Path integral derivations of K-theoretic Donaldson invariants

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Based on a work to appear with J. Manschot, G. Moore, R. Tao, X. Zhang

Motivation

Five-dimensional supersymmetric gauge theories provide important playgrounds in studying interaction between geometry and physics.

- They can be obtained by geometric engineering of M-theory on local Calabi-Yau threefolds. [Seiberg 96] [Morrison, Seiberg, Intriligator 96]...
- $\exists$ UV completion with 5d or 6d SCFTs [Seiberg, 96]
- Relation to 6d QFTs or lower dimensional QFTs by KK reductions

Key tools have been various exact computations of partition functions and indices, defined on compact manifolds such as S^5, $S^4 \times S^1$ or $\mathbb{C}^2_{\epsilon_1\epsilon_2} \times S^1$.

5d SYM and K-theoretic Donaldson invariants

I will discuss correlators of 5d $\mathcal{N}=1$ pure SU(2) gauge theory on $M_4 \times S^1$ with a topological twist on M_4, when M_4 is a smooth closed four-manifold.

$$\langle W_{\mathfrak{R}}(x_1)\cdots D(S_1)\cdots\rangle_{M_4\times S^1} = \int [dV]\ W_{\mathfrak{R}}(x_1)\cdots D(S_1)\cdots e^{-S[V]}\ .$$

Mathematically, they compute the K-theoretic Donaldson invariants on M_4, which can be schematically written as

$$Z_\mu[\mathcal{R},\{x_i,S_i\}] = \sum_n \mathcal{R}^{d(n)} \int_{\mathcal{M}_{n,\mu}} \hat{A}(T\mathcal{M}_{n,\mu})\,(\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x_1)\cdots e^{\mu(S_1)}\cdots\ ,$$

where $\mathcal{M}_{n,\mu}$ is the moduli space of instantons on M_4. This formula can be thought of as a natural 4d analogue of the Verlinde formula.

The goal of this talk is to establish the precise relation between these two expressions and provide their physical derivations.

Previous works

The K-theoretic Donaldson invariants have been considered in various physical and mathematical context.

- [Nekrasov] [Losev, Nekrasov, Shatashvili, 98] - toric localization, K-theoretic instanton partition function
- [Göttsche, Nakajima, Yoshioka, 06] - wall-crossing formula for algebraic toric surfaces
- [Göttsche, Kool, Williams 19] - Verlinde formulas for algebraic surfaces
- [Hosseini, Yaakov, Zaffaroni 18] [Crichigno, Jain, Willett 18] - toric localization and holography

However, a complete path integral derivation in a 5d gauge theory point of view is not available in the literature.

Two path integral derivations

In this talk, I will provide two different physical ways of computing the K-theoretic Donaldson invariants. This reproduces and generalizes the result of [Göttsche, Nakajima, Yoshioka, 06] and [Göttsche, Kool, Williams 19].

1. **Coulomb branch computation** ("4d IR")
   - via U-plane integral analysis
   - applicable for general class of M_4 ($b_2^+(M_4) > 0$ and $b_1(M_4) = 0$)
2. **Localization in $SU(2)$ gauge theory** ("4d UV")
   - from the perspective of $SU(2)$ instanton counting
   - restricted to toric M_4
   - useful in understanding geometric interpretation of the partition function via reduction to SQM

Wall-crossing formula

In particular, when $b_2^+(M_4) = 1$, the partition functions are expected to jump discontinuously as a function of metric on M_4.

We derive the wall-crossing formulas in the two approaches and show that they agree.

5d $\mathcal{N} = 1$ SU(2) Gauge Theory

5d $\mathcal{N}=1$ SU(2) gauge theory on $\mathbb{R}^5$

- 5d $\mathcal{N}=1$ vector multiplet

$$V = (A^{\mu}, \sigma, \lambda^{A}, D^{AB})$$

- Supersymmetric YMs action

$$S_{YM} = \frac{1}{g_{\text{YM}}^2} \int d^5x \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + |D_A \sigma|^2 + \frac{1}{2} D^{AB} D_{AB} + \text{(fermionic)} \right]$$

- Global symmetry group is $SU(2)_R \times U(1)_I$, where $U(1)_I$ is flavor symmetry associated to the current

$$j = * \operatorname{tr}(F \wedge F) \ ,$$

whose charged particles are instanton particles. In particular, the gauge coupling constant g_{YM} can be thought of as the VEV of the scalar field in the background vector multiplet $V_{(I)}$ for $U(1)_I$,

$$\sigma_{(I)} \sim \frac{1}{g_{\text{YM}}^2} \ .$$

Mixed CS term

- We also consider a mixed Chern-Simons term between $SU(2)$ and $U(1)_I$

$$S_{\text{mixed CS}} = \frac{\kappa}{8\pi^2} \int F_{(I)} \wedge \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A^3 \right)$$

Mixed CS term

- We also consider a mixed Chern-Simons term between $SU(2)$ and $U(1)_I$

$$
\begin{aligned}
S_{\text{mixed CS}} &= \frac{\kappa}{8\pi^2}\int F_{(I)}\wedge \mathrm{tr}\left(A\wedge dA+\frac{2}{3}A^3\right) \\
&+\frac{\kappa}{8\pi^2}\int d^5x\ \sigma_{(I)}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+|D_A\sigma|^2+\frac{1}{2}D^{IJ}D_{IJ}\right] \\
&+\frac{\kappa}{4\pi^2}\int d^5x\left[-\frac{1}{2}F^{\mu\nu}_{(I)}\mathrm{tr}(\sigma F_{\mu\nu})-D^\mu\sigma_{(I)}\mathrm{tr}(\sigma D_\mu\sigma)\right. \\
&\left.\qquad -\frac{1}{2}D^{IJ}_{(I)}\mathrm{tr}(\sigma D_{IJ})\right]+(\text{fermionic})
\end{aligned}
$$

It is useful to think of the YMs action as a part of the supersymmetrized mixed CS action.

