The complex geometry of topological string partition functions

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The partition function of the topological string is of interest both for physics

(effective sugergravity actions, Nekrasov partition functions,...)

and mathematics

(enumerative invariants: Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa,...)

There are various approaches to its computation

(Topological recursion, holomorphic anomaly, topological vertex, matrix model...)

Goal: **Find a geometric characterisation of non-perturbatively defined partition functions encoding the BPS-spectrum in a simple way**

The basic problem

The topological string free energy,

$$\mathcal{F}(Q,\lambda) = \mathcal{F}_0(\lambda) + \widetilde{\mathcal{F}}(Q,\lambda)\,,$$

is defined by a divergent series

$$\mathcal{F}_0(\lambda) = \sum_{g\geq 0} \lambda^{2g-2} F_0^g, \qquad F_0^g = \frac{\chi(X)(-1)^{g-1}\, B_{2g}\, B_{2g-2}}{4g(2g-2)\,(2g-2)!}\,, \quad g \geq 2\,,$$

$$\widetilde{\mathcal{F}}(Q,\lambda) = \sum_{g\geq 0}\sum_{\beta\in\Gamma} \lambda^{2g-2} [\mathrm{GW}]_{\beta,g}\, Q^{\beta},$$

where $\Gamma = \{\beta \in H_2(X,\mathbb{Z}); \beta \neq 0\}$, $[\mathrm{GW}]_{\beta,g}$: Gromov–Witten invariants.

Questions:

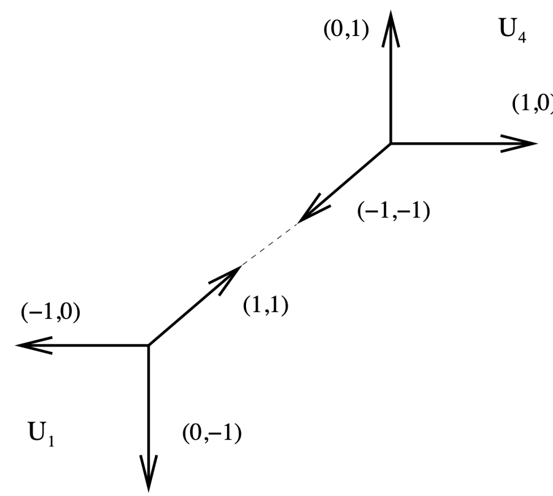
- Do there exist analytic functions with asymptotic expansion $\mathcal{F}(Q,\lambda)$?

- If there are several such functions, how are they related to each other?

A special case of the problem

Specialise to case of resolved conifold:

$$\widetilde{\mathcal{F}}(Q_F,t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q_F) + \frac{B_2}{2}\mathrm{Li}_1(Q_F) - \sum_{g=2}^{\infty} \lambda^{2g-2}\frac{(-1)^g B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(Q_F),$$



where $Q_F = e^{2\pi \mathrm{i}\, t}$, t: Kähler modulus.

Gopakumar-Vafa 1998, Faber-Pandharipande 2000

Borel summation

Consider a formal series $\sum_{k=0}^{\infty} a_n \lambda^n$, and assume that the **Borel transform**

$$G(\xi) := \sum_{k=1}^{\infty} a_k \frac{\xi^{k-1}}{(k-1)!}$$

is convergent in a neighbourhood of 0, and has an analytic continuation to a set $U \subset \mathbb{C}$ containing a ray $\rho = r\mathbb{R}_+$, $r \in \mathbb{C}$, such that

$$\int_\rho e^{-\xi/\lambda} G(\xi) d\xi$$

converges for λ in a non-empty open set V. Choosing a ray $\rho \subset U$ and setting

$$f_\rho(\lambda) := a_0 + \int_\rho e^{-\xi/\lambda} G(\xi) d\xi$$

defines a holomorphic function $f_\rho(\lambda)$ on V having asymptotic series $\sum_{k=0}^{\infty} a_k \lambda^k$.

Proof: Taylor-expand $G(\xi)$ around $\xi = 0$, and integrate term by term.

