# Koszul duality & twisted holography for asymptotically flat spacetimes

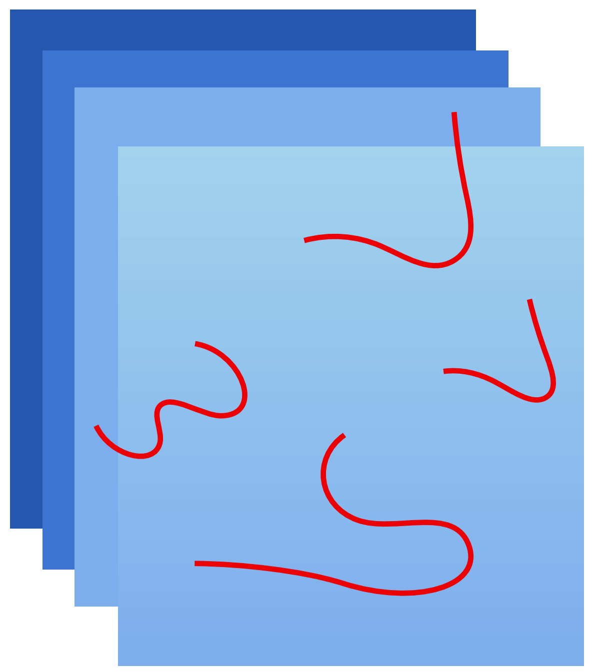
**Western Hemisphere Colloquium**

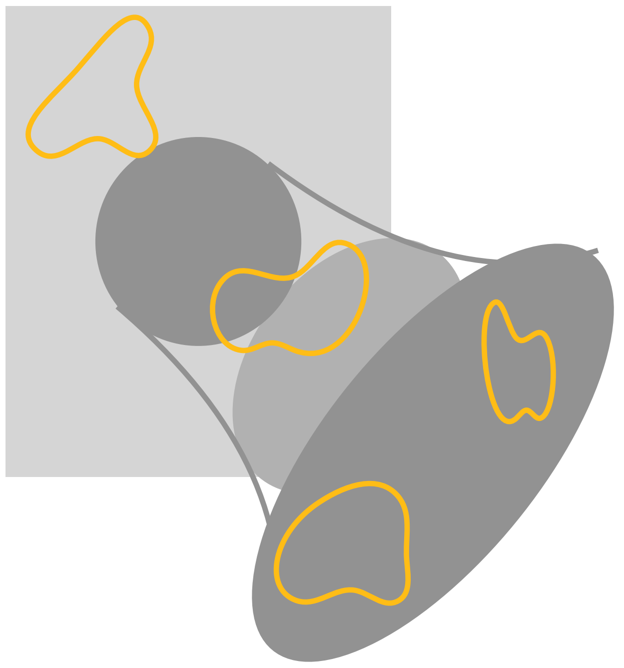
**Natalie M. Paquette**

# A confluence of progress in a few different subfields

points of contact: symmetry, universality

*Twisted holography*





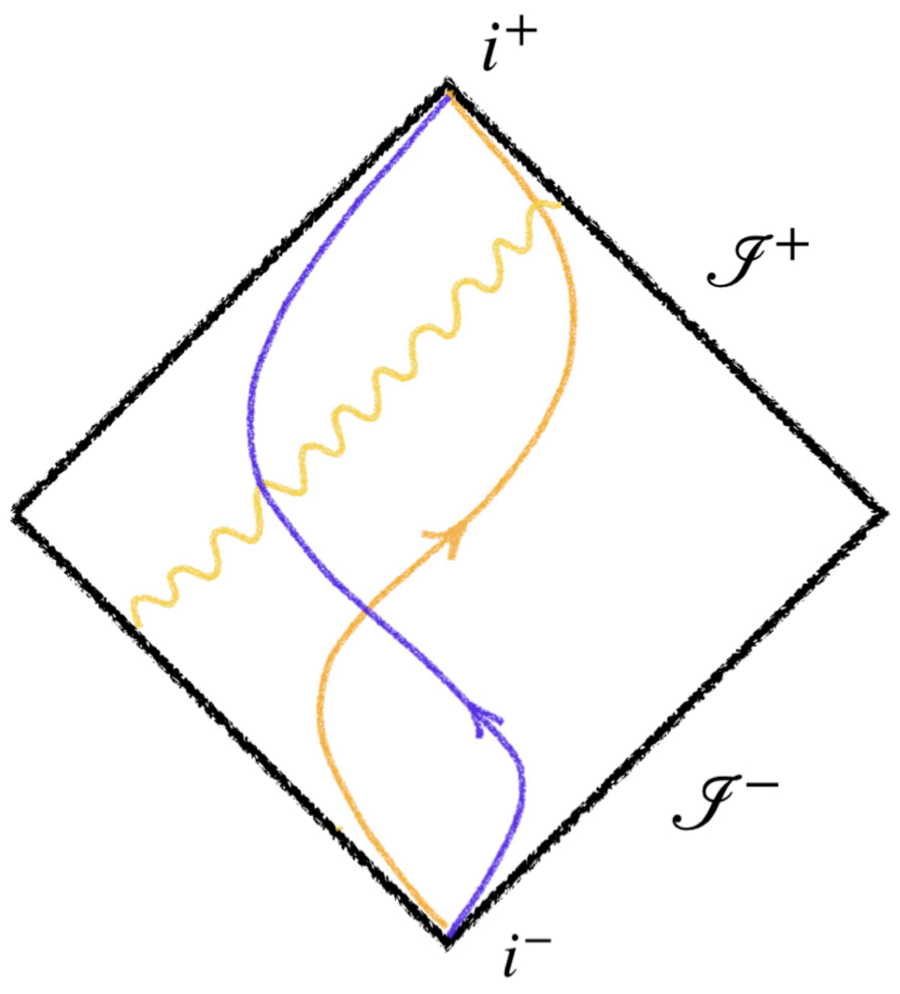
$$A = A_{\bar{z}} d\bar{z} + A_{\bar{w}_1} d\bar{w}_1 + A_{\bar{w}_2} d\bar{w}_2$$

$$\int \Omega \wedge CS(A)$$

$$\eta \in \Omega^{2,1}(M)$$

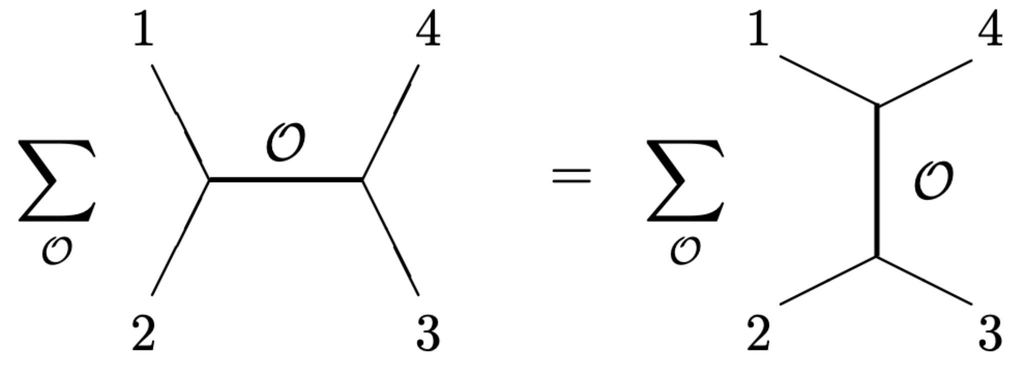
$$\frac{1}{2}\int (\partial^{-1}\eta)(\bar{\partial}\eta) + \frac{1}{6}\int \eta^3$$