BRST supercharge

We consider the theory on $M_4 \times S^1$, with a topological twist on M_4,

$$[SU(2)_- \times SU(2)_+] \times SU(2)_R \quad \leftarrow \quad SU(2)_- \times SU(2)' \ .$$

This gives a BRST supercharge $\bar{\mathcal{Q}}$ such that :

$$\begin{aligned}
&\bar{\mathcal{Q}} A_\mu = \psi_\mu \ , && \bar{\mathcal{Q}} \psi_\mu = F_{\mu 5} + i D_\mu \sigma \\
&\bar{\mathcal{Q}} (i A_5 + \sigma) = 0 \ , && \mathcal{Q} \chi_{\mu\nu} = F^+_{\mu\nu} + D_{\mu\nu} \\
&\bar{\mathcal{Q}} (i A_5 - \sigma) = \eta \ , && \bar{\mathcal{Q}} \eta = D_5 \sigma \\
&\bar{\mathcal{Q}} D_{\mu\nu} = D_5 \chi_{\mu\nu} + [\sigma, \chi_{\mu\nu}] \ ,
\end{aligned}$$

which satisfies $\bar{\mathcal{Q}}^2 \sim \partial_5$. This procedure gives a (partial) topological theory on $M_4 \times S^1$.

The topological reduction gives rise to 1d supersymmetric quantum mechanics on S^1 with one supercharge $\bar{\mathcal{Q}}$.

What does the partition function compute?

Reduction to SQM and moduli space of instantons

The $\bar{\mathcal{Q}}$-fixed point equations tell us that A is an anti-self-dual connection on M_4:

$$F_A^+ = 0 \ .$$

The effective supersymmetric quantum mechanics on S^1 is the 1d $\mathcal{N} = 1$ "sigma-model" into the moduli space of instantons on M_4,

$$S^1 \to \mathcal{M}_\mu \ ,$$

where

$$\mathcal{M}_\mu = \bigsqcup_{n=0}^{\infty} \mathcal{M}_{\mu,n} \ ,$$

where $\mathcal{M}_{\mu,n}$ is the moduli space of ASD connections with instanton number n with $w_2(P) = \mu$.

The Hilbert space $\mathcal{H}$ of QM is the space of sections of the spin bundle on $\mathcal{M}_\mu$,

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mu,n} \ .$$

Five-dimensional twisted index

The topologically twisted partition function on $M_4 \times S^1$ can be identified with the Witten index of the QM on S^1:

$$Z_\mu[\mathcal{R}] = \sum_{n=0}^{\infty} \mathcal{R}^{4n-3(1+b_2^+)/2} \, \mathrm{Tr}_{\mathcal{H}_{\mu,n}}(-1)^F \ .$$

From the standard argument of $\mathcal{N}=1$ SQM, the Witten index can be written as an intersection integral over $\mathcal{M}_\mu$ [Nekrasov]

$$Z_\mu[\mathcal{R}] = \sum_{n=0}^{\infty} \mathcal{R}^{4n-3(1+b_2^+)/2} \int_{\mathcal{M}_{\mu,n}} \hat{A}(T\mathcal{M}_{\mu,n}) \ ,$$

where

$$\hat{A}(T\mathcal{M}_{\mu,n}) = \prod_i \frac{x_i/2}{\sinh[x_i/2]} \ .$$

Topological observables

- Wilson loop along S^1

$$W_{\mathfrak{R}}(x) = \mathrm{Tr}_{\mathfrak{R}}\ \mathrm{P}\exp \int_{S^1} (iA_5 + \sigma) dx^5 \ ,$$

where $x \in H_0(M_4)$.

- Topological descents of Wilson loop
- 3D CS like defect on $S^1 \times S$, where $S \in H_2(M_4)$ [Baulieu, Losev, Nekrasov]

$$S_{\text{mixed CS}} = \int_{M_4 \times S^1} F_{(I)} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3 \right) + (\text{SYSY completion}) \ ,$$

for a closed two-form $F_{(I)} = PD(S^1 \times S)$.

- Other 3D TFTs

3D CS-like defect and and Donaldson map

Consider the 3D Chern-Simons like defect,

$$S_{\text{mixed CS}}[V] = \frac{1}{4\pi^2} \int F_{(I)} \wedge \text{Tr}\left(A \wedge dA + \frac{2}{3}A^3\right) + (\text{SYSY completion}) \ ,$$

with $\left[F_{(I)}/2\pi\right] = \mathfrak{n}_I$. This corresponds to a choice of line bundle L_I on M_4 with $c_1(L_I) = \mathfrak{n}_I$.