Stokes phenomena

Pasquetti-Schiappa found a formula for the Borel-transform of $\widetilde{\mathcal{F}}(Q,\lambda)$.
We found a useful alternative expression:

$$G(\xi,t) = \frac{1}{(2\pi)^2} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{m^3}\frac{1}{2\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{1-e^{-2\pi\mathrm{i}t+\xi/m}} - \frac{\xi^2}{1-e^{-2\pi\mathrm{i}t-\xi/m}}\right).$$

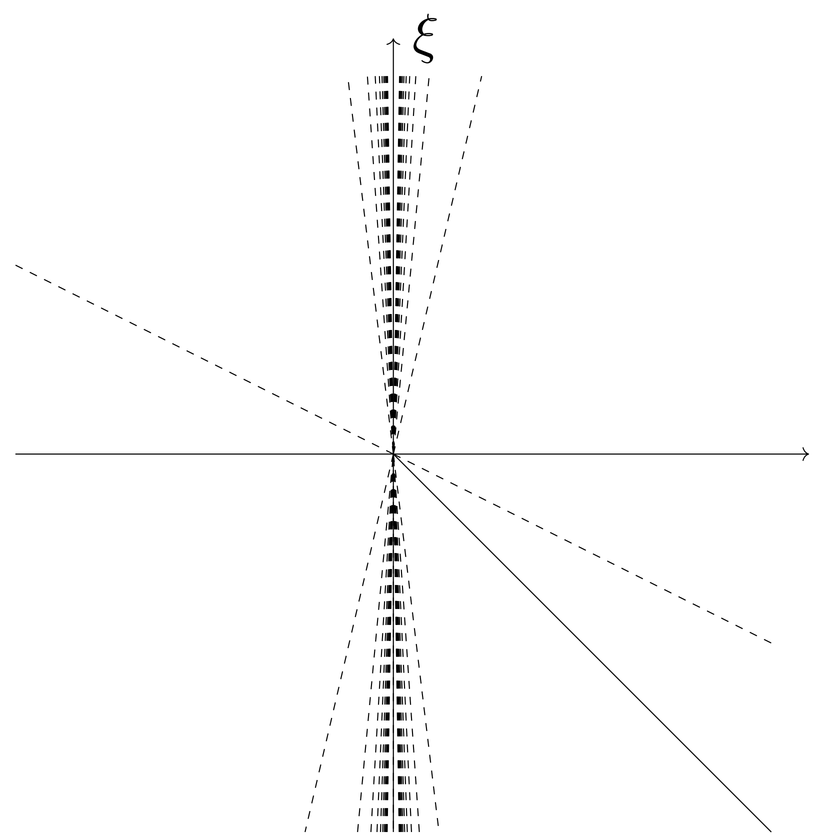
The poles are located on lines l_k, $k\in\mathbb{Z}$, through origin $\Rightarrow$ can take any ray ρ_k in wedges $\mathcal{W}_k$ between lines l_k and l_{k-1} as integration contour $\rightsquigarrow$ Functions $F_{\rho_k}(\lambda,t)$ on $\mathcal{W}_k$, $k\in\mathbb{Z}$.

Borel summations **jump** across lines l_k:

$$\begin{aligned}\phi_{\pm l_k}(\lambda,t) :&= F_{\pm\rho_{k+1}}(\lambda,t) - F_{\pm\rho_k}(\lambda,t)\\ &= \frac{1}{2\pi\mathrm{i}}\partial_{\check\lambda}\Big(\check\lambda\,\mathrm{Li}_2\big(e^{\pm 2\pi\mathrm{i}(t+k)/\check\lambda}\big)\Big)\,.\end{aligned}$$

Of special interest turns out to be

$$F_{\mathbb{R}_{>0}}(\lambda,t) = -\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{8u}\frac{e^{u(t-1/2)}}{\sinh(u/2)(\sinh(\check\lambda u/2))^2}\,.$$



Resulting picture

$$F_\rho(\lambda,t) = F_{\rm GV}(\lambda,t) + F_{\rm D}(\lambda,t;\rho), \qquad F_{\rm GV}(\lambda,t) := \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} k t}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2},$$

- $F_{\rm GV}(\lambda,t)$: Re-organisation of **perturbative** corrections a la Gopakumar-Vafa,
- $F_{\rm D}(\lambda,t;\rho)$: **Non-perturbative** part, functions of $Q' = e^{4\pi^2 \mathrm{i} t/\lambda}$, $q' = e^{4\pi^2 \mathrm{i}/\lambda}$.