*Celestial holography*

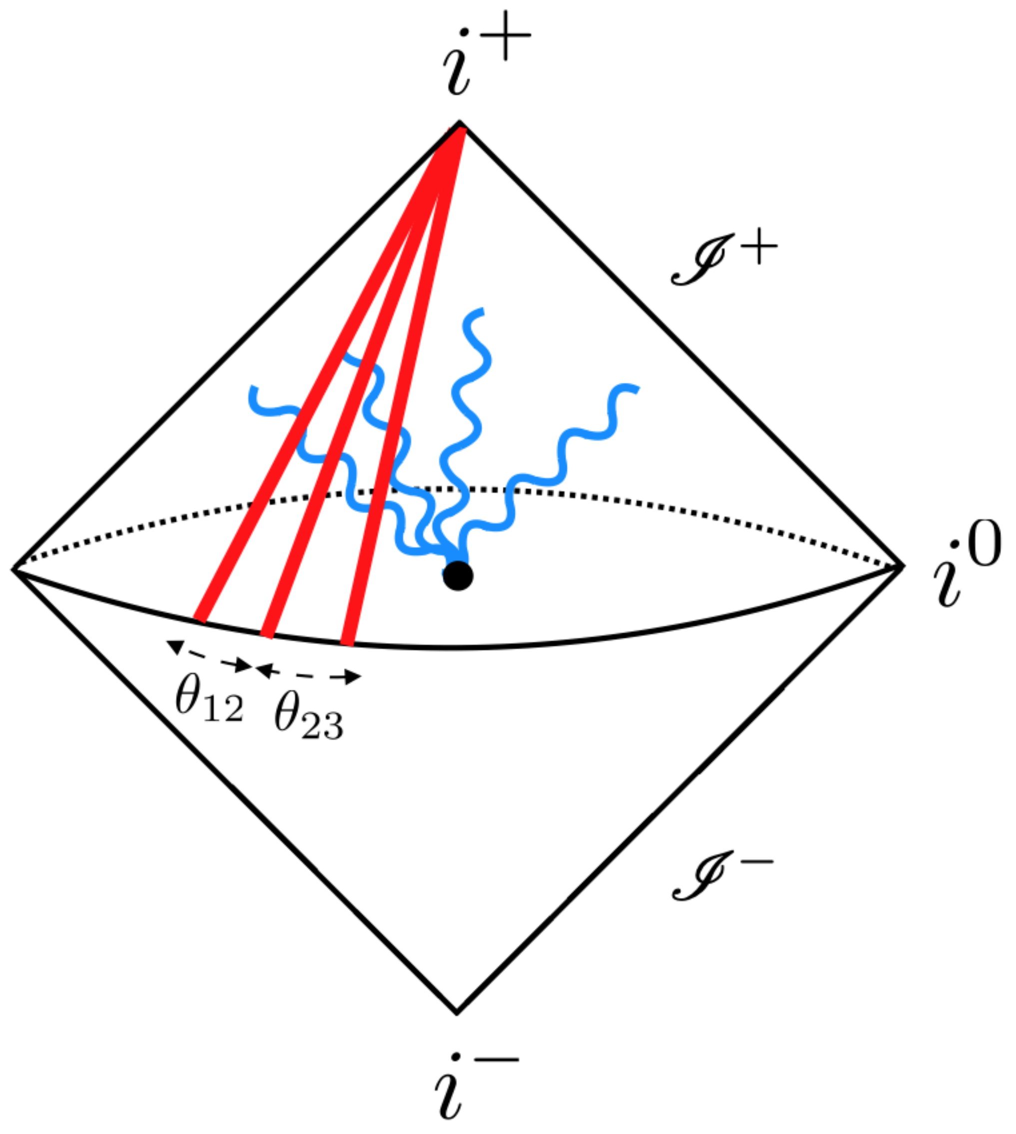



$$SO^+(1,3) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$$

*Bootstrap (CFTs, S-matrix,...)*

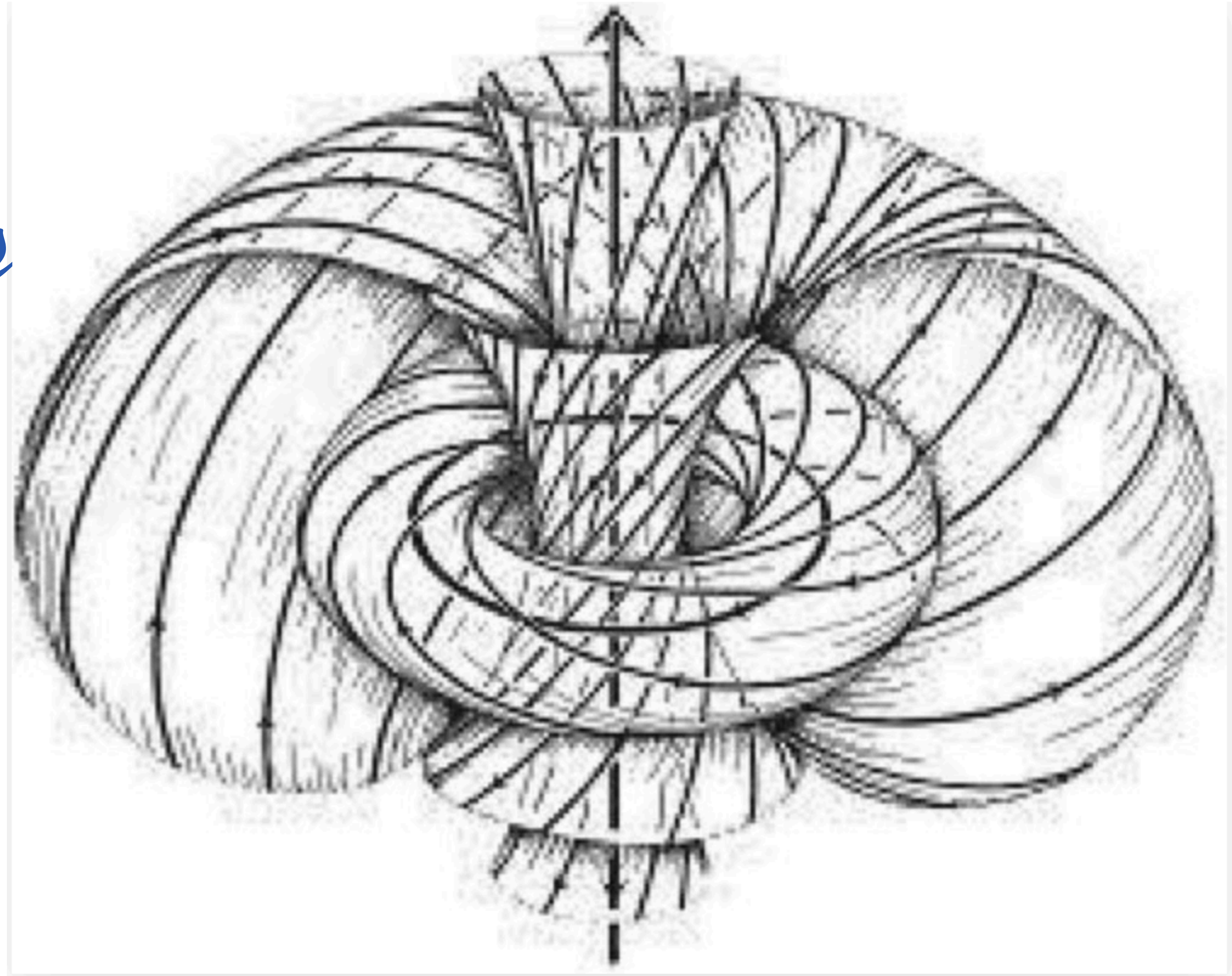



$$\mathcal{O}_1(z,\bar{z})\,\mathcal{O}_2(0,0) = \sum_{k_{\rm Schur}} \frac{\lambda_{12k}}{z^{h_1+h_2-h_k}}\,\mathcal{O}_k(0) + \{\mathbb{Q}, \ldots]$$



**Today:** we will start to flesh out some of those connections

*Holomorphic theories*
*on*
*Twistor space*



**6d bridge between 4d & 2d physics**

**From this perspective:**

form factors which reproduce some 4d (YM) scattering amplitudes ⟷ correlators of a certain 2d chiral algebra

**Ultimately, we will obtain a top-down example of (twisted) holography in an asymptotically flat spacetime, using open/closed duality of the topological string**

# A brief twistor primer

**Analogy for 2d:** $(z, \tilde{z}) \in \mathbb{C}^2 \simeq \mathbb{CM}_2$

$$\frac{\partial^2 \phi}{\partial z \partial \tilde{z}} = 0$$

complex analytic solution

$$\phi = f(z) + g(\tilde{z})$$

$\tilde{z} = \bar{z}$

solution to Laplace equation on $\mathbb{E}_2$

$z = u, \tilde{z} = v$
$(u, v) \in \mathbb{R}^2$

solution to 2d wave equation on $\mathbb{M}_2$

**Further:**

Fix $\phi, \nabla\phi$ on $S \subset \mathbb{CM}_2$, $D(S) = \left\{(z, \tilde{z}) : \text{ both null lines through } (z, \tilde{z}) \text{ meet } S\right\}$

$S \subset \mathbb{E}_2$

$D(S) \subset \mathbb{CM}_2$ to which $\phi$ can be analytically extended

$S \subset \Sigma$

$D(S) \subset \mathbb{M}_2$ usual domain of dpdce

# *Twistor space is this gadget for four dimensions*

dpds on ***conformal structure*** {
- In 2d space w/ inner product, choice of orientation → unique complex structure
- Every harmonic function on $\mathbb{E}_2$ the real part of a holomorphic function

- In 4d, orientation + conformal structure don't pick out a unique complex structure

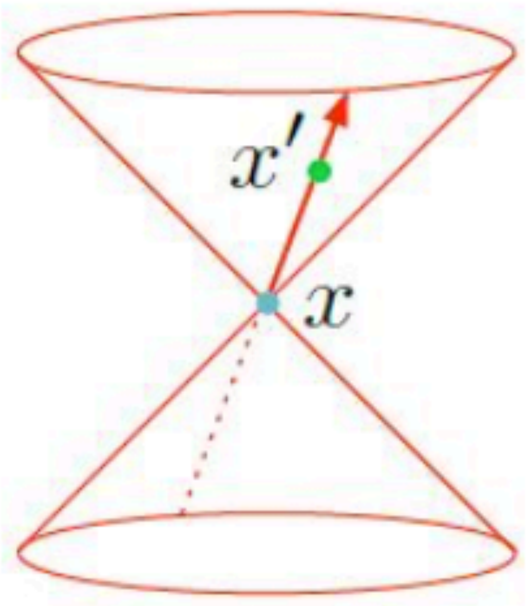
$x \in \mathbb{E}_4$

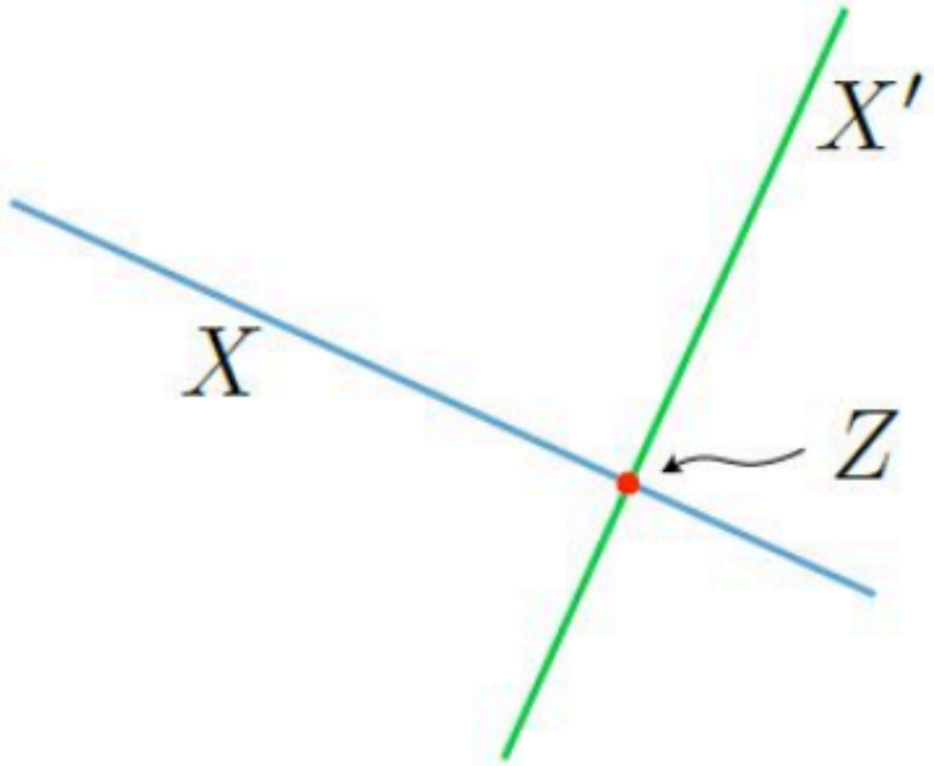
s.d. or a.s.d $\quad S_x^{2,\pm}$

$$\mathbb{PT} \simeq \mathbb{R}^4 \times \mathbb{CP}^1 \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$$

$$\downarrow$$

$z \in \mathbb{CP}^1 \qquad \mathbb{CP}^1$




Penrose transform $\quad H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))$

hol'c massless fields on $\mathbb{C}^4$

solutions to massless field equations

harmonic functions

(entire analytic functions, can pass to any signature)

[Penrose, Atiyah, Ward, Wodehouse, Mason, Lebrun,...]

# Twistor space is good for:

- computing classical solutions to nonlinear (massless) field equations
- making symmetries manifest (twistor gauge transformations ARE chiral mode algebra)
- computing amplitudes (esp. integrands, so we don't have to worry about symmetry-breaking regulators)

$$SO(4,\mathbb{C}) \simeq (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\mathbb{Z}_2$$

Spinor helicity variables

$$P^\mu, (P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2 = 0 \quad \longrightarrow \quad P^{\alpha\dot{\alpha}} = \begin{pmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{pmatrix} =: \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

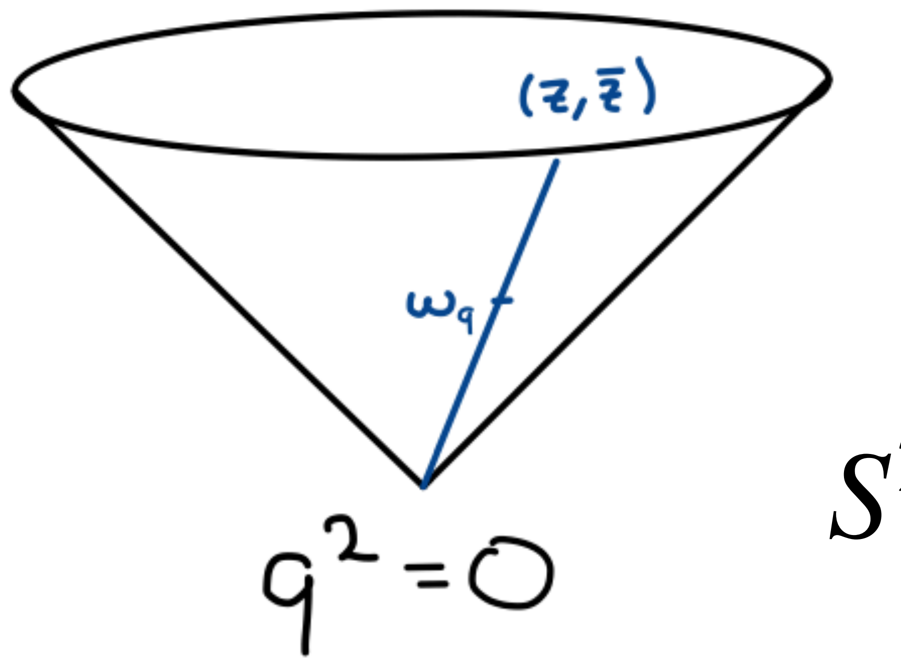
$$\lambda^\alpha = \left(1, \frac{P^1 + iP^2}{P^0 + P^3}\right) \equiv (1, z)$$

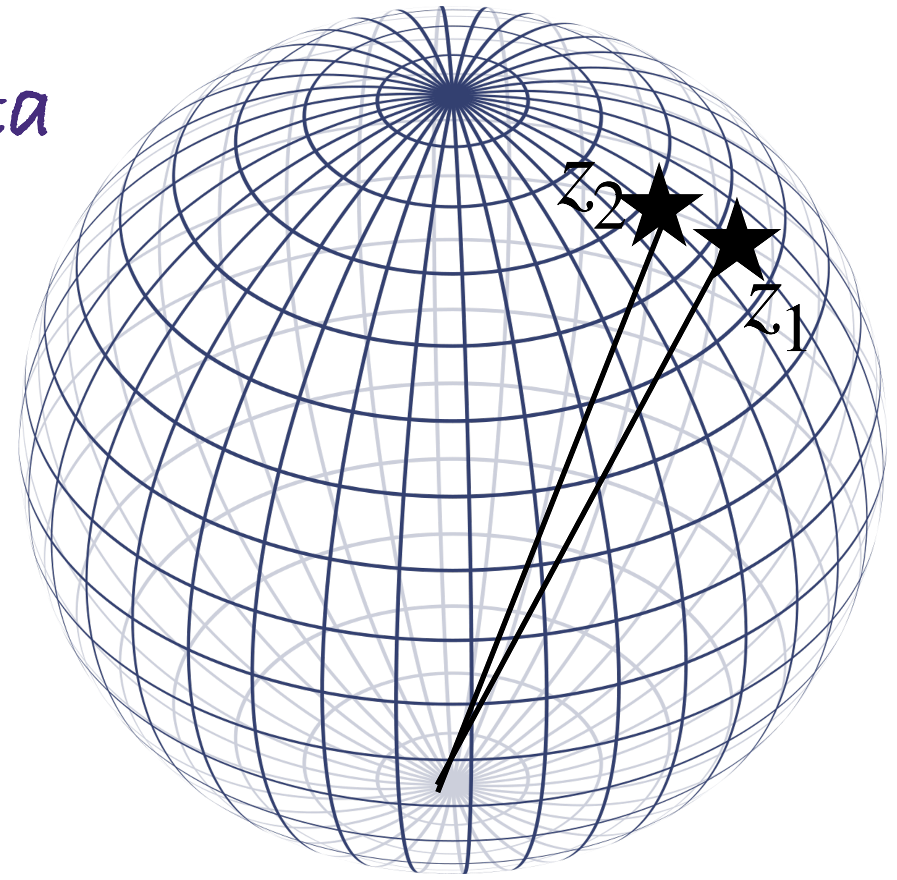
incidence relation:

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$$

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha$$

hol'c coord. of celestial sphere, "space" of null momenta

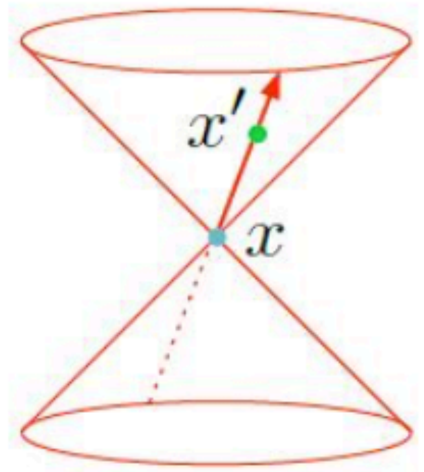


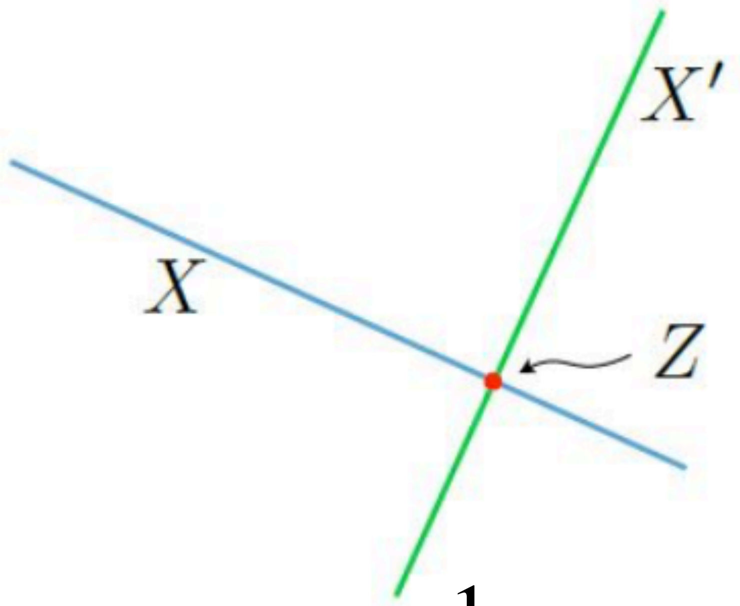

$$S^2 \simeq \mathbb{CP}^1_z$$

**2d OPE limits ↔ 4d collinear limits**

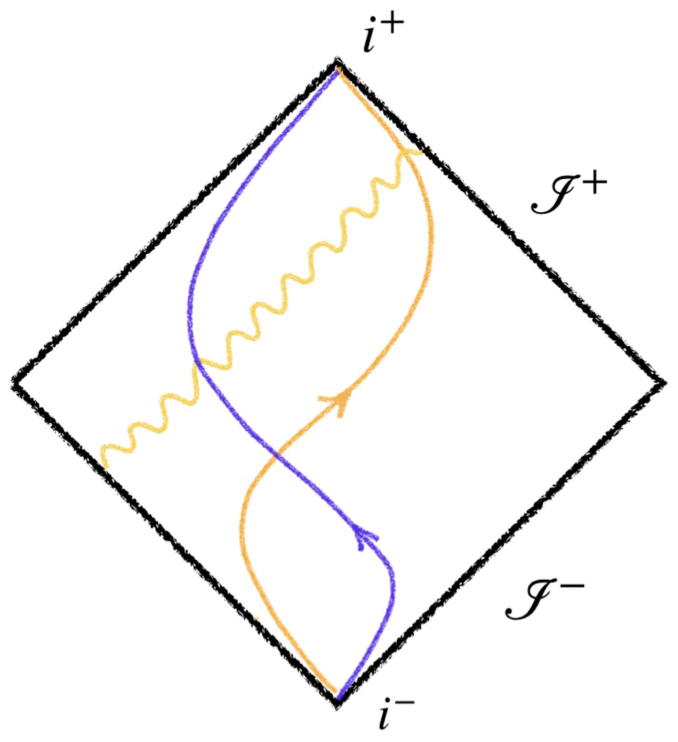
Today I'd like to discuss work on connections between 4d physics and 2d chiral algebras




$x \in \mathbb{C}^4$




$\mathbb{CP}^1_x \in \mathbb{PT}$

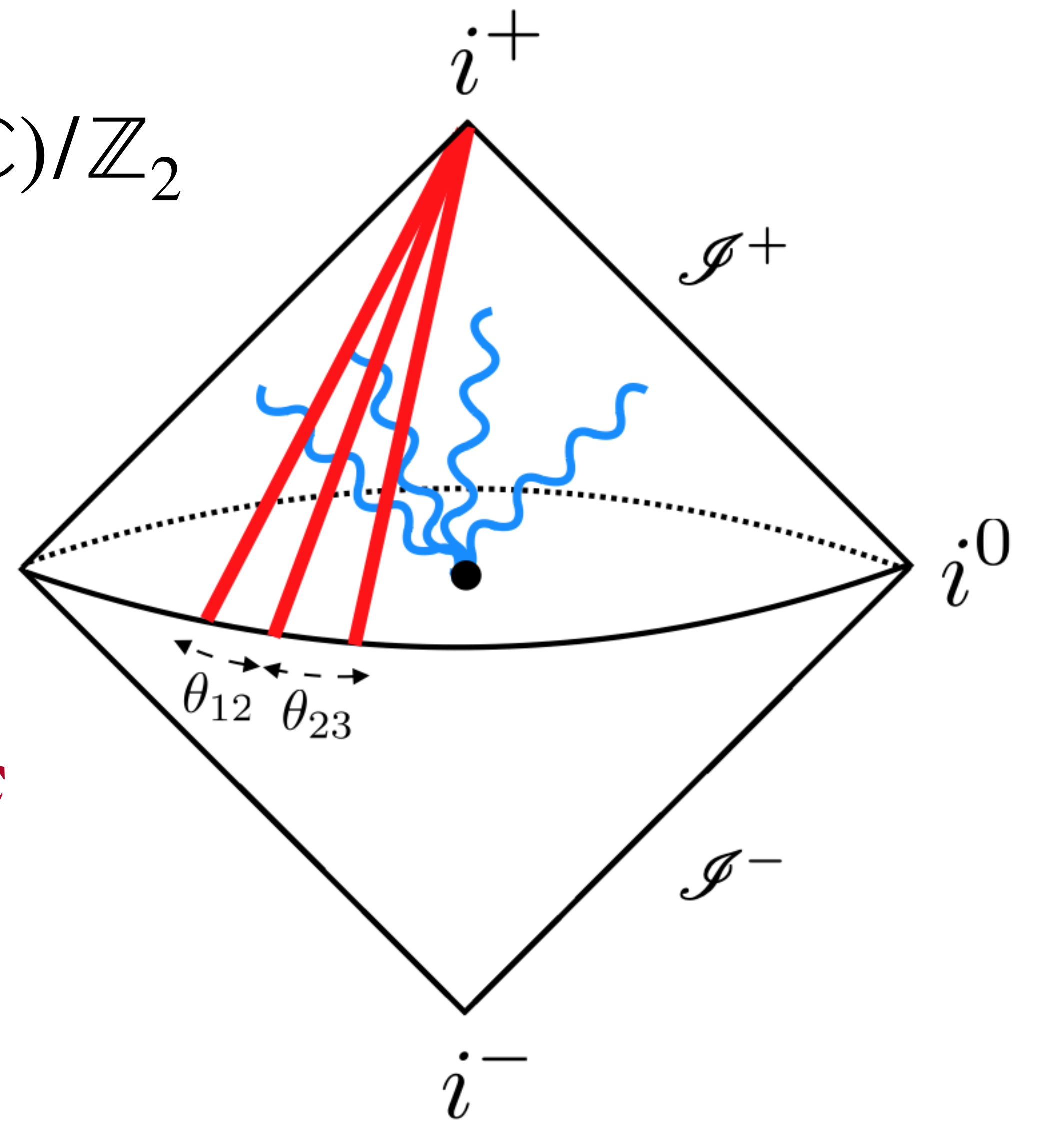



$SO^+(1,3) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$

[Guevara-Himwich-Pate-Strominger]

[Strominger]

***chiral algebras of asymptotic symmetries at tree-level***




A chiral algebra is a hol'c structure which can appear as algebra of symmetries of a full-fledged 2d CFT
Is that what's going on here?

[Costello, Costello-Li]

In work with Costello (2204.05301, 2201.02595), we showed that if a 4d theory admits a lift to a local holomorphic theory on twistor space, a chiral algebra can also control collinear singularities in its scattering amplitudes at loop-level

Failures of associativity in the chiral algebra at the quantum level are tied to gauge anomalies in twistor space
The 4d theory isn't inconsistent: this is like an obstruction to integrability

We focused on self-dual Yang-Mills, coupled to an axion with a quartic kinetic term
Similar considerations apply to self-dual gravity, or to, e.g., SD SU(Nc) YM w/ Nf=Nc flavors.

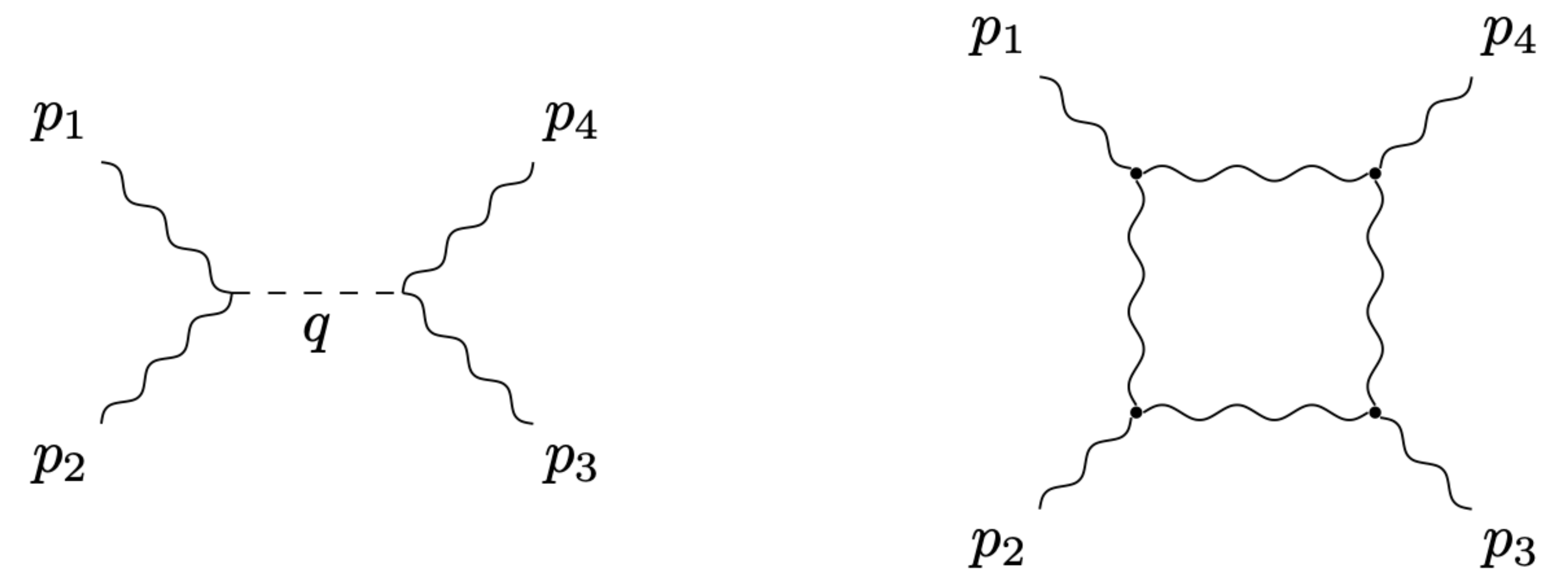
$$\int_{\mathbb{PT}} \mathrm{Tr}(\mathscr{B}\mathscr{F}^{(0,2)}(\mathscr{A})) \mapsto \int_{\mathbb{R}^4} \mathrm{Tr}(BF(A)_{-})$$

[Costello, Costello-Li]

$\mathscr{B} \in \Omega^{3,1}(\mathbb{PT}, \mathfrak{g})$
$\mathscr{A} \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$