Let Z^J be the coordinates on $\mathcal{M}_{\mu,n}$. The mixed CS action $S_{\text{mixed CS}}[V]$ induces a coupling in the effective QM,

$$S_{\text{mixed CS}}[V] \underset{\text{reduction on } M_4}{\longrightarrow} \int_{S^1} \widetilde{\mathcal{A}}_J \dot{Z}^J dt \ ,$$

where $\widetilde{\mathcal{A}}$ is the connection on a line bundle $\mathcal{L}_{(I)}$ on $\mathcal{M}_\mu$ with

$$\mathcal{F} = d\widetilde{\mathcal{A}} = \frac{1}{4\pi^2} \int_{M_4} F_{(I)} \wedge \text{Tr}\left(\mathbb{F} \wedge \mathbb{F}\right) \ ,$$

where $\mathbb{F}$ is the curvature of the universal bundle $\mathbb{E}$ on $\mathcal{M}_{n,\mu} \times M_4$.

3D CS-like defect and the moduli space

In the presence of the coupling

$$\int_{S^1} \widetilde{\mathcal{A}}_J \dot{Z}^J dt \ ,$$

the Hilbert space of the QM is now the space of sections of $S \otimes \mathcal{L}_{(I)}$ on $\mathcal{M}$. This leads to the expression

$$Z_\mu[\mathcal{R}, \mathfrak{n}_I] = \sum_{n=0}^{\infty} \mathcal{R}^{4n-3(1+b_2^+)/2} \int_{\mathcal{M}_{\mu,n}} \hat{A}(T\mathcal{M}_{\mu,n}) \wedge \exp c_1(\mathcal{L}_{(I)}) \ .$$

Note that

$$\mathcal{F} = d\widetilde{\mathcal{A}} = \frac{1}{4\pi^2} \int_{M_4} F_{(I)} \wedge \mathrm{Tr}\left(\mathbb{F} \wedge \mathbb{F}\right) \ ,$$

can be thought of as a Donaldson map,

$$\mu : H_2(M_4) \to H^2(\mathcal{M}) \ ,$$

for a co-dimension two locus $S = PD(c_1(L_I)) \in H_2(M_4)$. Therefore we can also write $\exp c_1(\mathcal{L}_{(I)}) = \exp \mu(S)$.

Wilson loop

We also consider an insertion of a Wilson loop at $x \in H_0(M_4)$.

$$W_{\mathfrak{R}}(x) = \mathrm{Tr}_{\mathfrak{R}}\ \mathrm{P}\exp\int_{S^1}(iA_5+\sigma)dx^5\ .$$

This leads to the insertion of the Chern character in representation $\mathfrak{R}$,

$$\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x\ ,$$

where $/ : H^{\bullet}(\mathcal{M}_{n,\mu}\times M_4)\times H_i(M_4)\to H^{\bullet-i}(\mathcal{M})$.

To summarize,

$$Z_\mu[\mathcal{R},\{x_{\mathfrak{R}},S\}] = \sum_n \mathcal{R}^{4n-3(1+b_2^+)/2}\int_{\mathcal{M}_{\mu,n}}\hat{A}(T\mathcal{M}_{\mu,n})\,(\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x)\,e^{\mu(S)}\ .$$

Path integral derivation of partition functions

Our approach to computation of the partition functions on $M_4 \times S^1$:

- The 5d $\mathcal{N} = 1$ theory on S^1 can be thought of as an effective 4d $\mathcal{N} = 2$ theory on M_4, with infinitely many Kaluza-Klein particles with mass $m_{\mathrm{KK}} \sim n/R$, for $n \in \mathbb{Z}$.
- Using the topological invariance on M_4, we scale the metric on M_4 as

$$g_{\mu\nu} \to t g_{\mu\nu} \ , \qquad t \to \infty.$$

- The physics is replaced by a low energy effective theory on the Coulomb branch of 4d $\mathcal{N} = 2$ theory.
- The Coulomb branch is described by the Seiberg-Witten geometry.
- Computation of the partition function is reformulated into that of Seiberg-Witten invariants, with contributions from the U-plane integral.

Seiberg-Witten Geometry of Effective 4d $\mathcal{N} = 2$ Theory

Coulomb branch of 4d effective theory

- Classical Coulomb branch of effective 4d theory is parametrized by

$$a = \frac{1}{R}\int_{S^1}(\sigma + iA_5)\,dx^5 \qquad \in \mathbb{R}\times S^1 \ .$$

- The Coulomb branch effective theory is determined by the prepotential

$$F(a,\Lambda) = \frac{2}{R^2}\Big(Li_3(e^{-Ra}) - \zeta(3)\Big) + a^2\left(\log(\mathcal{R}) - \frac{\pi i}{2}\right) + \mathcal{O}(R\Lambda)\ ,$$

where Λ is a dynamically generated scale of the 4d theory, which is related to the 5d gauge coupling

$$4\log(R\Lambda) = -\frac{8\pi^2 R}{g_{\mathrm{YM}}^2}$$

It is useful to define a dimensionless parameter $\mathcal{R} = R\Lambda$, which is the instanton counting parameter.