Varying ρ interpolates between $F_{\rm D}(\lambda,t;\mathrm{i}\mathbb{R}_+) = 0$ and

$$F_{\rm D}(\lambda,t;\mathbb{R}_+) = \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial\lambda} \lambda F_{\mathrm{N}S}\big(\tfrac{4\pi^2}{\lambda}, \tfrac{2\pi}{\lambda}(t-\tfrac{1}{2})\big), \quad F_{\rm NS}(g,t) := \frac{1}{2\mathrm{i}} \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} k t}}{k^2 \sin\big(\frac{gk}{2}\big)},$$

the Nekrasov-Shatshvili limit of the refined version of $F_{\rm GV}(\lambda,t)$.

This proves, in particular, the conjecture of Hatsuda-Okuyama that

$$F_{\mathbb{R}_+}(\lambda,t) = F_{\rm np}(\lambda,t) := F_{\rm GV}(\lambda,t) + F_{\mathrm{N}S}(\lambda,t),$$

with $F_{\rm np}(\lambda,t)$ being the **non-perturbative completion** of the topological string partition function proposed by Hatsuda-Marino-Moriyama-Okuyama.

Stokes jumps encode BPS-spectrum

- Charge lattice Γ_c (trivial local system) $\Gamma_c = \mathbb{Z}\cdot\delta \oplus \mathbb{Z}\cdot\beta$; $\delta \sim \mathrm{D0}$, $\beta \sim \mathrm{D2}$
- Central charge functions $Z_{n\beta+m\delta}(t) = 2\pi\mathrm{i}(nt+m)$, for $n, m \in \mathbb{Z}$.
- BPS-indices[1]

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm\beta + n\delta \quad \text{for} \quad n \in \mathbb{Z}\,, \\ -2 & \text{if } \gamma = k\delta \quad \text{for } \; k \in \mathbb{Z}\setminus\{0\}\,, \\ 0 & \text{otherwise.} \end{cases}$$

One-to-one correspondence between **jumps** and **BPS-states**:

$$\phi_\gamma(\lambda, t) = \frac{\Omega(\gamma)}{2\pi\mathrm{i}}\frac{\partial}{\partial\lambda}\Big(\lambda\,\mathrm{Li}_2\big(e^{Z_\gamma(t)/\lambda}\big)\Big) \quad \text{represents jump along } \; \mathbb{R}_+ Z_\gamma.$$

Non-perturbative contributions $F_{\mathrm{D}}(\lambda, t; \rho)$:

Sum over all **stable** D-branes with charge γ having positive imaginary part of $Z_\gamma(t)$.

[1] Joyce-Song, Banerjee-Longhi-Romo

Ingredients of a more general picture (incomplete[2], sorry)

Quantum mirror symmetry, integrability, quantum curves

Aganagic-Dijkgraaf-Klemm-Marino-Vafa 2006

Dijkgraaf-Hollands-Sulkowski-Vafa 2007

Aganagic-Cheng-Dijkgraaf-Krefl-Vafa 2012

Geometry of hypermultiplet moduli spaces

Alexandrov-Persson-Pioline 2010

Alexandrov-Persson-Pioline 2011

Alexandrov-Pioline 2012

Non-perturbative partition functions, quantum curves

Hatsuda-Marino-Moriyama-Okuyama 2013

Grassi-Hatsuda-Marino 2014

Bonelli-Grassi-Tanzini 2017

Geometry of spaces of stability conditions

T. Bridgeland 2016

T. Bridgeland 2017

Bridgeland-Strachan

Tau-functions, quantum curves, spectral coordinates

Gamayun-Iorgov-Lisovyy, Iorgov-Lisovyy-T.

Coman-Pomoni-T., Coman-Longhi-T.

Alim-Saha-T.-Tulli, ...

