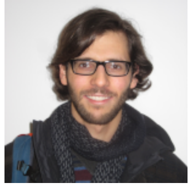

Appel


Balagovic


Ben-Zvi


Brochier


Cooke


Ganev


Gunningham


Le

? ?
Ma


Pym


Safronov


Schrader


Shapiro


Snyder


Vazirani


Weelinck

Outline

I) Character varieties, character stacks and their quantizations

II) Recent approach via factorization homology & cobordism hypothesis

III) Applications in quantum topology and representation theory

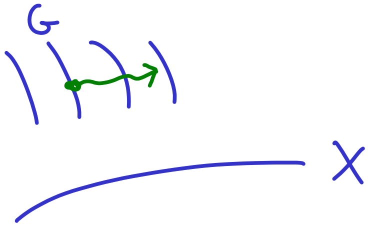
Broader goals:

- Underscore utility and convenience of stacks and higher categories
- Introduce stratified factorization homology, and its zoo of defects
- Emphasize 4D KW viewpoint vs 3D CS viewpoint on skeins
- Formulate S–duality/Langlands duality in skein theory

Character stacks

Fix G reductive (e.g. GL(N), SL(N)).

Def: A G-local system on a space X is a principal G-bundle E over X, together with a flat connection.



Our primary object of study is the character stack:

$$\mathrm{Ch}(X) = \left\{ \begin{matrix} G\text{-local systems} \\ \text{on } X \end{matrix} \right\} \Big/ _{\text{isom}} \simeq \left\{ \rho : \pi_1(X) \longrightarrow G \right\} \Big/ _{\substack{\text{conj by} \\ G}}$$

Via gauge-fixing, we can present it as a quotient stack:

$$\mathrm{Ch}^{fr}(X) = \left\{ \begin{matrix} G\text{-local system } E \text{ on } X \\ \text{w/ trivialization } E_x \cong G \\ \text{at } x \in X \end{matrix} \right\} \Big/ _{\text{isom}} \cong \left\{ \rho : \pi_1(X) \longrightarrow G \right\}$$

$$\mathrm{Ch}(X) = \mathrm{Ch}^{fr}(X) / G$$

Ch(X) is (2−dim(X))−shifted symplectic (Pantev−Toen−Vaquie−Vezzosi).

- When dim(X)=2, given by the Atiyah−Bott/Goldman/Fock−Rosly bracket.
- When dim(X)=3, given by the Batalin−Vilkovisky structure on the critical locus of the Chern−Simons action functional.

Appearances of quantum character stacks in physics

Kapustin–Witten twist of 4D $N=4$ theory

- has a dependence on a parameter ψ.
- On surfaces, it should attach a category $\mathrm{Fuk}(T^*\mathrm{Bun}_G(\Sigma))$, which is mathematically ill-defined.
- Varying ψ <=> q–deformation.

$$\begin{tikzcd}
 & \mathrm{Fuk}(T^*\mathrm{Bun}_G(\Sigma)) \arrow[dl, rightsquigarrow, swap, "\text{deRham (Beilinson-Drinfeld)}"] \arrow[dr, rightsquigarrow, "\text{Betti (Ben-Zvi-Nadler}"] & \\
\mathcal{D}\text{-mod}(\mathrm{Bun}_G(\Sigma)) \arrow[d, rightsquigarrow] & & \mathrm{QC}((\mathrm{Ch}(\Sigma)) \arrow[d, rightsquigarrow] \\
\mathcal{D}^{\psi}\text{-mod}(\mathrm{Bun}_G(\Sigma)) & & \mathrm{DQ}(\mathrm{Ch}(\Sigma))
\end{tikzcd}$$

4D $N=2$ "Theories of class S", from a Riemann surface.

- Compactifying on cirlce gives "Sicilian" 3D $N=4$ theory.
- Its Coulomb branch is the Hitchin system (Gaiotto–Moore–Neitzke), whose hyper–Kahler rotation is $\mathrm{Ch}(\Sigma)$.
- Omega–background <=> q–deformation.

Character stacks vs. character varieties

We study a quotient stack through its category of sheaves:

- $\mathcal{O}(X)$ gets an action of the group G
- $QC(X/G)$ = G-equivariant $\mathcal{O}(X)$-modules
- $QC(\overline{X/G})$ = $\mathcal{O}(X)^G$-modules
- We have a functor $\Gamma: QC(X/G) \to QC(\overline{X/G})$, but is destructive.

Who cares about stacks?

- Character stacks glue, their quotient varieties don't!
- Example: Compute $Ch(S^1)$ and hence $Z(S^1) = QC(Ch(S^1))$.

  $QC(pt/G)$ = G-equivariant vector spaces = $Rep(G)$.