Coulomb branch and BPS particles

- The prepotential F determines the gauge coupling of the effective theory

$$a_D = -\frac{1}{2\pi i}\frac{\partial F}{\partial a}\ , \qquad \tau = \frac{\partial a_D}{\partial a}$$

- There are BPS particles on the Coulomb branch:
  - W-boson : a
  - Magnetic string wrapping S^1 : a_D
  - Instanton particle : $m_I = \frac{2}{R}\log(\mathcal{R})$
  - Kaluza-Klein particle : $m_{KK} = \frac{2\pi i}{R}$

Seiberg-Witten curve

The order parameter for the quantum corrected Coulomb branch is the VEV of the fundamental Wilson loop,

$$U = \langle \mathrm{Tr}_F \; \mathrm{P} \exp(Ra) \rangle = e^{Ra} + e^{-Ra} + \mathcal{O}(R\Lambda) \; .$$

The Seiberg-Witten curve is parametrized by $U \in \mathbb{C}$, [Ganor, Morrison, Seiberg 96] [Eguchi, Sakai 02]

$$-\mathcal{R}^2 X \left(\omega + \frac{1}{\omega} \right) = P(X)^2 \; , \qquad P(X) = X^2 + UX + 1 \; .$$

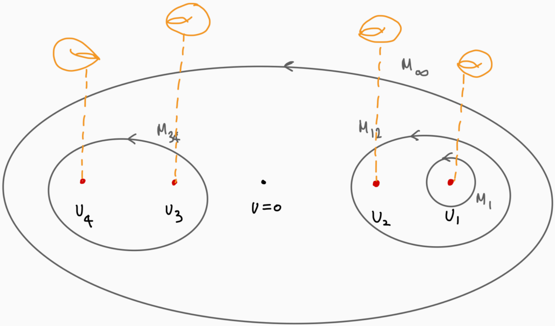
with

$$\lambda = \frac{1}{2\pi i R} \log(X) \frac{d\omega}{\omega} \; .$$

Then

$$a = \int_A \lambda \; , \qquad a_D = \int_B \lambda \; .$$

Seiberg-Witten geometry can be understood as an elliptic fibration over the "**U-plane**". For generic $\mathcal{R}$, it is parametrized by $U \in \mathbb{C}$ with four singularities, where a BPS particle becomes massless.



U_1 : A monopole becomes massless $\quad$ U_2: A dyon becomes massless

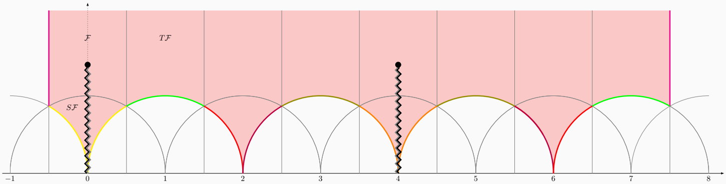
U_3, U_4 : A dyonic instanton becomes massless

See also [Closset-Magureanu, 21]

From the Seiberg-Witten curve, we obtain the relation

$$U(\tau) = \pm \left(-4\mathcal{R}^2 \frac{\theta_2(\tau)^4 + \theta_3(\tau)^4}{\theta_2(\tau)^2\theta_3(\tau)^2} + 4\mathcal{R}^4 + 4 \right)^{1/2} .$$

- The fundamental domain $\mathcal{F}_{\mathcal{R}}$ is a double copy of $\mathbb{H}/\Gamma^0(4)$ of 4d SYM.
- For $\mathcal{R} = 1$, $U(\tau)$ is modular invariant for $\Gamma^0(8) \subset SL(2,\mathbb{Z})$. Therefore, $\mathcal{F}_{\mathcal{R}=1} = \mathbb{H}/\Gamma^0(8)$.
- For generic $\mathcal{R}$, $\exists$ branch points at $U(\tau) = 0$, whose position depends on $\mathcal{R}$.



Fundamental domain $\mathcal{F}_R$.

See also [Aspman, Furrer, Manschot, 21]

U-plane integral

U-plane integral

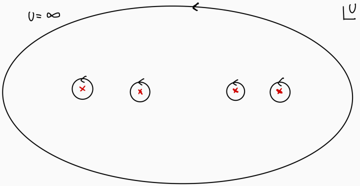
Following [Moore, Witten 97], the partition function of effective 4d theory for $b_2^+(M_4) > 0$ can be written as

$$Z_{J,\mu}[\mathcal{R}, \mathfrak{n}_I] = \Phi_{J,\mu}[\mathcal{R}, \mathfrak{n}_I] + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i}[\mathcal{R}, \mathfrak{n}_I] \ ,$$

where Φ is the so-called "U-plane integral" contribution and Z^{SW} is the Seiberg-Witten contribution at the four singular points in the U-plane, where a BPS particle becomes massless.

- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on M_4.
- For $b_2^+ > 1$, $\Phi_{J,\mu}$ identically vanishes.
- For $b_2^+ = 1$, the metric dependence comes through J, the period point. $J \in H^2(M_4, \mathbb{R})$ with $J = *J$ and $J^2 = 1$).

$$Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z_{J,\mu,i}^{SW}$$



- For $b_2^+ = 1$, $Z_{J,\mu}$ is a piecewise constant function of J. The dependence on J only comes from the region $U \to \infty$.
- The J-dependence of Φ around the singularities at finite $U = U_i$ are also non-trivial, but they are canceled with the wall-crossing of $Z_{J,\mu,i}^{SW}$.
- We can utilize this fact to compute $Z_{J,\mu,i}^{SW}$ for $b_2^+ > 1$.

We will focus on $b_2^+ = 1$ in this talk.