[2] Exact WKB missing, for example. Listed are the ingredients most relevant for the following slides.

Which function? ... Which space?

Compare with counting functions for D0-D2-D6 bound states,

$$\mathcal{Z}^{\mathcal{C}}_{\mathrm{DT}}(u, \mathbf{Q}(\mathbf{t})) := \sum_{n\in\mathbb{Z}} \sum_{\beta\in H_2(X,\mathbb{Z})} [\mathrm{DT}]^{\mathcal{C}}_{n\delta+\beta+\check{\delta}}\, u^n\, e^{2\pi \mathrm{i}\langle\beta,\mathbf{t}\rangle},$$

where BPS indices $[\mathrm{DT}]^{\mathcal{C}}_{n\delta+\beta+\check{\delta}}$ are locally constant w.r.t. Kähler moduli, but jump across walls of marginal stability $\rightsquigarrow$ dependence on choice of chamber $\mathcal{C}$.

Maulik-Nekrasov-Okounkov-Pandharipande:

In infinite volume chamber $\mathcal{C}_\infty$ one has $\boxed{\mathcal{Z}^{\mathcal{C}_\infty}_{\mathrm{DT}} \propto \mathcal{Z}_{\mathrm{top}}.}$

When there are compact 4-cycles it turns out to be more convenient to consider:

Dual partition functions (counting D0-D2-D4-D6 bound states) specialise to[3]

$$\mathcal{T}^{\mathcal{C}}_{\mathrm{DT}}(\boldsymbol{\eta}, \mathbf{t}, \lambda) := \sum_{n\in\mathbb{Z}} \sum_{\beta\in H_2,\check{\beta}\in H^2} [\mathrm{DT}]^{\mathcal{C}}_{n\delta+\beta+\check{\beta}+\check{\delta}}\, u^n\, e^{2\pi \mathrm{i}(\langle\beta,\mathbf{t}\rangle - \langle\check{\beta},\eta\rangle)}$$

$$\mathcal{T}^{\mathcal{C}_\infty}_{\mathrm{DT}}(\boldsymbol{\eta}, \mathbf{t}, \lambda) = \sum_{\check{\beta}\in H^2} e^{-2\pi \mathrm{i}\langle\check{\beta},\boldsymbol{\eta}\rangle}\, \mathcal{Z}_{\mathrm{top}}\big(\mathbf{t} + \check{\beta}\lambda, \lambda\big).$$

[3] Dijkgraaf-Hollands-Sulkowski-Vafa

Relation to geometry of spaces of stability conditions I (Bridgeland)

Consider the local system of (charge-)lattices Γ with intersection pairing $\langle -, - \rangle$, fibered over the space of stability conditions $\mathcal{B}$ such that

- there is an atlas of local charts for $\mathcal{B}$ admitting local bases for Γ generated by elements denoted as $\gamma_0, \dots, \gamma_d, \check{\gamma}^0, \dots, \check{\gamma}^d$ satisfying $\langle \gamma_r, \check{\gamma}^s \rangle = \delta_r{}^s$.

- A local chart should have local complex coordinates for $\mathcal{B}$ given by the central charge functions $Z = (Z^0, \dots, Z^d)$, $\check{Z} = (\check{Z}_0, \dots, \check{Z}_d)$.

Consider total space $\mathcal{M} := \mathcal{T}\mathcal{B}$ with coordinates $\theta = (\theta^0, \dots, \theta^d)$, $\check{\theta} = (\check{\theta}_0, \dots, \check{\theta}_d)$ on the tangent fibres.

Case of Conifold: $d = 1$, $\delta \sim D0$, $\beta \sim D2$,

$$\Gamma = \mathbb{Z} \cdot \delta \oplus \mathbb{Z} \cdot \beta \oplus \mathbb{Z} \cdot \beta^\vee \oplus \mathbb{Z} \cdot \delta^\vee, \qquad \begin{aligned} \delta &\equiv \gamma_0, \qquad & Z_\delta &\equiv Z^0, \\ \beta &\equiv \gamma_1, \qquad & Z_\beta &\equiv Z^1, \end{aligned}$$

with $\check{Z}_0$ and $\check{Z}_1$ determined by special geometry (prepotential).