  $Ch^{fr}([0,1]) = pt$. $Ch([0,1]) = pt/G$. $Ch([0,1) = pt$.

$$Z(\bigcirc) = Z(\subset) \boxtimes_{Z(=)} Z(\supset) = RepG \boxtimes_{Rep_G^2} RepG \doteq G\text{-eq. } \mathcal{O}(G)\text{-modules}$$

$$\simeq QC\left(\frac{G}{G}\right) \text{ as expected.}$$

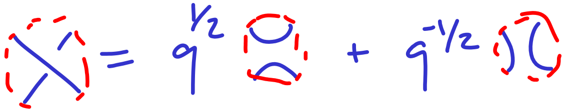
$Ch^{fr}(S^1) = G$. $Ch(S^1) = \frac{G}{G}$. $\overline{Ch(S^1)} = \mathrm{Spec}(\mathcal{O}(G)^G) = H/W$

Quantizations of character varieties of surfaces

Historically, how it was done:

- Skein algebras (Przytycki, Turaev, many others...) for $G=SL(2)$ mostly:

  $Sk(M^3) = \mathbb{C}\cdot\{\text{links}\}/_{\text{skein rel}^{\text{ns}}}$ $Sk(\Sigma\times I)$ is an algebra via stacking in I direction.

  Thm (Bullock–Frohman–Kania–Bartoszynska): At $q=1$, $Sk(S) = \mathcal{O}(Ch(S))$, with AB/G Poisson bracket.

  g: genus
  r: # punctures.

- Moduli algebras (Alekseev–Grosse–Schomerus) – any G, $\Sigma_{g,r}$ punctured

  $Ch^{fr}(\Sigma_{g,r}) \cong G^{2g+r-1}$, if $r \geq 1$

  $\mathcal{O}(Ch^{fr}(\Sigma_{g,r})) \cong \mathcal{O}(G)^{\otimes 2g+r-1}$

  $\xrightarrow{AGS} \mathcal{A}(\Sigma_{g,r}) := \underbrace{\mathcal{O}_q(G)\,\tilde{\otimes}\cdots\tilde{\otimes}\,\mathcal{O}_q(G)}_{2g+r-1}$

  cross rel$^{\text{ns}}$ use quantum R matrices, after Fock-Rosly

- Quantum cluster algebras (Fock–Goncharov):
  - Consider decorated character variety (B–reductions near boundaries)
  - Triangulations/Ptolemy flips give toric log–canonical charts/mutations.

  $\underline{\Delta}$ of $\Sigma \rightsquigarrow \mathcal{U}(\underline{\Delta}) \subseteq Ch^{dec}(\Sigma)$, $\mathcal{U}(\underline{\Delta}) \cong (\mathbb{C}^\times)^r$, $\{z_i, z_j\} = a_{ij} z_i \cdot z_j$

  "log canonical"

  $\xrightarrow[\text{quantization}]{FG} X_i X_j = q^{\frac{a_{ij}}{2}} X_j X_i.$

Topological factorization homology

(Ben-Zvi-Francis-Nadler) Ch(X) is a (2-shifted) sigma model, Ch(X) = Maps(X, BG), hence it satisfies <u>excision</u>:

$$Ch(S_1 \underset{P\times I}{\cup} S_2) = Ch(S_1) \underset{Ch(P\times I)}{\times} Ch(S_2)$$

$$\Rightarrow Z(S_1 \underset{P\times I}{\cup} S_2) \simeq Z(S_1) \underset{Z(P\times I)}{\otimes} Z(S_2)$$

(Lurie, Ayala-Francis) Factorization homology is the universal assignment,

$$(M, A) \longmapsto Z_A(M) = \int_M A \longleftarrow \text{category}$$

n-mfld

E_n-algebra

(or vector space, or chain complex, or dg-cat...)

satisfying excision.

The En algebra are the local observables, $Z_A(M)$ is the global observables.
A symmetric monoidal category like Rep(G) is an En-algebra for any n.

Quantum character stacks (BZBJ1, BZBJ2)

In the case n=2, E2-algebras are braided tensor categories.

Defn: The quantum character stack is:

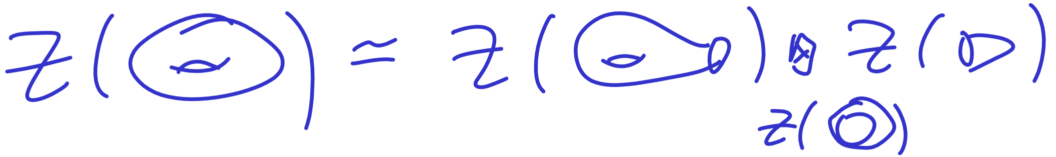
$$Z(\Sigma) := Z_{\mathrm{Rep}_q}(\Sigma) = \int_{\Sigma} \mathrm{Rep}_q G$$

Computations:

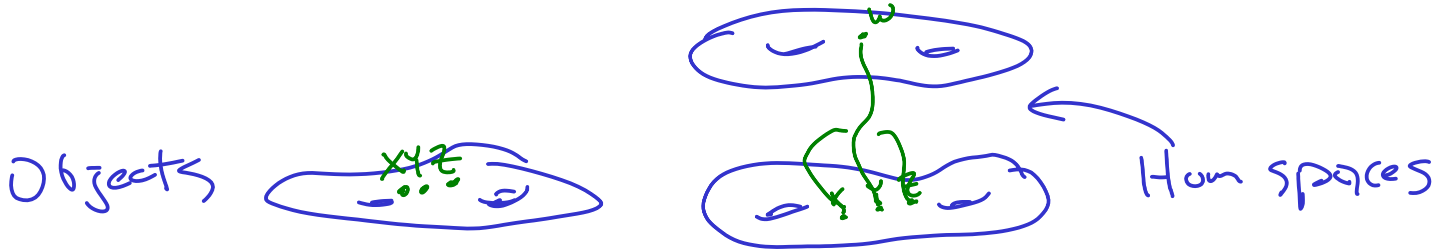
1) (BZBJ1) For punctured surfaces with a gate, we recover AGS algebras:

$$Z(\Sigma) \simeq U_q(\mathfrak{g})\text{-equivariant } A(\Sigma)\text{-modules}$$

2) (BZBJ2, BJ, GJV) For G=GL(N), on the closed torus we recover the q=t specialization of the sph. DAHA, via quantum Hamiltonian reduction.

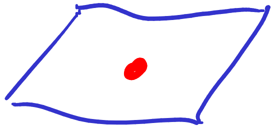
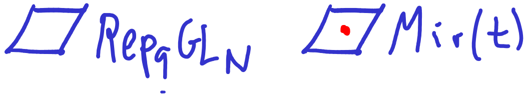
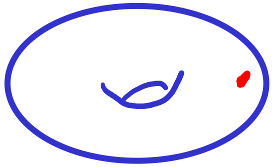
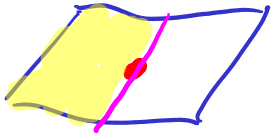
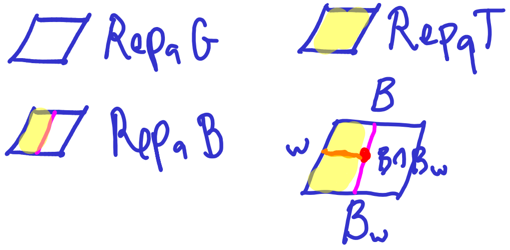
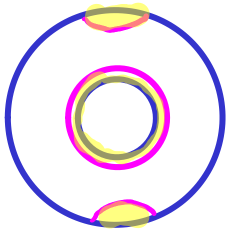
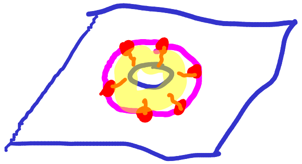
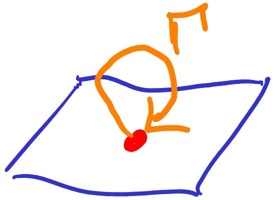
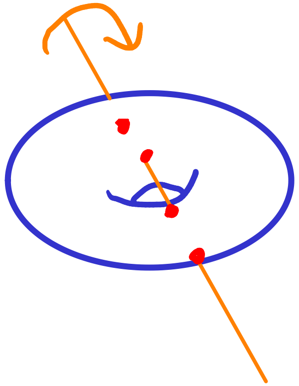


3) (Cooke) For any surface, we recover Walker's "skein category":



Stratified factorization homology (Ayala–Francis–Tanaka)

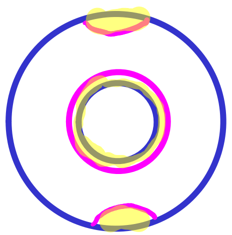
- This allows manifold with defects, retaining functoriality and excision.
- Locality replaced by richer algebraic structures:

Local data	Example coefficients	Example output
 Braided module categories	Conjugacy classes, G–Hamiltonian spaces  	 ⟼ Type A DAHA for $q \neq t$. (J, BJ, Varagnolo–Vasserot)
 Central tensor & bimodule cat's	Parabolic induction, flag varieties, Stokes phenomena  	 ⟼ Fock–Goncharov style quantization (JLSS, next slide)  ⟼ Boalch-style quantization (AJP)
 Quantum symmetric pairs & K-matrices	Symmetric spaces, orbifold character stacks  $\theta : G \to G$ $\theta^2 = \mathrm{id}$	 ⟼ Type C DAHA (JM, Weelinck, JW)

More details on Fock–Goncharov cluster quantization

We consider the decorated character stack, the moduli space of:

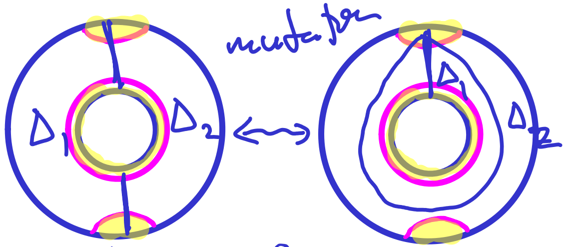
- G–local systems in white G–region
- T–local system in yellow T–region
- B–reduction of the resulting G×T–local system along red defects.