Coulomb branch effective action

- The effective theory on Coulomb branch can be thought of as a $\mathcal{N}=2$ $U(1)_G \times U(1)_I$ theory.
- When $b_2^+ = 1$, only the zero mode contributes to the U-plane integral. [Moore-Witten 97] The Coulomb branch effective action restricted to zero modes reads

$$\begin{aligned} S \quad = \quad & \int_{M_4} \frac{i}{16\pi}\left(\bar{\tau}_{ab} F_+^a \wedge F_+^b + \tau_{ab} F_-^a \wedge F_-^b\right) - \frac{1}{8\pi}\mathrm{Im}(\tau_{ab}) D^a \wedge D^b \\ & + \frac{i\sqrt{2}}{16\pi} \bar{F}_{abc} \eta^a \chi^b \wedge (D + F_+)^c \ . \end{aligned}$$

- The couplings τ_{ab} for $a = 1, 2$ are

$$\tau = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial a^2}\ , \quad v = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial a \partial m_I}\ , \quad \xi = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial m_I^2}\ ,$$

where $m_I = \frac{2}{R}\log(\mathcal{R})$.

The U-plane integral

Integrating over the zero modes of D, η, χ and summing over gauge fluxes, we arrive at a finite-dimensional integral

$$\Phi_{J,\mu}[\mathcal{R}, \mathfrak{n}_l] = K_U \int da \wedge d\bar{a} \, \frac{d\bar{\tau}}{d\bar{a}} A^{\chi} B^{\sigma} C^{\mathfrak{n}^2} \Psi^J_\mu(\tau, \bar{\tau}, v\mathfrak{n}_l/2, \bar{v}\mathfrak{n}_l/2) \, .$$

where

$$C = \exp \frac{1}{8} \frac{\partial^2 F}{\partial m_l^2} \, , \qquad v = -\frac{1}{2\pi i} \frac{\partial^2 F}{\partial a \partial m_l} \, .$$

Similar couplings are discussed in [Shapere,Tachikawa 08][Manschot,Moore 21]

Ψ^J_μ is the sum over fluxes

$$\begin{aligned} \Psi^J_\mu(\tau, \bar{\tau}, z, \bar{z}) \quad = \quad & e^{-2\pi y b_+^2} \sum_{k \in H^2(M_4, \mathbb{Z}) + \mu} \partial_{\bar{\tau}}(\sqrt{2y} B(k+b, J)) \\ & \cdot (-1)^{B(k, \mathcal{K}_{M_4})} q^{-k_-^2/2} \bar{q}^{k_+^2/2} e^{-2\pi i B(k_-, z) - 2\pi i B(k_+, \bar{z})} \, , \end{aligned}$$

with $b = \text{Im } z/y$. We defined $B(k_1, k_2) = \int_{M_4} k_1 \wedge k_2$ and $q = e^{2\pi i \tau}$.

Integral over $\mathcal{F}_R$

It is natural to change the variable to the fundamental domain of τ.

$$\Phi_{J,\mu}[\mathcal{R}, \mathfrak{n}_I] = K_U \int_{\mathcal{F}_R} d\tau \wedge d\bar{\tau}\ \nu_R(\tau) C^{\mathfrak{n}_I^2} \Psi_\mu^J(\tau, \bar{\tau}, v\mathfrak{n}_I/2, \bar{v}\mathfrak{n}_I/2).$$

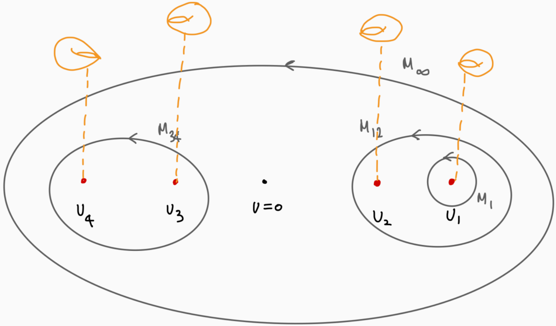
From the Seiberg-Witten geometry, the couplings can be written as q-expansions [Göttsche, Nakajima, Yoshioka]

$$\frac{\theta_1(\tau, v/2)}{\theta_4(\tau, v/2)} = -\mathcal{R}\ , \qquad C = e^{2\pi i \xi} = \frac{\theta_4(\tau, v/2)}{\theta_4(\tau)}\ .$$

We also have

$$\nu_R(\tau) = -\frac{i}{4} \frac{\theta_4^{13-b_2}(\tau)}{\eta(\tau)^9} \frac{1}{\sqrt{8\mathcal{R}^2 \mathfrak{u} + 4\mathcal{R}^2 + 4}}\ , \qquad \mathfrak{u}(\tau) = \frac{\theta_2(\tau)^4 + \theta_3(\tau)^4}{2\theta_2(\tau)^2\theta_3(\tau)^2}$$

Monodromies



- It is highly non-trivial that the integrand is single-valued under various monodromies in the U-plane.
- A proper choice of $\bar{\mathcal{Q}}$-exact terms is crucial in establishing the single-valuedness.
- This condition determines further quantization of the background flux $\mathfrak{n}_l$.