Relation to geometry of spaces of stability conditions II

Darboux coordinates on $\mathcal{M}$: Solutions to BPS Riemann-Hilbert problem

(Gaiotto-Moore-Neitzke; Bridgeland)

Define ζ-deformed complex structures by atlas of coordinates on $\mathcal{Z} \simeq \mathcal{M} \times \mathbb{C}^\times$ with charts $\{\mathcal{U}_\imath; \imath \in \mathbb{I}\}$, Darboux coordinates $(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = (x_\imath^0, \dots, x_\imath^d, \check{x}_0^\imath, \dots, \check{x}_d^\imath)$,

$$\Omega = \sum_{r=0}^{d} dx_\imath^r \wedge d\check{x}_r^\imath, \qquad \text{such that}$$

- changes of coordinates across $\{\zeta \in \mathbb{C}^\times; a_\gamma/\zeta \in \mathrm{i}\mathbb{R}_-\}$ represented as

$$X_{\gamma'}^\jmath = X_{\gamma'}^\imath (1 - X_\gamma^\imath)^{\langle \gamma', \gamma \rangle \Omega(\gamma)}, \qquad X_\gamma^\jmath = e^{2\pi \mathrm{i} \langle \gamma, x_\imath \rangle} = e^{2\pi \mathrm{i} (p_r^\imath x_\imath^r - q_\imath^r \check{x}_r^\imath)},$$
$$\text{if } \gamma = (q_\imath^0, \dots, q_\imath^d ; p_0^\imath, \dots, p_d^\imath),$$

  determined by **BPS-indices** $\Omega(\gamma)$ satisfying Kontsevich-Soibelman-WCF.

- asymptotic behaviour

$$\mathsf{x}_\imath^r \sim \frac{1}{\zeta} Z_\imath^r + \theta_\imath^r + \mathcal{O}(\zeta), \qquad \check{\mathsf{x}}_\imath^r \sim \frac{1}{\zeta} \check{Z}_r^\imath + \check{\theta}_r^\imath + \mathcal{O}(\zeta).$$

Relation to geometry of spaces of stability conditions III

Upshot:

Complex structure on $\mathcal{B}$ + **BPS-data**	$\Leftrightarrow$	Family of complex (Poisson-) structures on $\mathcal{Z} = T\mathcal{B} \times \mathbb{C}^\times$

Originally: Z and $\check{Z}$ independent complex coordinates.

In string theory: Need to cut out Lagrangian by imposing special geometry relations

$$\check{Z}_r^\imath = \frac{\partial}{\partial Z_\imath^r}\mathcal{F}_\imath(Z), \qquad \mathcal{F}_\imath : \quad \text{prepotential.}$$

We call data $\{\mathcal{F}_\imath; \imath \in \mathcal{I}\}$ **compatible** with BPS-data if there exists a solution to the BPS-RH-problem supplemented by $\check{Z}_r^\imath = \frac{\partial}{\partial Z_\imath^r}\mathcal{F}_\imath(Z)$.

We expect this to be the case in string theory, in which case $\mathcal{Z}$ should be related to the **hypermultiplet moduli space** $\mathcal{M}_{\mathrm{H}}$ in the conformal limit[4].

[4] Gaiotto 2014, Alexandrov-Pioline 2021

Role of dual partition functions

Conjecture: (Coman-Longhi-T.) The dual partition functions

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}_\imath, \lambda) = \sum_{\mathbf{n} \in H^2(X,\mathbb{Z})} e^{2\pi \mathrm{i} \langle \mathbf{n}, \check{\mathsf{x}}_\imath \rangle} \mathcal{Z}_\imath(\mathsf{x}_\imath + \mathbf{n}\lambda, \lambda) \tag{1}$$

are holomorphic sections of a canonical line bundle on $\mathcal{Z}$, with

transition functions $\sim$ **generating functions** of changes of Darboux coordinates.

BPS-spectrum encoded most simply in $\{\mathcal{T}_\imath\,;\, \imath \in \mathcal{I}\}$.