Defn (JLSS): The quantum decorated character stack is the factorization homology with coefficients in the triple (Repq(G), Repq(B), Repq(T)).

Thm(JLSS): Let G=SL(2). Each triangulation Δ of S determines a subcategory $Z(\Delta)$ of $Z(S)$, an equivalence $Z(\Delta)=A(\Delta)$–mod, and and isomorphism, $A(\Delta) \cong \mathbb{C}_q[X1^{\pm1}, \dots, Xr^{\pm1}]$ (a quantum torus).

triangulation $\underline{\Delta}$ ⟷ excision prescription ⟶ computation of A



Recovers, extends quantum cluster A–, X–, and P– varieties of Fock–Goncharov and Goncharov–Shen.

This page was in case of questions...

Skein theory is intrinsically 4-dimensional => KW, not CS

Most approaches to skein theory focus on 3-dimensional aspects.

But, braided tensor categories naturally form a 4-category (Haugseng, Johnson-Freyd-Scheimbauer)

- Objects are braided tensor categories A, B, ...
- Hom(A,B) are central tensor categories C,D, ...
- Hom(C,D) are central bimodule categories M,N, ...
- Hom(M,N) are bimodule functors F,G, ...
- Hom(F,G) are natural transformations.

Results: Applying cobordism hypothesis n-dualizability => n-dim TFT's.

1) (Calaque-Scheimbauer) All braided tensor categories are 2-dualizable.
2) (BJS) Rigid braided tensor categories are 3-dualizable.
3) (BJS, after Freed-Teleman, Walker) Braided fusion categories are 4-dualizable.
4) (BJSS) (non-semisimple) modular tensor categories are invertible.

- MTC's give Crane-Yetter theories, hence 3D WRT/CS theories.

Applications to skein theory

Theorem (GuJS, conj. by Witten) Suppose q is generic, and M is an arbitrary closed, oriented 3-manifold. Then the skein module of M has finite dimension.

Theorem (GaJS, asked by Bonahon-Wong, see Frohman-Kania-Bartoszynska-Le) Suppose instead q is a root of unity. Then the skein algebra of Σ is Azumaya (i.e. locally Morita-trivial) over the smooth locus of the character variety.

Main ideas:

- First lift the problem from quantum character variety to quantum character stack (c.f. Walker's skein category TFT).
- Given a Heegard splitting of $M = M_1 \underset{\Sigma}{\bigcup} M_2$, write $Z(M) = Z(M_1) \circ Z(M_2)$, where $Z(M_1): Vect \rightarrow Z(\Sigma)$, $Z(M_2): Z(\Sigma) \rightarrow Vect$.
- $Z(\Sigma)$ is the quantum Hamiltonian reduction of $Z(\Sigma_0)$. We express $Z(M_1) \circ Z(M_2)$ instead as $Z(M_1) \underset{A(\Sigma)}{\otimes} Z(M_2)$.
- $A(\Sigma)$ is a quantization of a smooth variety $Ch^{fr}(\Sigma^o)$. $Z(M_1)$, $Z(M_2)$ are holonomic modules, i.e. they quantize Lagrangians $G^g \subset G^{2g}$.
- Finite-dimensionality follows from Kashiwara-Schapiro theory of def. quantization and holonomicity.

Langlands/S-duality for skein modules

At $q=1$, the TFT $Z(X)$ recovers the character variety:

- For $X=S$, it recovers the monoidal category $QC(\frac{G}{G})=QC(Ch(S^1))$
- For $X=\Sigma$ a surface, it recovers the category $QC(Ch(\Sigma))$.
- For $X=M$ a 3-manifold it recovers the vector space $O(Ch(M))$.

These are precisely the state spaces of Kapustin-Witten equations in each dimension, for $\psi = \infty$.

Expectation: For generic values of q, and corresponding generic values of ψ, $Z(M)$ recovers the Kapustin-Witten state space of M.

Conj. (BZGuJS): The dimension of skein module at generic values of q coincide, for G and its Langlands dual G (e.g. $SL(2)$ and $PGL(2)$).

Evidence: computations for $SL(N)$ and $PGL(N)$ on T^3 and lens spaces.