Wall-crossing

Suppose that we have two period points J and J'. One can show that [Korpas, Maschot 17] [Korpas, Manschot, Moore, Nidaiev 19]

$$\Psi_\mu^J(\tau,\bar\tau,z,\bar z) - \Psi_\mu^{J'}(\tau,\bar\tau,z,\bar z) = \partial_{\bar\tau}\widehat\Theta_\mu^{J,J'}(\tau,\bar\tau,z,\bar z)\ ,$$

$$\begin{aligned}\widehat\Theta_\mu^{J,J'}(\tau,\bar\tau,z,\bar z) \quad = \quad & \sum_{k\in H^2(M_4,\mathbb{Z})+\mu} \frac{1}{2}\left[E(\sqrt{2y}B(k+b,J)) - E(\sqrt{2y}B(k+b,J'))\right] \\ & (-1)^{B(k,\mathcal{K}_{M_4})} q^{-k^2/2} e^{-2\pi i B(k,z)}\ ,\end{aligned}$$

where $E(x) = \mathrm{Erf}(\sqrt{\pi}x)$.

Therefore the metric dependence of the partition function is determined by the contribution from the boundary of $\mathcal{F}_\mathcal{R}$ at $\tau \to i\infty$

$$Z_\mu^J[\mathcal{R},\mathfrak{n}_l] - Z_\mu^{J'}[\mathcal{R},\mathfrak{n}_l] = \lim_{Y\to\infty}\int_{iY-1/2}^{iY+15/2} d\tau\ \nu_R(\tau) C^{\mathfrak{n}^2}\widehat\Theta_\mu^{JJ'}(\tau,\mathfrak{n}v/2)\ ,$$

Wall-crossing formula

We arrive at the wall-crossing formula

$$Z_\mu^J[\mathcal{R},\mathfrak{n}_I] - Z_\mu^{J'}[\mathcal{R},\mathfrak{n}_I] = \sum_{k\in\mathcal{W}_{J,J'}} 8\left[\nu_R(\tau) C^{\mathfrak{n}^2}(-1)^{B(k,K)} q^{-k^2/2} e^{-2\pi i B(k,\mathfrak{n}v/2)}\right]_{q^0} ,$$

where

$$\mathcal{W}_{J,J'} = \left\{ k \mid B(k-\mathfrak{n}_I/4, J) > 0 \ \text{ and } B(k-\mathfrak{n}_I/4, J') < 0 \right\} .$$

Comparison to GNY

There are two ways to evaluate this expression.

(1) Expand the integrand in small $\mathcal{R}$ first and evaluate the q^0 term. This procedure reproduces the formula of [Göttsche, Nakajima, Yoshioka 06]. For example, $Z_{\mu,n}$ for $M_4 = \mathbb{P}^2$ can be computed via the WC formula, based on a blowup and existence of vanishing chamber.

$$Z^{\mathbb{P}^2}_{0,\mathfrak{n}}(\mathcal{R}) = \begin{cases} \mathcal{R} + \mathcal{R}^5 + \mathcal{R}^9 + \cdots & \mathfrak{n} = 0 \\ 3\mathcal{R} + 6\mathcal{R}^5 + 10\mathcal{R}^9 + \cdots & \mathfrak{n} = 1 \\ 6\mathcal{R} + 21\mathcal{R}^5 + 56\mathcal{R}^9 & \mathfrak{n} = 2 \\ 10\mathcal{R} + 56\mathcal{R}^5 + 230\mathcal{R}^9 + \cdots & \mathfrak{n} = 3 \\ \cdots & \end{cases}$$

which agrees with the interpretation

$$Z^{\mathbb{P}^2}_{0,n}(\mathcal{R}) = \sum_{d \geq 0} \chi\left(\mathcal{M}^{\mathbb{P}^2}_{0,d}, \mathcal{O}(\mu(H^{\otimes n}))\right) \mathcal{R}^d \ .$$

A puzzle

However, in the five-dimensional theory point of view, it is more natural to

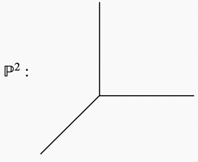
(2) Keep $\mathcal{R}$ finite and expand the integrand in small q first, and take the q^0 term. Surprisingly, we find that the result does not agree with (1). In particular, for $b_2^+ = 1$, the result of the U-plane integral contains arbitrary negative powers of $\mathcal{R}$.

A puzzle : How do we interpret meaning of the $\mathcal{R}^{-1}$ dependence? Existence of another branch? Subtleties in the UV completion?

Supersymmetric Localization

Toric localization

There is an alternative path integral derivation when M_4 is a smooth toric four-manifold, which can be construted by gluing copies of $\mathbb{C}^2$. For example,



The partition function can be computed from the perspective of instanton counting in the $SU(2)$ gauge theory. This is the approach adopted by many authors including [Nekrasov][GNY][Hosseini et al.][Crichigno et al.][Bonelli et al.]...

Note that $b_2^+ = 1$ for a toric four-manifold. The partition function undergoes a wall-crossing.

Localization to the Q-fixed locus

The Q-fixed point equations are

$$F^{+}_{\mu\nu} + \frac{i}{2}\xi^{I}\Gamma_{\mu\nu}\xi^{J}D_{IJ} = 0\ , \qquad \iota_v F = 0\ , \qquad D_\mu \sigma = 0\ .$$

When $\sigma \neq 0$, the gauge bundle splits into a sum of line bundles

$$E = L \oplus L^{-1}\ ,$$

and the moduli space reduces to a product of of abelian instantons, $\mathfrak{N}$, which can be described as a product of Hilbert scheme of points on M_4.