$B \in \Omega^2_{-}(\mathbb{R}^4, \mathfrak{g})$

$\mathfrak{g} = su(2), su(3), so(8), e_{6,7,8}$

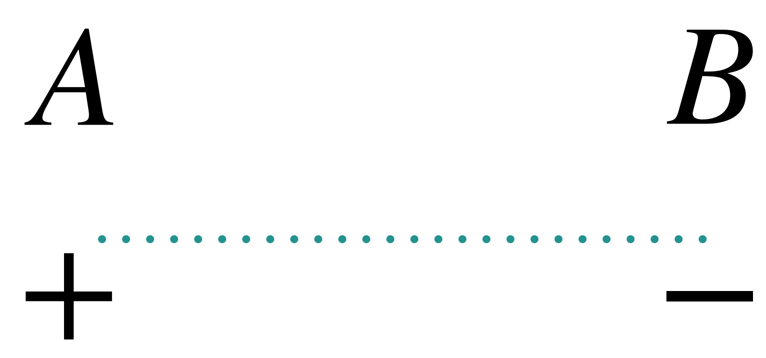


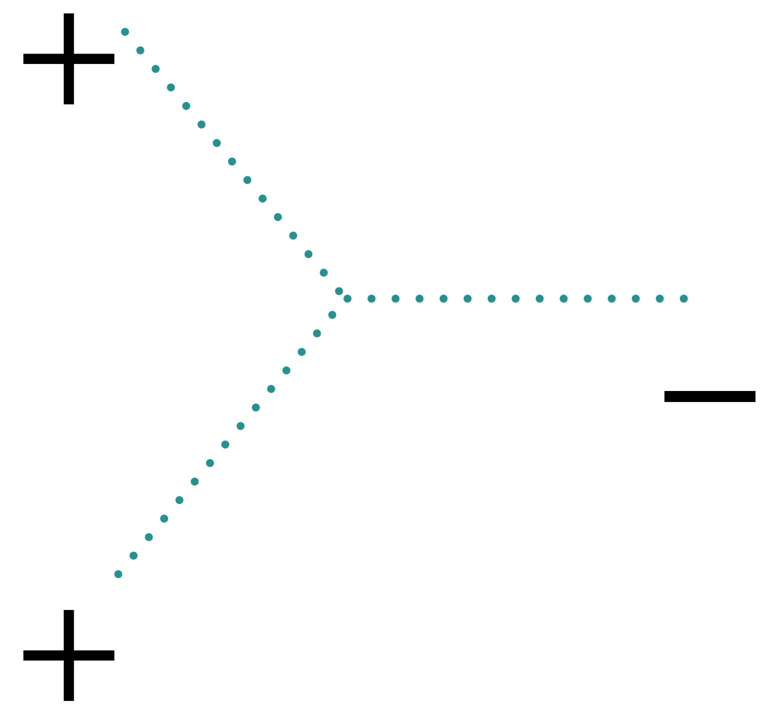

$$\frac{1}{2}\int (\partial^{-1}\eta)(\bar{\partial}\eta) + k\hat{\lambda}_g \int \eta \, \mathrm{tr}(\mathscr{A}\partial\mathscr{A}) \mapsto$$

$$\frac{1}{2}\int (\Delta\rho)^2 + k'\hat{\lambda}_g \int \rho(F \wedge F)$$

6d: free "closed string" (BCOV) sector

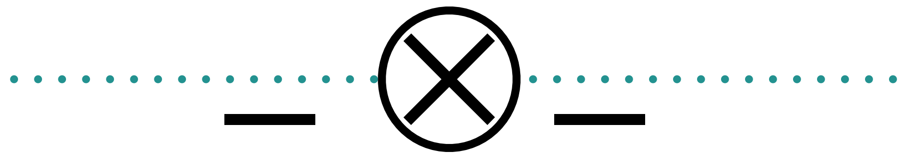
## Self-dual YM



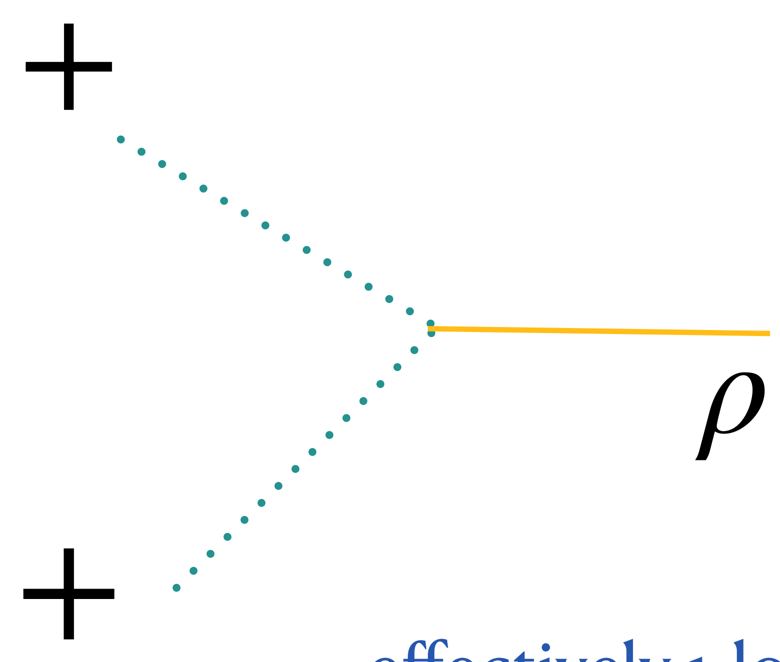

**form factors:**

$\mathrm{tr}(B^2)(x)$




L loops, N insertions →
N-L+1 (-) helicity, arbitrary (+) helicity gluons
in QCD (integrand)

## + axion




effectively 1-loop
by Green-Schwarz

## 4d OPE:

$$\mathrm{Tr}B^2(0)\mathrm{Tr}B^2(x_1)\ldots\mathrm{Tr}B^2(x_{n-1}) \sim \sum_i F_i(x_1,\ldots,x_{n-1})\mathcal{O}^i(0)$$

rational,
constrained by associativity

$$\mathrm{tr}(B^2)(0)\,\mathrm{tr}(B^2)(x) \sim \frac{1}{\|x\|^2} B^a_{\alpha_1\beta_1} B^b_{\alpha_2\beta_2} B^c_{\alpha_3\beta_3} f_{abc}\epsilon^{\beta_1\alpha_2}\epsilon^{\beta_2\alpha_3}\epsilon^{\beta_3\alpha_1}.$$

The associated chiral algebra (conformally soft modes on celestial sphere, governing collinear singularities) can be obtained from Koszul duality approaches on twistor space

conformal primary states on twistor space of neg. weight (on-shell gauge theory states)

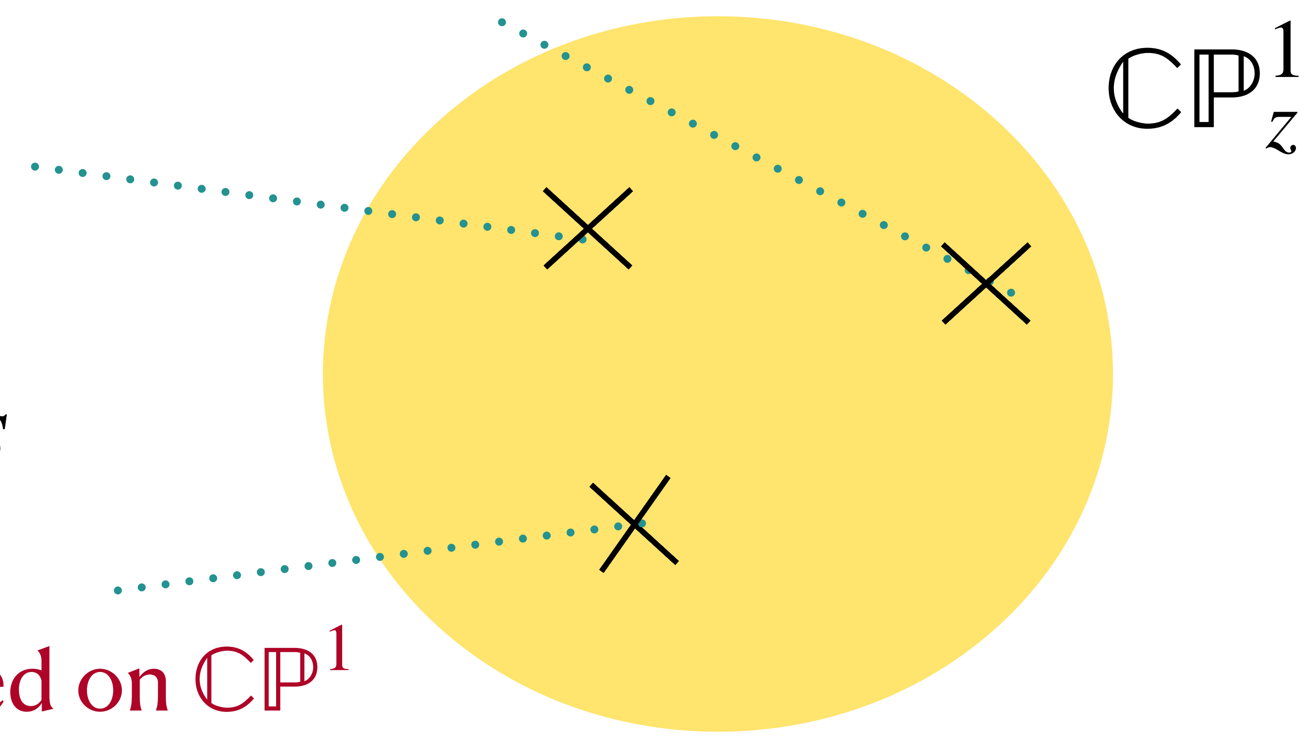
Penrose transform ↕

4d basis of conformal primary states w/ neg. weight

[Pasterski-Shao-Strominger]

$$J[r,s](z_i) \leftrightarrow \mathscr{A} = \delta_{z=z_i}(\tilde{\lambda}^{\dot 1})^r(\tilde{\lambda}^{\dot 2})^s$$

state in vacuum module = on-shell background field localized on $\mathbb{CP}^1$




it is a very large, non-unitary algebra

| Generator | Spin | Weight | $SU(2)_+$ representation | Field | Dimension |
|---|---|---|---|---|---|
| $J[m,n]$, $m,n \geq 0$ | $1-(m+n)/2$ | $(m-n)/2$ | $(m+n)/2$ | $A$ | $-m-n$ |
| $\widetilde{J}[m,n]$, $m,n \geq 0$ | $-1-(m+n)/2$ | $(m-n)/2$ | $(m+n)/2$ | $B$ | $-m-n-2$ |
| $E[m,n]$, $m+n>0$ | $-(m+n)/2$ | $(m-n)/2$ | $(m+n)/2$ | $\rho$ | $-m-n$ |
| $F[m,n]$, $m,n \geq 0$ | $-(m+n)/2$ | $(m-n)/2$ | $(m+n)/2$ | $\rho$ | $-m-n-2$ |

Table 1: The generators of our 2d chiral algebra and their quantum numbers. Dimension refers to the charge under scaling of $\mathbb{R}^4$.

There is a prescription for obtaining the OPEs, not just classically but including possible deformations

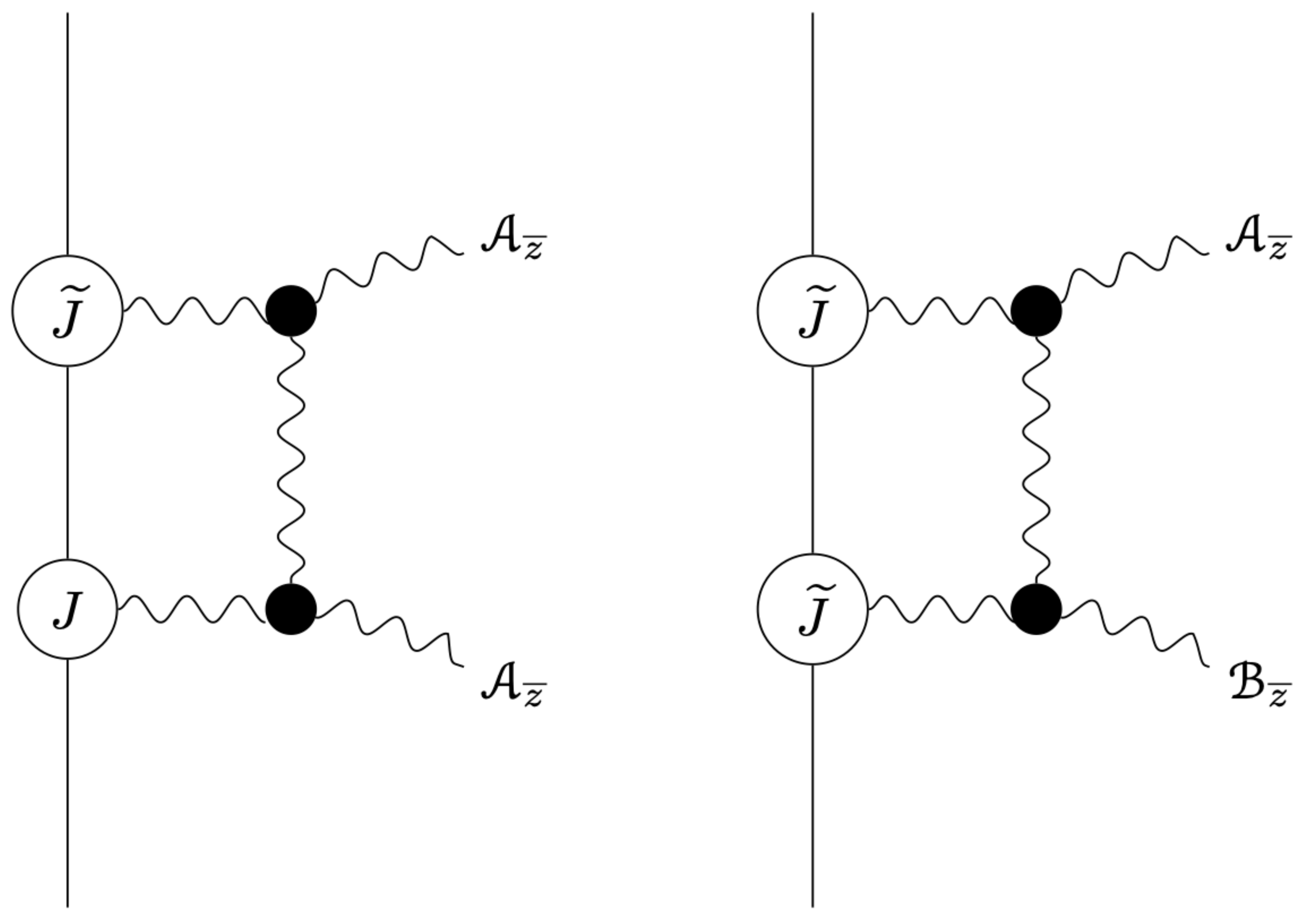
$$PExp \sum_{r,s\geq 0} \int_{\mathbb{CP}^1_z} (\partial^r_{\tilde{\lambda}^{\dot{1}}} \partial^s_{\tilde{\lambda}^{\dot{2}}} \mathscr{B}^a_{\bar{z}}) \tilde{J}_a[r,s](z)$$