By inverting relations (1) one can represent the changes of Darboux coordinates as **integral transformations**

$$\mathcal{Z}_\imath(\mathsf{x}_\imath, \lambda) = \int d\mathsf{x}_\imath \; K(\mathsf{x}_\imath, \mathsf{x}_\jmath) \mathcal{Z}_\jmath(\mathsf{x}_\jmath, \lambda)$$

So $\mathcal{Z}_{\mathrm{top}} \equiv \mathcal{Z}_\imath$ are **wave functions**, after all!

Relation to hypermultiplet moduli spaces and S-duality

In the approach to the quaternionic Kaehler (QK) geometry of hypermultiplet moduli spaces $\mathcal{M}_{\mathrm{H}}$ developed in a series of papers by Alexandrov, Pioline and collaborators, one can use a BPS-RH problem similar to the one studied by Bridgeland to describe D-instanton corrections. Sections of a canonical hyper-holomorphic line bundle define an additional coordinate α needed for twistorial descriptions of the QK-geometry.

Conjecture: $\mathcal{T}_\imath \sim \alpha_\imath$ in the conformal limit (up to specific changes of trivialisation)[5]

S-duality: $\mathcal{M}_{\mathrm{H}}$ was conjectured in work of Alexandrov, Pioline and others to have an isometric action of S-duality by isometries represented on the twistor space by simple holomorphic changes of Darboux coordinates. Conformal limit of the basic S-duality transformation represented by $x_\imath^0 \mapsto -1/x_\imath^0$, and $\mathsf{x}_\imath^r \mapsto \mathsf{x}_\imath^r/x_\imath^0$, and $\alpha_\imath \mapsto \check{x}_0^\imath + \dots$.

Taken together: $\boxed{\mathcal{T}_\imath \sim \alpha_\imath \sim \check{x}_0^\imath}$. This makes a lot of sense:

- There is evidence that weak coupling jumps of α and $\mathcal{T}_\imath$ coincide.
- Strong coupling jumps of $\check{x}_0^\imath$ and $\mathcal{T}_\imath$ coincide.[6]

[5] For Conifold: Alexandrov-Pioline (unpublished); for relation of transition functions see Coman-Longhi-T., Appendix H.
[6] J.T. in preparation

S-duality and non-perturbative completion

Conjecture: There are distinguished chambers $\mathcal{C}_\infty$ such that

$$\mathcal{Z}_{\rm np}^{\mathcal{C}_\infty} = \mathcal{Z}_{\rm NS}^{\mathcal{C}_\infty} \cdot \mathcal{Z}_{\rm DT}^{\mathcal{C}_\infty},$$

with $\mathcal{Z}_{\rm NS}^{\mathcal{C}_\infty}$ determined in terms of $\mathcal{Z}_{\rm DT}^{\mathcal{C}_\infty}$, and vice-versa by the HMMO-cancellation mechanism.[7] For local CY there exists representations of the corresponding functions $\mathcal{T}_{\rm np}^{\mathcal{C}_\infty}$ in terms of spectral determinants (Grassi-Hatsuda-Marino).

Remark: Grassi-Hatsuda-Marino proposed spectral determinants as a background-independent representation in terms of entire functions.

We prefer to keep a background dependence as it encodes the spectrum of BPS states through the changes of Darboux coordinates.

[7] Hatsuda-Marino-Moriyama-Okuyama

Conclusions

Non-perturbative effects encoded in the perturbative expansion of the topological string partition function (resurgence).

Mathematical coding of non-perturbative effects $\sim$ BPS-spectrum:

Weak coupling Stokes jumps $\rightsquigarrow$ canonical line bundles on hypermultiplet moduli space ($\hookleftarrow$ Bridgeland's RH problems, generalises proposal of Coman-Longhi-J.T.).

Strong coupling Stokes jumps $\rightsquigarrow$ **S-dual** RH problem of Bridgeland type
($\hookleftarrow$ S-duality on hypermultiplet moduli space).

S-duality **automatic**, different types of framed wall-crossing at weak/strong coupling.

All this was fully worked out in the case of the resolved conifold, but there is evidence that the essential ingredients of the resulting picture can hold in larger generality.