The path integral reduces to a finite-dimensional integral over the zero modes of effective 1d $\mathcal{N} = 2$ vector multiplet

$$(a\ , \bar{a}\ , \lambda\ , \bar{\lambda}\ , h)\ ,$$

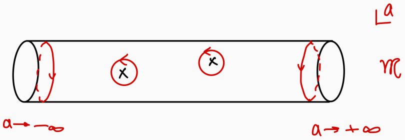
and a sum over abelianized fluxes

$$\left[\frac{F}{4\pi}\right] = k \in H^2(M_4, \mathbb{Z}) + \mu\ .$$

Integral over zero modes

The supersymmetric localization process gives

$$
\begin{aligned}
\widetilde{Z}_{J,\mu}[\mathcal{R},\mathfrak{n}_I] &= \sum_k \int da\, d\bar{a}\, d\lambda\, d\bar{\lambda}\, dh\; Z_{k,\mu,J}(a,\bar{a},\lambda,\bar{\lambda},h) \\
&= \sum_k \int \frac{dh}{h} \int_{\mathfrak{M}} da\, d\bar{a}\; \partial_{\bar{a}} Z_{k,\mu,J}(a,\bar{a},0,0,h) \\
&= \sum_k \oint_{C_J} da\; Z_{k,\mu}(a)
\end{aligned}
$$



The contour C_J is determined by the choice of metric on M_4.

Integrand

The toric action on M_4 induces a $(\mathbb{C}^*)^2$-action on the moduli space $\mathfrak{N}$. The path integral of the gauge theory localizes to its fixed points $\mathfrak{N}^T$.

For the $SU(2)$ gauge theory, the fixed loci are labeled by

$$\mathfrak{N}^T \longleftrightarrow \quad 2\chi\text{-tuples of Young-tableaux } \{Y^{a=1,2}_{i=1,\cdots,\chi}\}$$

and one can show that

$$\begin{aligned} Z_{k,\mu}(a) &= \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda) \\ &= \prod_{i=1}^{\chi} Z_{\text{pert}}(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda) Z_{\text{inst}}(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda) \ , \end{aligned}$$

where $Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda)$ is the K-theoretic Nekrasov partition function on $S^1 \times \mathbb{C}^2_{\epsilon_1^i \epsilon_2^i}$, localized at i-th fixed locus on M_4.

Non-equivariant limit

Let us focus on the non-equivariant limit $\epsilon_1, \epsilon_2 \to 0$ of the partition function. The Nekrasov conjecture tells us that

$$\begin{aligned} \log Z(a, \epsilon_1, \epsilon_2, R, \Lambda) &= \frac{1}{\epsilon_1 \epsilon_2} F(a, R, \Lambda) + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} H(a, \Lambda) \\ &\quad + \log A(a, \Lambda, R) + \frac{\epsilon_1^2 + \epsilon_2^2}{3\epsilon_1 \epsilon_2} \log B(a, \Lambda, R) + \cdots \end{aligned}$$

Summing over contributions from all the fixed loci, we obtain the finite non-equivariant limit,

$$\begin{aligned} Z_{k,\mu}(a, \mathcal{R}, \mathfrak{n}_I) &= \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda) \\ &= \exp\left[\frac{1}{2} \frac{\partial^2 F}{\partial a^2} k^2 + \frac{R}{4} \frac{\partial^2 F}{\partial a \partial \log \mathcal{R}} k \mathfrak{n}_I + \frac{R^2}{32} \frac{\partial^2 F}{\partial \log \mathcal{R}^2} \mathfrak{n}_I^2 \right. \\ &\quad \left. + \frac{\partial H}{\partial a} k \mathcal{K}_{M_4} + \chi A + \sigma B \right] + \mathcal{O}(\epsilon_1, \epsilon_2) \ . \end{aligned}$$

Contour

Let us go back to the contour integral formula.

$$
\begin{aligned}
\widetilde{Z}_{J,\mu}[\mathcal{R}, \mathfrak{n}_I] &= \sum_k \int \frac{dh}{h} \int_{\mathfrak{M}} da d\bar{a}\ \partial_{\bar{a}} Z_{k,\mu,J}(a, \bar{a}, 0, 0, h) \\
&= \sum_k \oint_C da\ Z_{k,\mu}(a)
\end{aligned}
$$

- The contour C depends on the details of the function $Z_{k,\mu,J}(a, \bar{a}, 0, 0, h)$, especially on its h-dependence.
- This expression in general gets contribution from non-BPS modes.
- Using the fact that $b_2^+ = 1$, we can argue that

  $$Z_{k,\mu}(a, \bar{a}, 0, 0, h) = \exp\left(\frac{-\mathrm{Im}\tau}{\pi}\left[h^2 + 2\pi i h B\left(k + \frac{1}{2}\frac{\mathrm{Im}(v)}{\mathrm{Im}(\tau)}\mathfrak{n}, J\right)\right]\right) Z_{k,\mu}(a)\,.$$

  in non-equivariant limit.