$$PExp \sum_{r,s\geq 0} \int_{\mathbb{CP}^1_z} (\partial^r_{\tilde{\lambda}^{\dot{1}}} \partial^s_{\tilde{\lambda}^{\dot{2}}} \mathscr{A}^a_{\bar{z}}) J_a[r,s](z)$$

gauge inv't couplings to arbitrary defect

↔ Hom from Koszul dual algebra into defect algebra

**Quantum deformations:**




**OPEs among currents on defect by imposing gauge (BV-BRST) invariance**

**→ universal or "Koszul dual" algebra**
**Tree level: recover current algebra for gauge symmetry**

**BRST variation of all diagrams at given loop order must be zero**

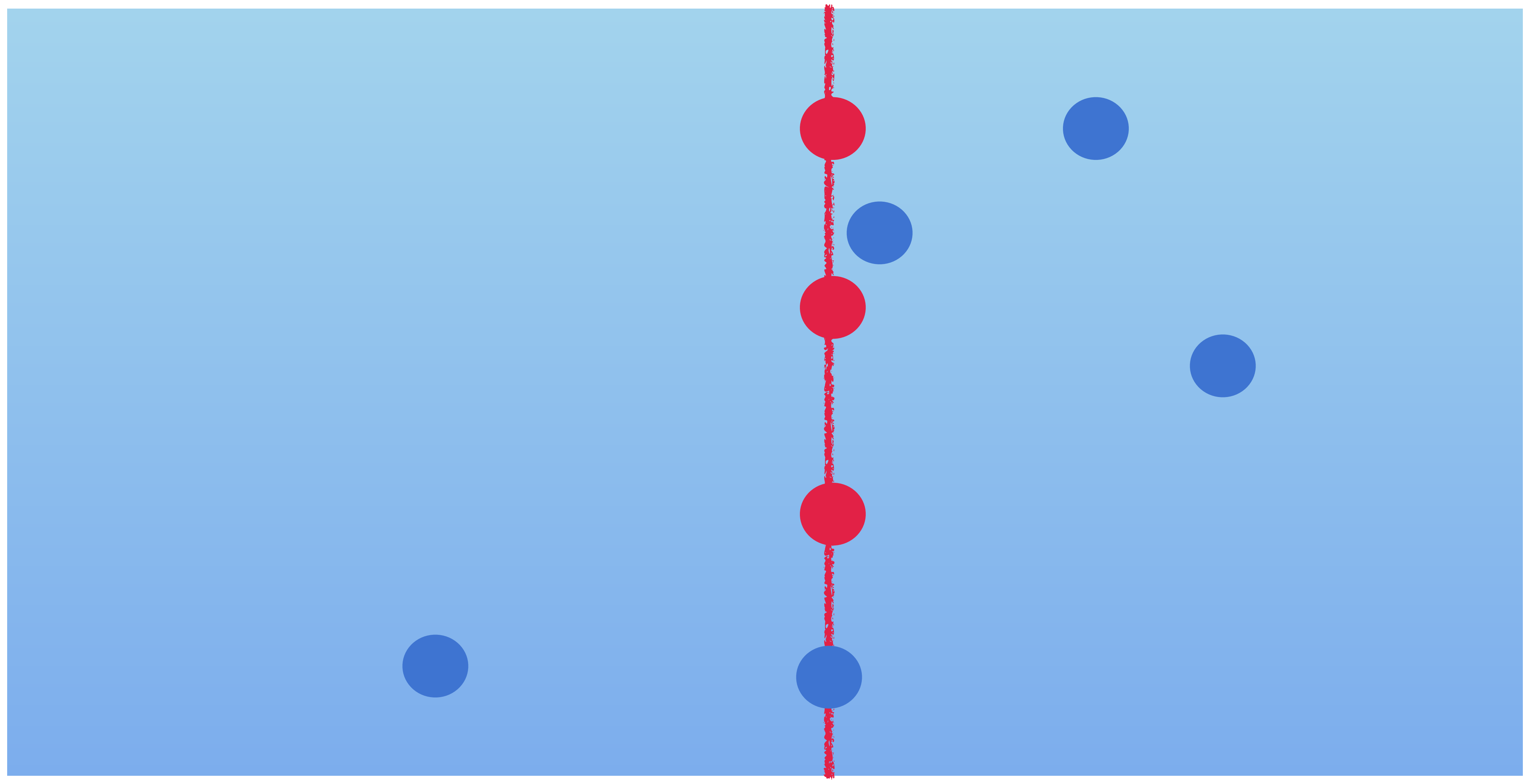
## A brief interlude on Koszul duality
## (for associative algebras)

*for more details, see NMP-Williams: 2110:10257*

# Koszul duality: Coupling an order-type defect to a Lagrangian theory
## Focus on line defects

$S = S_A + S_B + S_{AB}$

generalized Wilson line



*differential graded algebras*

$\mathscr{A} \otimes \mathscr{B}$

$\delta = \delta_A + \delta_B$

What are the constraints on $S_{AB}$ (actually PExp) imposed by gauge (BRST) invariance?

$S_{AB} \leftrightarrow \alpha \in \mathrm{MC}(\mathscr{A} \otimes \mathscr{B})$

Koszul dual $\alpha : \mathscr{A}^! \to \mathscr{B}$

$\delta\alpha + \alpha \star \alpha = 0$

Claim: $\mathscr{A}^!$ is the algebra of operators on the "universal line defect" for theory A

# The concrete setting: twisted theories

Begin w/ a supersymmetric field theory on flat space, Euclidean signature

General twisted theories: $\delta \to \delta + Q =: \delta_Q$

$$Q^2 = 0$$ choice of nilpotent supercharge

Compute cohomology with respect to the twisted BRST differential

cf.

$$\phi : Spin(d) \to G_R$$

$$T^\phi = \{Q, ..\}$$

to obtain topological field theories

$\to E_d$-algebras

[Lurie,…]

$$\{Q, Q_\mu\} = iP_\mu$$

- A- and B-twist of 2d $\mathcal{N} = (2,2)$ theories
- Half-twist of 2d $\mathcal{N} = (0,2)$ theories (CDOs, cdR,..)
- Holomorphic twist of 4d $\mathcal{N} = 2$ theories (chiral algebra)
- Holomorphic-topological twist of 3d $\mathcal{N} = 2$ theories
- $\vdots$

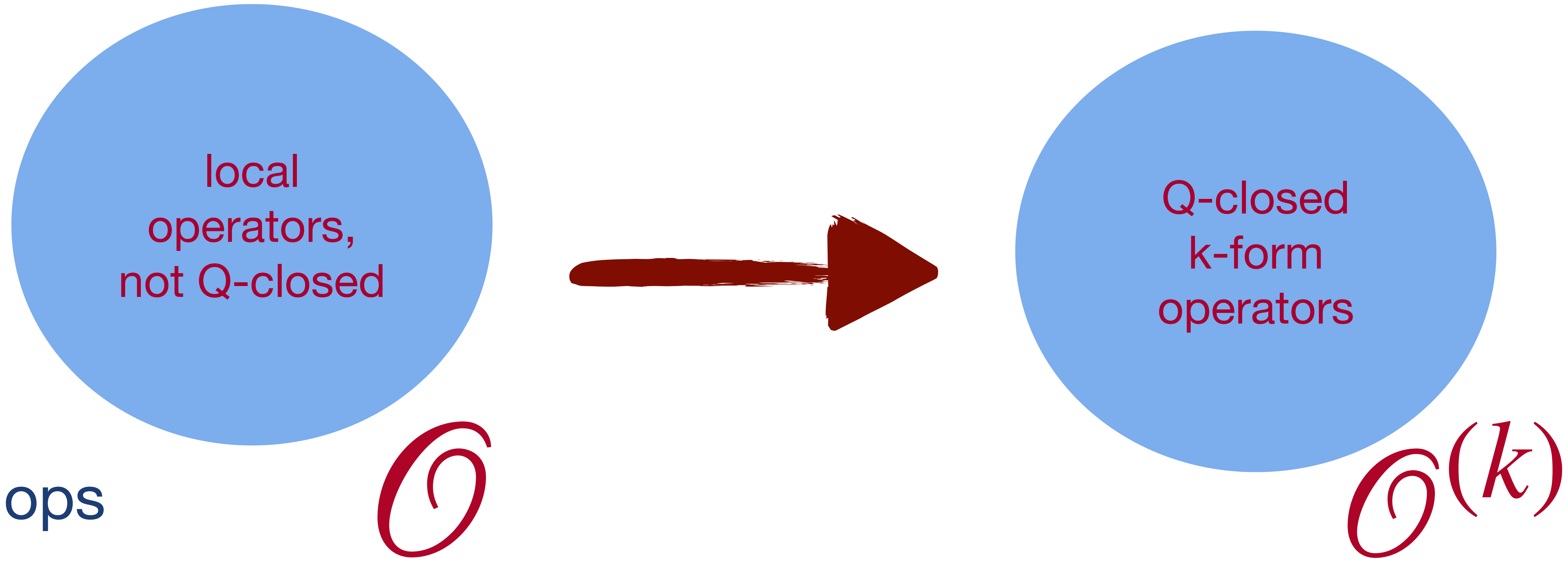
$E_1/A_\infty$-algebra: topological theory in 1d

$L_\infty, E_2$-algebra: topological theory in 2d

vertex/chiral algebra: holomorphic theory in 2d

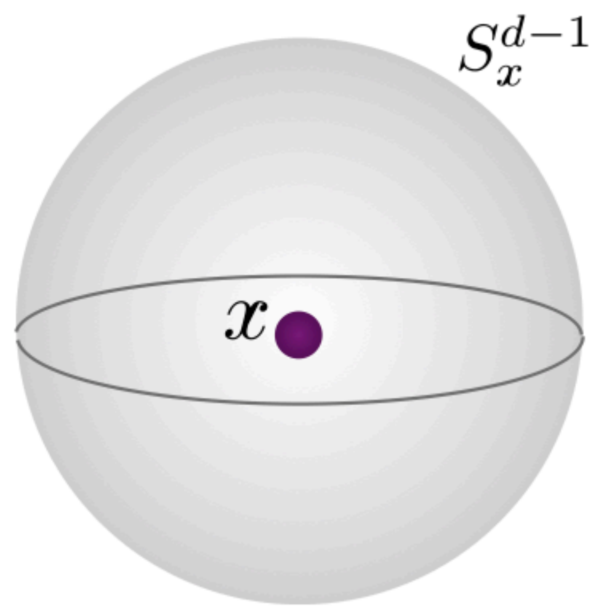
# Descent

[Witten, Moore-Witten,…]




- higher brackets/products on algebras of local ops
- generate couplings/deformations

$$\left\{Q, Q_\mu\right\} = iP_\mu$$



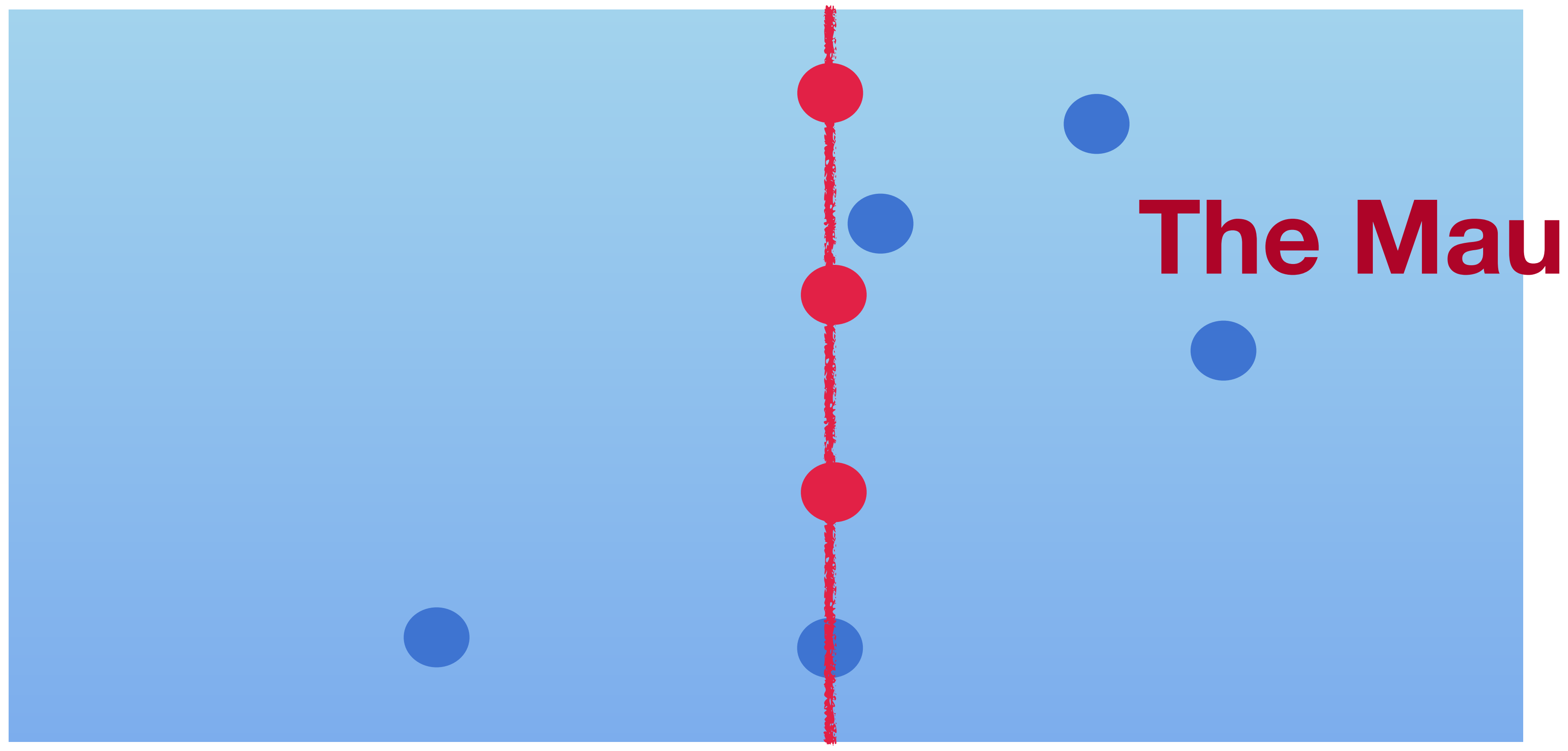

[Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke]

$$\mathcal{O}^{(k)} := \frac{1}{k!} Q_{\mu_1}(\ldots Q_{\mu_k}(\mathcal{O}))dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$$

$$Q(O^{(k)}) = d\mathcal{O}^{(k-1)} \rightarrow \int_{\gamma_k} \mathcal{O}^{(k)}$$ is Q-closed for $\gamma_k$ a cycle

($d \rightarrow d_A$ in gauge thys)

Hol'c-top'l: [Yagi, Costello-Dimofte-Gaiotto]



# The Maurer-Cartan equation & bulk/defect couplings

$$\left\{Q,\hat{Q}\right\} = iP_t$$

translations along $\mathbb{R}_t$ are cohomologically trivial

$\therefore\ \mathscr{A}$ (homotopy) associative algebra

Add in to a top'l QM, thy B with op alg $\mathscr{B}$ (homotopy associative)

$$S_A + S_B$$

$$\mathrm{PExp}\left(\int_{\mathbb{R}_t} \mathcal{O}^{(1)}\right)$$

$$\mathcal{O} \in \mathscr{A} \otimes \mathscr{B}$$

local operator
coh. degree/ghost number 1

Claim: This ``generalized Wilson line"/bulk-defect coupling is BRST invariant to all orders in perturbation theory iff

$$\delta_Q \mathcal{O} + \mathcal{O} \star \mathcal{O} = 0$$ [Costello-Paquette, Paquette-Williams]

$$\delta_Q \mathcal{O} + \mu_2(\mathcal{O},\mathcal{O}) + \mu_3(\mathcal{O}^3) + \ldots = 0$$ [Gaiotto-Oh]

# Koszul Duality & the Universal Top'l Line Defect

$$\alpha \in \mathrm{MC}(\mathscr{A} \otimes \mathscr{B})$$

s.t.

$$(\epsilon \otimes 1_{\mathscr{B}})\alpha = 0 \in \mathscr{B}$$

$$\longleftrightarrow$$

$$\phi : \mathscr{A}^! \to \mathscr{B}$$

$$\mathscr{A}^! := \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathbb{C}_\epsilon, \mathbb{C}_\epsilon)$$

[Loday-Vallette, Lurie, …]

$\mathscr{A}, \mathscr{A}^!$ mutually commuting symmetries of vacuum

$$\epsilon : \mathscr{A} \to \mathbb{C}$$

augmentation
(choice of massive vacuum
s.t. compactified thy trivial)

$$\mathbb{R}_t \times M_\epsilon \to \mathbb{R}_t$$

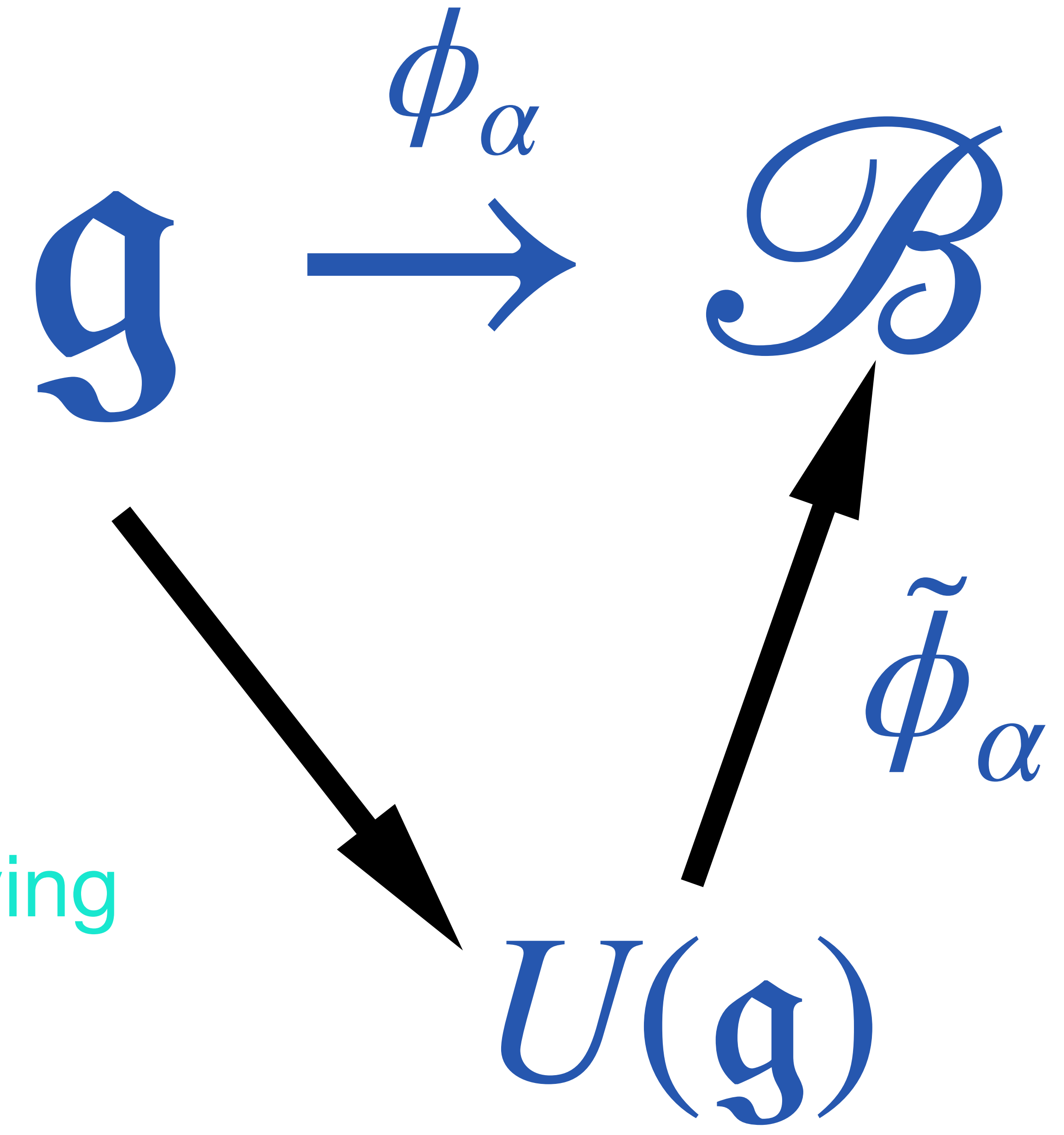
Require coupling **preserves** vacuum

$$\alpha \in g^* \otimes \mathscr{B} \subset C^*(\mathfrak{g}) \otimes \mathscr{B}[1]$$

$$\alpha_{univ} \in \mathscr{A} \otimes \mathscr{A}^!$$

$$\alpha = \tilde{\phi}_\alpha(\alpha_{univ})$$

any other coupling can be obtained from the
universal coupling by application of an algebra-preserving
map $\tilde{\phi}_\alpha : \mathscr{A}^! \to \mathscr{B}$




example commuting diagram

# Curved Koszul duality & (Twisted) AdS/CFT

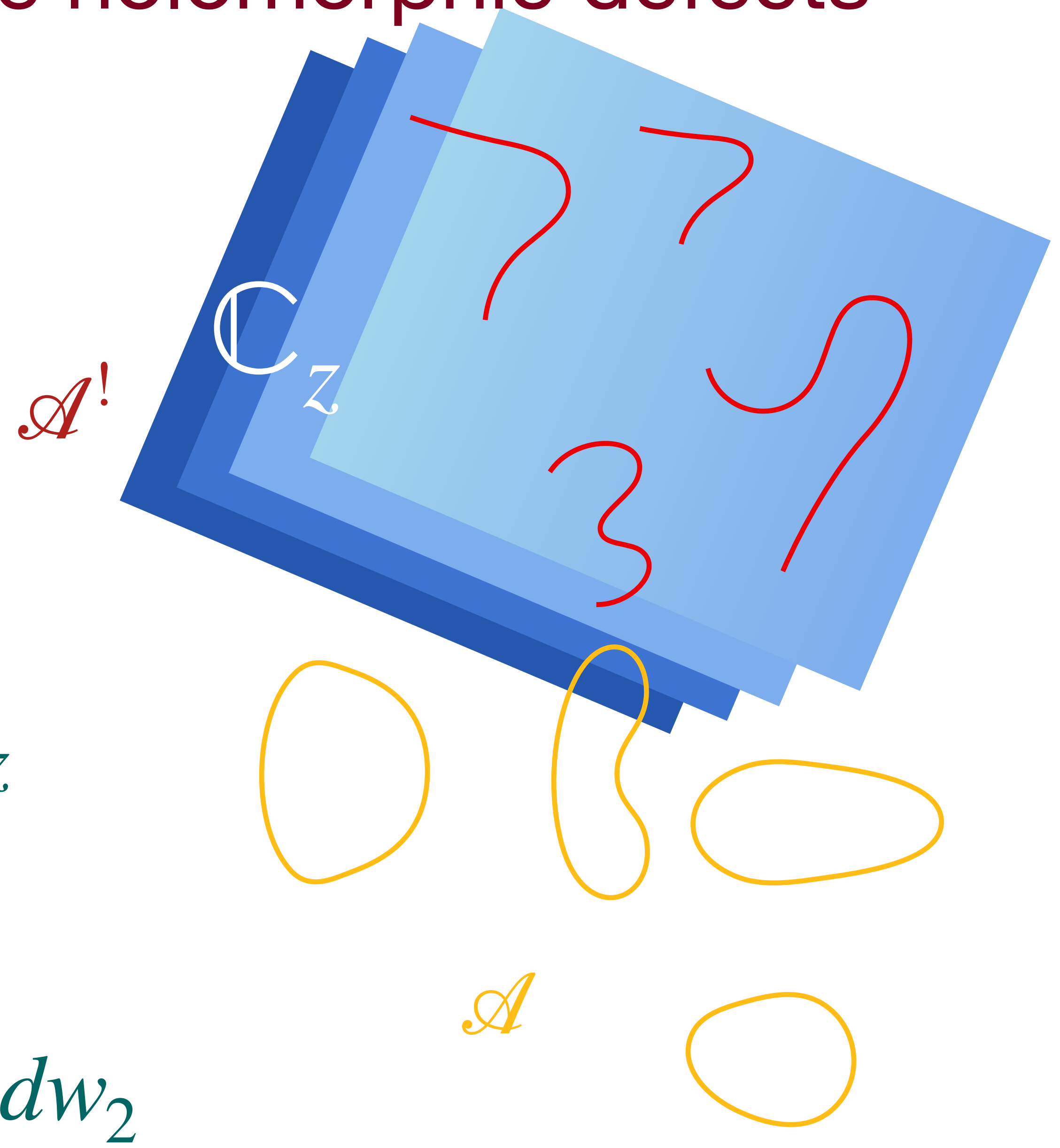
[Costello-Paquette]

For vertex algebras: no math formalism, but proceed via BRST invariance of order-type holomorphic defects

D-branes, even twisted, can **backreact** on the geometry.
In at least some special examples, one can fully incorporate this effect.
e.g. B-model string theory:
-holomorphic CS theory
-Kodaira-Spencer "supergravity"

$$S \to S + S_{BR}$$

$\delta_{gauge} S_{BR} \neq 0$ **on** $\mathbb{C}_z$




$$S_{BR} = \frac{1}{2}\int_{\mathbb{C}^3} \mu\mu_{BR}\mu dz dw_1 dw_2 \sim N \int_{\mathbb{C}^3} \frac{\epsilon^{ij}\bar{w}_i d\bar{w}_j}{(w_1\bar{w}_1 + w_2\bar{w}_2)^2} \mu\partial_z\mu dz dw_1 dw_2$$



- coupling to identity element of vertex algebra
- total BRST variation, including this contribution, must cancel
- curved Koszul duality (source term in MC equation)

In classical ($N \to \infty$) limit, this was used to compute central extensions of boundary chiral algebra in an example of twisted holography (AdS3/CFT2)

Loop corrections for these theories are in progress [Costello-Paquette-Williams]

see also [Costello, Costello-Li, Costello-Gaiotto,…]

# End of Interlude

A more bottom-up perspective: the OPEs are related to 4d collinear singularities, which are known in detail in Yang-Mills.