The metric dependence is encoded in the contour at large $|a|$ region. The h-integral gives the error function,

$$\begin{aligned}
&\int \frac{dh}{h}\ e^{\left(\frac{-\mathrm{Im}\tau}{\pi}\left[h^2+2\pi i h B\left(k+\frac{1}{2}\frac{\mathrm{Im}(v)}{\mathrm{Im}(\tau)}\mathfrak{n},J\right)\right]\right)} Z_{k,\mu}(a) \\
= \quad & -i\pi E\left[\sqrt{2\mathrm{Im}(\tau)}B\left(k+\frac{1}{2}\frac{\mathrm{Im}(v)}{\mathrm{Im}(\tau)}\mathfrak{n},J\right)\right] Z_{k,\mu}(a)\ .
\end{aligned}$$

which gives the wall-crossing formula,

$$\widetilde{Z}_{J,\mu}[\mathcal{R},\mathfrak{n}_I] - \widetilde{Z}_{J',\mu}[\mathcal{R},\mathfrak{n}_I] = \sum_{k\in\mathcal{W}_{J,J'}}\left(\underset{a=\infty}{\mathrm{res}}\, da + \underset{a=-\infty}{\mathrm{res}}\, da\right) Z_{k,\mu}(a,\mathcal{R},\mathfrak{n}_I)\ ,$$

where

$$\mathcal{W}_{J,J'} = \left\{k \mid B(k-\mathfrak{n}_I/4, J) > 0 \ \text{ and } B(k-\mathfrak{n}_I/4, J') < 0\ \right\}\ .$$

In the non-equivariant limit,

$$\begin{aligned}\widetilde{Z}_{k,\mu}(a,\mathcal{R},\mathfrak{n}_I) \quad &= \quad \exp\left[\frac{1}{2}\frac{\partial^2 F}{\partial a^2}k^2 + \frac{R}{4}\frac{\partial^2 F}{\partial a \partial \log \mathcal{R}}k\mathfrak{n}_I + \frac{R^2}{32}\frac{\partial^2 F}{\partial \log \mathcal{R}^2}\mathfrak{n}_I^2 \right. \\ &\qquad \left. + \frac{\partial H}{\partial a}k\mathcal{K}_{M_4} + \chi A + \sigma B\right]\end{aligned}$$

Changing the variable from a to $q = \exp(2\pi i \tau)$, we find

$$\begin{aligned}\widetilde{Z}_{J,\mu}[\mathcal{R},\mathfrak{n}_I] - \widetilde{Z}_{J',\mu}[\mathcal{R},\mathfrak{n}_I] \quad &= \quad \sum_{k\in B^+}\left(\operatorname*{res}_{a=\infty} da + \operatorname*{res}_{a=-\infty} da\right) Z_{k,\mu}(a,\mathcal{R},\mathfrak{n}_I) \\ &= \quad 8\sum_{k\in B^+}\operatorname*{res}_{q=0} dq\frac{da}{dq}\; Z_{k,\mu}(a(q),\mathcal{R},\mathfrak{n}_I) \\ &= \quad 8\sum_{k\in B_+}\left[Z_{k,\mu}(a(q),\mathcal{R},\mathfrak{n}_I)q\frac{da}{dq}\right]_{q^0} \\ &= \quad Z_{J,\mu}[\mathcal{R},\mathfrak{n}_I] - Z_{J',\mu}[\mathcal{R},\mathfrak{n}_I]\ ,\end{aligned}$$

which agrees with the U-plane integral analysis.

$b_2^+(M_4) > 1$ and the Seiberg-Witten contribution

K-theoretic Donaldson invariants for $b_2^+(M_4) > 1$

From

$$Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i} \ ,$$

and the fact that the wall-crossing at the strong coupling singularities cancels between Φ and Z^{SW},

$$\Phi_{J,\mu}|_{U_i} - \Phi_{J',\mu}|_{U_i} = -\left(Z^{SW}_{J,\mu,i} - Z^{SW}_{J',\mu,i} \right) \ ,$$

it is possible to derive the K-theoretic Donaldson invarinat for $b_2^+(M_4) > 1$. We find that the result reproduces [Göttsche, Kool, Williams 19].

Conclusion

Summary

- The topological correlator of 5d $\mathcal{N}=1$ $SU(2)$ gauge theory computes the K-theoretic Donaldson invariant

$$Z_{J,\mu}[\Lambda, R, \{x, S\}] = \sum_n \Lambda^{4n-3(1+b_2^+)/2} \int_{\mathcal{M}_{n,\mu}} \hat{A}_R(T\mathcal{M}_{n,\mu})\,(\mathrm{Ch}_R(\mathbb{E})/x)\, e^{R\mu(S)}\,.$$

- In particular, we derived the metric dependence of $Z_{J,\mu}$ for $b_2^+=1$ in the perspective of the U-plane integral, and also in the $SU(2)$ gauge theory perspective. We show that they agree in a non-trivial way.
- Using the U-plane integral approach, the computation can be generalized to $b_2^+>1$. This reproduces the result of [Göttsche, Kool, Williams 19]

Future directions

- Extension to general rank 1 E_N theories and 5d $\mathcal{N} = 1^*$ theory.
- Computation on more general five-dimensional manifolds (e.g., S^5) using the KK-symmetry.
- Uplift to 6d $\mathcal{N} = (1, 0)$ gauge theories.
- Extension to $b_1(M_4) > 0$ and dimensional reduction to 3d $\mathcal{N} = 4$ theories.
- More general topological defects.