Compute the associator (say at tree or 1-loop order) and see if it vanishes!

$$J^a[r,s](0)J^b[t,u](z) \sim \frac{1}{z} f^{ab}_c J^c[r+t, s+u](0)$$
$$J^a[r,s](0)\widetilde{J}^b[t,u](z) \sim \frac{1}{z} f^{ab}_c \widetilde{J}^c[r+t, s+u](0)$$

**Tree-level**

**Includes level-0 Kac-Moody algebra for $\mathrm{Maps}(\mathbb{C}^2, \mathfrak{g})$**

[Guevara-Himwich-Pate-Strominger]

$$J^a[r,s](0)E[t,u](z) \sim \frac{1}{z}\frac{(ts-ur)}{t+u}\widetilde{J}^a[t+r-1, s+u-1](0)$$
$$J^a[r,s](0)F[t,u](z) \sim -\frac{1}{z}\partial_z \widetilde{J}^a[r+t,s+u](0) - \frac{1}{z^2}(1+\frac{r+s}{t+u+2})\widetilde{J}^a[r+t,s+u](0)$$
$$J^a[r,s](0)J^b[t,u](z) \sim \frac{1}{z}K^{ab}(ru-st)F[r+t-1,s+u-1](0)$$
$$-\frac{1}{z}K^{ab}(t+u)\partial_z E[r+t,s+u](0) - \frac{1}{z^2}K^{ab}(r+s+t+u)E[r+t,s+u](0).$$

**Failure of associativity in pure SDYM theory in one-loop**

**Axion field necessary for its restoration**

[Bern-Dixon-Kosower]

$$\mathrm{Split}^{[1]}_{+}(a^+, b^+) = -\frac{N_c}{96\pi^2}\frac{[ab]}{\langle ab\rangle^2}$$

$$J_a[1,0](0)J_b[0,1](z)$$
$$= -\frac{1}{2\pi i z} C K^{fe}(f^c_{ae}f^d_{bf} + f^d_{ae}f^c_{bf}) : J_c[0,0]\widetilde{J}_d[0,0] :$$
$$+ \frac{1}{2\pi i z}\tfrac{1}{2} D f^c_{ab}\partial_z \widetilde{J}_c(0) + \frac{1}{2\pi i z^2} D f^c_{ab}\widetilde{J}_c(0).$$

**Quantum deformation**

C, D are known & fixed by ***anomaly coefficient in 6d***

For self-dual YM, no further collinear singularities at higher loops.
Costello and I further showed that a 4d theory with a twistorial uplift has form factors which are isomorphic to chiral correlators of the 2d theory.
In the case of Yang-Mills, this leads to some cute 2d expressions for certain 4d amplitudes

$$P^{\alpha\dot{\alpha}} =: \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

$$\lambda^\alpha \equiv (1,z)$$

| **2d chiral algebra** | **4d theory** |
|---|---|
| conf. primary generators | conf. primary states (boost eigenbasis) |
| OPEs | collinear limits |
| conformal blocks (cf. CS/WZW) | local operators |
| correlation functions | form factors |

$$\langle tr(B^2)\,|\,\tilde{J}^a(z_1)\tilde{J}^b(z_2)J^c(z_3)\rangle = \frac{z_{12}^3}{z_{13}z_{23}} f^{abc}$$

Momentum eigenbasis:

$$\langle ij\rangle = z_i - z_j$$

$$J(\tilde{\lambda}, z) = \sum_{r,s} \omega^{r+s} \frac{(\tilde{\lambda}^{\dot{1}})^r (\tilde{\lambda}^{\dot{2}})^s}{r!s!} J[r,s](z)$$

$$\langle tr(B^2)\,|\,J^{a_1}(z_1)\ldots\tilde{J}^{a_2}(z_i)\ldots\tilde{J}^{a_j}(z_j)\ldots J^{a_n}(z_n)\rangle = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} tr(t^{a_1}\ldots t^{a_n}) + \text{ permutations}$$

# *4d form factors as computed by 2d chiral correlators*

$P^{\alpha\dot{\alpha}} =: \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$

$\lambda^\alpha \equiv (1, z)$

**tree-level**

$\langle tr(B^2) \,|\, \tilde{J}(z_1)\tilde{J}(z_2)J(z_3)\ldots J(z_n)\rangle \leftrightarrow$ color ordered MHV amps [Parke-Taylor]

$\frac{1}{|x|^2}\langle tr(B^3) \,|\, \tilde{J}(z_1)\tilde{J}(z_2)\tilde{J}(z_3)J(z_4)\ldots J(z_n)\rangle \leftrightarrow$ NMHV amps in CSW form [Cachazo-Svrcek-Witten]

**1-loop (axion comes in)**

$\langle (\Delta\rho)^2 \,|\, J_{a_1}(\tilde{\lambda}_1, z_1)\ldots J_{a_n}(\tilde{\lambda}_n, z_n)\rangle \leftrightarrow$ all-(+) one-loop in SDYM/QCD [Mahlon, Bern et. al.,...]

$$\langle tr(B^2) \,|\, \tilde{J}_{a_1}(\tilde{\lambda}_1, z_n) J_{a_2}(\tilde{\lambda}_2, z_2)\ldots J_{a_n}(\tilde{\lambda}_n, z_n)\rangle =$$

$$\frac{1}{192\pi^2}\sum_{2\le i<j\le n}\frac{[ij]\langle 1i\rangle^2\langle 1j\rangle^2}{\langle ij\rangle\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\mathrm{Tr}(t_1\ldots t_n) + \text{ perms in } S_{n-1}$$

[Costello-NP]

cf. [Dixon-Glover-Khoze] for QCD

SDYM + axion isn't the only 4d theory with a nice twistorial uplift…
Self-dual gravity version of the previous analysis, see recent work by Bittleston
WZW4 with G = SO(8) + quartic "Kahler scalar" field
Costello-Li show this follows from a topological string analogue of Green-Schwarz mechanism for type I string

Recall [Donaldson, Nair, Losev-Moore-Nekrasov-Shatashvili]:

$g : M \to SO(8)$

$$\mathscr{L} = \frac{N}{8\pi^2}\int_M \partial\bar{\partial}K \wedge \mathrm{tr}(g^{-1}\partial g \wedge g^{-1}\bar{\partial}g) - \frac{N}{24\pi^2}\int_{M\times[0,1]} \partial\bar{\partial}K \wedge \mathrm{tr}(\tilde{g}^{-1}d\tilde{g})^3$$

$N \in \mathbb{Z}_+$

5d analogue of WZ term

4d analogue of KM level

Classically, a gauge-fixed formulation of SDYM w/ $A = -\bar{\partial}g g^{-1}$

``Closed string/gravitational" sector is the theory of a scalar controlling perturbations of the Kahler potential
(See Costello's Strings 2021 talk)

*e.o.m.:* $R(K+\rho) = 0$

$K \mapsto K + \rho$

Again, a special, integrable 4d theory. But I claim this is the seed for a nice toy model of holography in asymptotically flat spacetimes!
(Work w/ Costello & Sharma, 2208.14233 + to appear)

## Let's use the origin of the 6d anomaly cancellation from topological type I open+closed string theory on twistor space

Add **additional** N D1-branes on top of $\mathbb{CP}^1$ and ``backreact'', study resulting open/closed duality a la **twisted holography**

$\mu^a = 0$

In type I Kodaira-Spencer theory, ``backreaction''is a deformation of complex structures

$$Z_0 = \mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathbb{CP}^1$$

$\mu^1$ $\mu^2$ $z$

$$z \mapsto \frac{1}{z} \implies \mu^\alpha \mapsto \frac{\mu^\alpha}{z}$$

$$\bar\partial = \mathrm{d}\bar z \frac{\partial}{\partial \bar z} + \mathrm{d}\bar\mu \cdot \frac{\partial}{\partial \bar\mu}$$

$$\bar\partial V + \frac{1}{2}[V, V] = (2\pi)^2 N \,\bar\delta^2(\mu)\, z^2 \frac{\partial}{\partial z}$$

$$V = N \frac{\bar\mu^{\bar 1}\,\mathrm{d}\bar\mu^{\bar 2} - \bar\mu^{\bar 2}\,\mathrm{d}\bar\mu^{\bar 1}}{||\mu||^4}\, z^2 \frac{\partial}{\partial z}$$

$$\mathcal{L}_V \Omega_0 = 0 \quad \text{away from } \mu^\alpha = 0$$

$$\Omega_0 = \frac{\mathrm{d}z\,\mathrm{d}\mu^1\,\mathrm{d}\mu^2}{z^2}$$

4d: **Burns metric**

$$u^\alpha = (u^1, u^2) \in \mathbb{C}^2\,, \qquad ||u||^2 = |u^1|^2 + |u^2|^2$$

$$ds^2 = ||du||^2 + \frac{|u^1 du^2 - u^2 du^1|}{||du||^4}$$

Note: Burns space is **asymptotically flat**

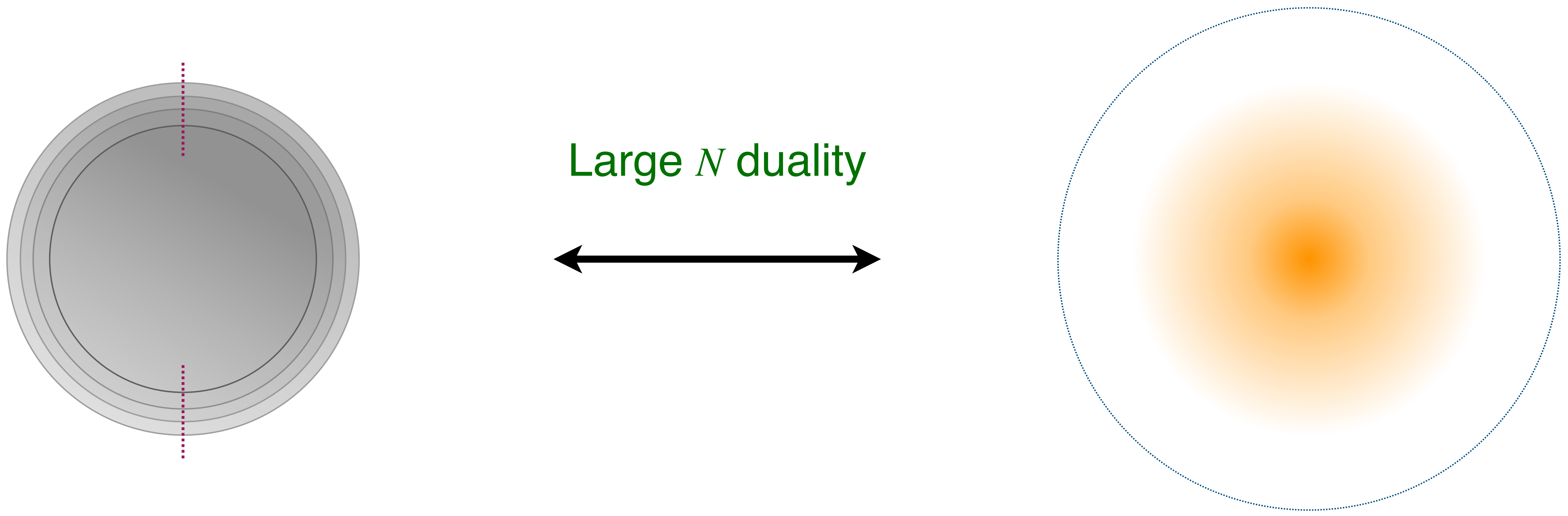
Moreover, worldvolume theories on D1-branes in the topological string are well known.
Proceeding a little more carefully and putting the pieces together ultimately gives us:

$\mathbb{CP}^1$

$N$ D1 branes in twistor space

$\mathscr{B}_N$

Burns space
4d, Euclidean signature, asymptotically flat




$\beta\gamma$ system with
Sp$(N)$ gauge symmetry
+ SO$(8)$ flavour
+ defects at $0, \infty$

WZW4 for G=SO$(8)$
+
self-dual scalar-flat gravity

Here: dual chiral algebra understood at **finite N**
In principle, exact description of collinear limits of 4d theory at finite coupling, from 2d chiral algebra at finite N

## To start, we have checked 2 & 3-pt funs in this proposed duality when $N \to \infty$
## For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:

**Open string-sector states in chiral algebra**

$$X_{\alpha ij} \in \mathbb{C}^2 \otimes \textstyle\bigwedge^2_{\text{t.f.}} \mathbb{C}^{2N}$$

$$I_{ri} \in \mathbb{C}^8 \otimes \mathbb{C}^{2N}$$

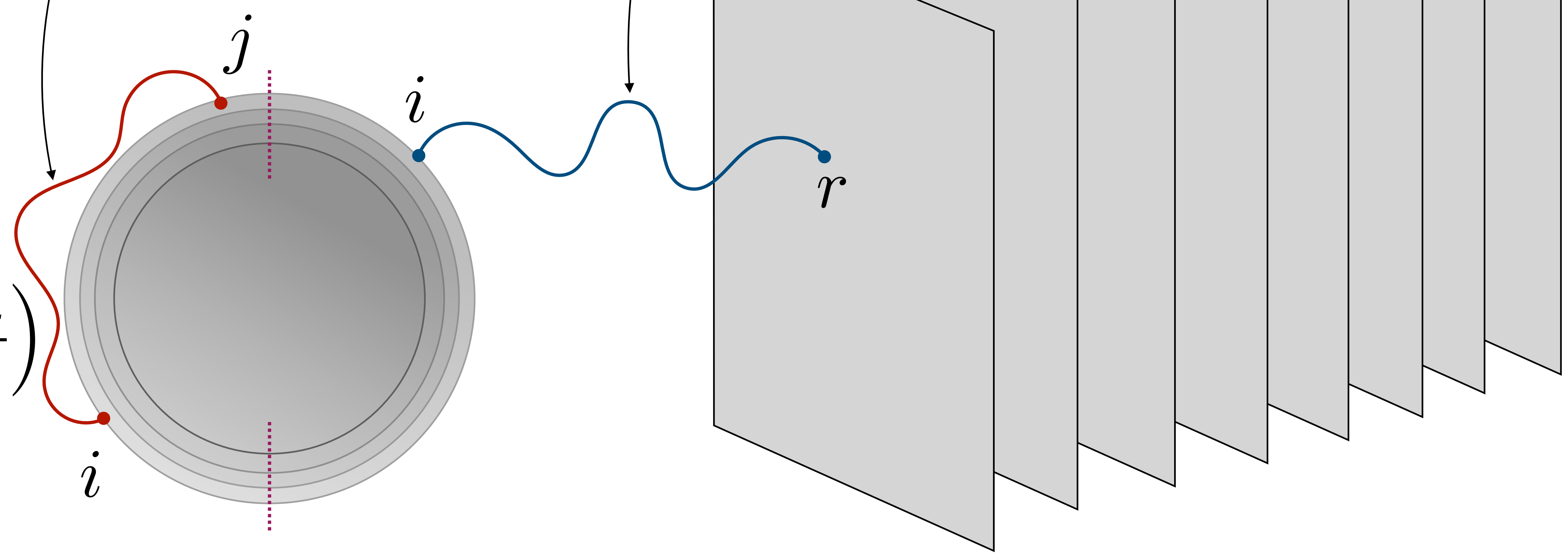
**+ ghosts for BRST reduction**

$$I_{im}(z_1) I_{jn}(z_2) \sim \frac{\delta_{ij}\omega_{mn}}{z_{12}}$$

$$X_{\alpha mn}(z_1) X_{\beta rs}(z_2) \sim \frac{\epsilon_{\alpha\beta}}{z_{12}} \left( \omega_{m[r|}\omega_{n|s]} - \frac{\omega_{mn}\omega_{rs}}{2N} \right)$$

Give rise to $X_{\alpha ij}$

Give rise to $I_{ri}$




**States in WZW4 on Burns space**

$$g = \exp(\phi\, \mathfrak{t})$$

**linearized field eqn**

$$\triangle \phi = 0$$

**closed form**
**sol'n for "momentum eigenstates"**

**(Recovers $e^{ik\cdot x}$ when $N = 0$)**

$N$ D1 branes at $\mu^\alpha = 0$ 4 space-filling ``D5" branes (+ O-plane)

Chiral worldvolume actions follow from Witten's prescriptions. [Witten '95]
[Costello, Gaiotto '18]

- Dictionary between soft modes of states and symmetry currents in the dual CFT

$$\tilde{\lambda}_\alpha = \omega(1, \tilde{z})$$

Soft expansion

$$\phi(\omega, z, \tilde{z}) = \frac{1}{z} \sum_{p=0}^{\infty} (\mathrm{i}\,\omega)^p \sum_{k+l=p} \frac{\tilde{z}^l}{k!\,l!}\, \phi[k,l](z)$$

Dictionary

$$\frac{1}{z}\,\phi[k,l](z)\ \mathrm{t}_{rs} \longleftrightarrow \langle I_r, X_1^{(k} X_2^{l)} I_s\rangle(z)$$

Examples of soft modes

$$\phi[0,0] = 1\,, \qquad \phi[1,0] = u^1 - z\,\bar{u}^{\bar{2}}\,, \qquad \phi[0,1] = u^2 + z\,\bar{u}^{\bar{1}}$$

$$\phi[1,1] = \phi[0,1]\phi[1,0] + \frac{Nz}{2}\,\frac{|u^1|^2 - |u^2|^2}{|u^1|^2 + |u^2|^2}$$

- chiral algebra OPE computations easily done in planar limit by Wick contractions

- In bulk, Euclidean amplitudes computed via on-shell effective action, as in standard AdS/CFT computations

$$J_a[\tilde{\lambda}_1](z_1)\, J_b[\tilde{\lambda}_2](z_2) \sim \frac{f_{ab}{}^c}{z_{12}}\, J_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_2) - \frac{[1\,2]\, f_{ab}{}^c}{z_{12}^2} \int_0^1 \mathrm{d}\omega_1 \int_0^1 \mathrm{d}\omega_2\ J_c[\omega_1\tilde{\lambda}_1 + \omega_2\tilde{\lambda}_2](z_2)$$

$$\updownarrow$$

$$\phi_1 \cdot \phi_2 \sim \frac{f_{a_1 a_2}{}^c}{z_{12}}\, \phi_c(z_2, \tilde{\lambda}_1 + \tilde{\lambda}_2) - \frac{[1\,2]\, f_{a_1 a_2}{}^c}{z_{12}^2} \int_0^1 \mathrm{d}\omega_1 \int_0^1 \mathrm{d}\omega_2\ \phi_c(z_2, \omega_1\tilde{\lambda}_1 + \omega_2\tilde{\lambda}_2) + O([1\,2]^2)\,. \quad ($$

Our toy top-down example of asymptotically-flat holography has passed standard checks.
Next: Go beyond the planar limit, study states of dim'n $\mathcal{O}(\sqrt{N})$, $\mathcal{O}(N)$, fully flesh out embedding in physical string, etc.

This gives a concrete toy-model of a ``celestial holography''-type correspondence,
analogous to the supersymmetric sectors of AdS/CFT we have been studying in twisted holography program

Unfortunately, I only know how to build associative chiral algebras using these methods for theories that are integrable/self-dual in 4d. Relatedly, the closed-string sector of the bulk theory, which only captures Kahler potential fluctations, means our gravitational sector is rather poor.

Perhaps the first step towards 4d asymptotically flat holography in more physically interesting setups would be to find a chiral algebra dual for self-dual Einstein gravity (perhaps coupled to SDYM).

We know how to cancel the twistorial anomaly there, thanks to work of Bittleston-Sharma-Skinner.

Note that Burns space can be viewed as an Einstein-Maxwell instanton…

non-unitarity, operator product associativity, integrability, etc. are all connected,
insight for how to move beyond the twisted realm?


Thank you!

To start, we have checked 2 & 3-pt funs in this proposed duality when $N \to \infty$

For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:

$$g = \exp(\phi\, \mathfrak{t})$$

$$\triangle \phi = 0$$

**“Momentum eigenstates”**

$$\phi = \frac{1}{z}\, \mathrm{e}^{\frac{\mathrm{i}}{2}([u\,\tilde{\lambda}]+z[\hat{u}\,\tilde{\lambda}])} \left( \cos\frac{\psi}{2} + \frac{\mathrm{i}([u\,\tilde{\lambda}]+z[\hat{u}\,\tilde{\lambda}])}{\psi} \sin\frac{\psi}{2} \right)$$

$\downarrow$

$k \cdot x$

$$\xrightarrow{N \to 0} \quad \frac{1}{z}\, \mathrm{e}^{\mathrm{i}k\cdot x}$$

$$\psi = \sqrt{([u\,\tilde{\lambda}]+z[\hat{u}\,\tilde{\lambda}])^2 + \frac{4N}{||u||^2}[u\,\tilde{\lambda}][\hat{u}\,\tilde{\lambda}]}$$

$$[u\,\tilde{\lambda}] = \tilde{\lambda}_1\, u^1 + \tilde{\lambda}_2\, u^2\,, \quad [\hat{u}\,\tilde{\lambda}] = \tilde{\lambda}_2\, \bar{u}^{\bar{1}} - \tilde{\lambda}_1\, \bar{u}^{\bar{2}}$$

$$X_{\alpha ij} \in \mathbb{C}^2 \otimes {\textstyle\bigwedge}^2_{\text{t.f.}} \mathbb{C}^{2N}$$

$$I_{ri} \in \mathbb{C}^8 \otimes \mathbb{C}^{2N}$$

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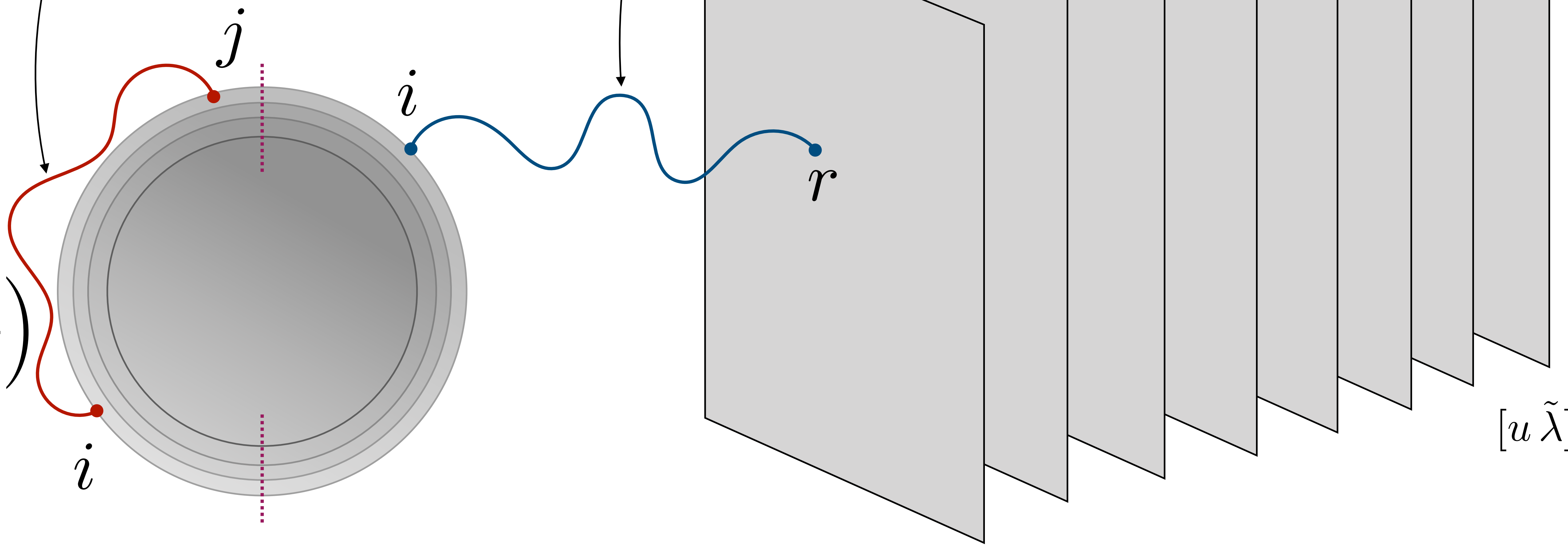
Give rise to $X_{\alpha ij}$

Give rise to $I_{ri}$




*N* D1 branes at $\mu^\alpha = 0$ 4 space-filling ``D5'' branes (+ O-plane)

Chiral worldvolume actions follow from Witten’s prescriptions. [Witten ’95] [Costello, Gaiotto ’18]