WHCGP   FRAMING DUALITY   w/ GUS SCHRADER & LINHUI SHEN

3/1/22

Goal: Calculate conjectural OGW invts of class of Lagrangians in $\mathbb{C}^3$, using cluster theory.

Overlaps work of many others

Physics:

Partition fn of 3d $N=2$ thy from M5 brane $\mathbb{R}^3 \times L$, $L \subset \mathbb{C}^3$

CCV
CEHRV $L \xrightarrow{2:1} B$ branched over tangle T

DGnGu
DGnGo   Depends on ideal triangulation

Basic case: $T =)($ $\longleftrightarrow$ tetrahedron

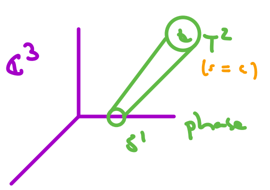
3d Abel $N=2$ CS thy + chiral multiplet $Z(x)$

Z depends on moduli of thy (brane, M

A-V D6 brane $\mathbb{R}^4 \times L \Rightarrow$ sup pot W of 4d $N=1$, disk integrality
O-V D4 brane $\mathbb{R}^2 \times L \Rightarrow$ couplings of 2d $(2,2)$ thy, $g>0$ integ
$\updownarrow$
M5 brane $\mathbb{R}^3 \times L$

A-V study mirror B-branes, W from HCS

Ex: W computed from how M sits in period domain



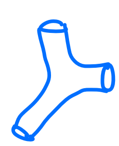
Mirror $ab = x + y - 1$

Brane $\{b=0\}$ $M: x + y = 1 \subset (\mathbb{C}^*)^2$

$x = e^u \quad y = e^v$

$v = -\ln(1-e^u)$
$= \partial_u \mathrm{Li}_2(x)$
W

$L_0: (re^{is}, re^{it}, re^{-is-it})$

M 

+

AKV many examples, dependence on framing, phase

AENV $L \subset$ resolved conifold
quantized A-poly for knot conormals
through large-N duality

$\Rightarrow$ Higher genus OGW

$\uparrow$

Augmentations = Sheaves = Fukaya alg's

Ex:

$$x + y = 1$$

$$x = e^u$$
$$y = e^{-v} = e^{i\hbar\partial_u}$$
$$x \cdot f(x) = x f(x)$$
$$y \cdot f(x) = f(e^{u+i\hbar})$$
$$= f(qx), \quad q = e^{i\hbar}$$

WITTEN
CEHNV
AENV

Wavefunction

[in quantization of moduli space]

$$y = 1 - x$$
$$\Psi(qx) = (1-x)\Psi(x)$$

$$\Psi = \prod_{n \geq 0} \frac{1}{1-q^n x} = \Phi(x)$$

(Framing p: $x y^{1-p} + y = 1$)

$$\log \Psi = \sum_{nm} \frac{q^{nm} x^m}{m} = \sum_m \frac{x^m}{m} \frac{1}{1-q^m}$$
$$= \sum \frac{(xq^{1/2})^m}{m \sin(m\hbar/2)}$$

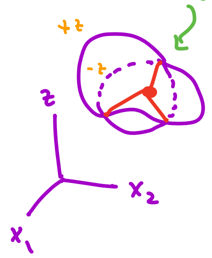
OV integrality

Plan: 1. Define a large class of branes labeled by Γ
+ phases, framings

2. Determine $\mathcal{M} \subset P$

3. Exploit cluster structure to compute wavefunctions

4. Corrections

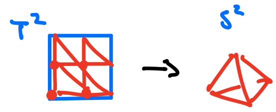


1. $\Gamma \subset S^2$ cubic plane graph, locally

$\Lambda_\Gamma \subset S^2 \times \mathbb{R}^3 \subset S^5$ Legendrian surface, locally in $\mathbb{R}^5$ $\quad y_i = \dfrac{\partial z}{\partial x_i}$

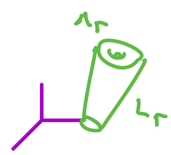
$V(\Gamma)$ = branch pts

H-L T^2 $(r=1)$ $z = \sin 2s + \sin 2t - \sin 2(s+t)$



$L_\Gamma \subset \mathbb{C}^3$ any Lagrangian filling $\partial^\infty L_\Gamma = \Lambda_\Gamma$

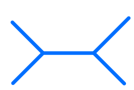
ker π = phase, e.g.



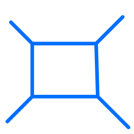
$$H_1(\Lambda_\Gamma) \underset{f}{\overset{\pi}{\rightleftarrows}} H_1(L_\Gamma)$$

ker labels collapsing S^1 $\longleftrightarrow$ small coord v in period domain

Splitting of π = framing
labels other coord $u + pv$



$3g+3$ edges $\hookrightarrow$ branch cuts generate $H_1(\Lambda_\Gamma)$



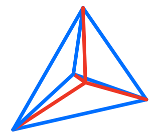
$g+3$ faces $\hookrightarrow$ relations $\sum_{e \in \partial f} e \sim 0$

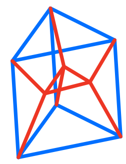
Prop. $\mathbb{Z}^{E_\Gamma} / \mathbb{Z}^{F_\Gamma} \simeq H_1(\Lambda_\Gamma)$ $\quad (3g+3-(g+3)=2g)$

Δ_2 tri'n $\qquad\qquad$ Δ_3 tet'n ideal

Constructing L_Γ : Extend $\Gamma \subset S^2$ to "foam" $F \subset B_3$ ideal

Ex: Γ



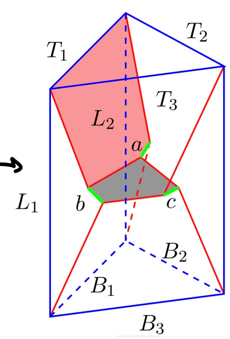


singular tangle (CEHNW)
+ sheets

Corresponding L is singular.
Deform foam with arcs

internal face relation

$$\sum_{a\in\partial f} \sigma(f,a)\cdot a = 0$$

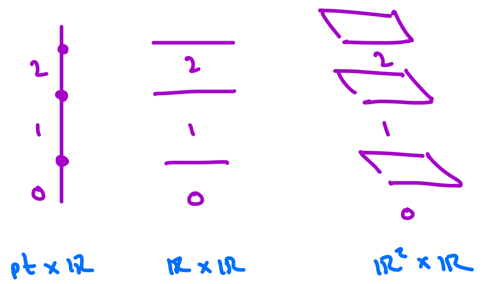


ext relation

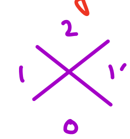
$$\pi(e) + \sum_{a\in\partial f} \sigma(f,a)\cdot a = 0$$

2. $Sh^1_{\Lambda_\Gamma}(\mathbb{R}^3)_0 \simeq Fuk_{\Lambda_\Gamma}(\mathbb{C}^3)$

Front of Λ_Γ is jumping locus for stalks of loc const divs
1 means require rk jumps up by 1

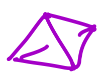


Crossing



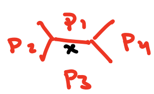
TRANSVERSE!

$\mathcal{M} = \{ \text{flag colorings} \} / PGL_{n=2}$
$\subset (\mathbb{P}^1)^{g+3} / PGL_2$

Ex:  = 

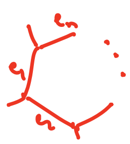
p.o.p. as in A-V  No M.S.!

$Sh_\Lambda \longrightarrow Loc(\Lambda)$



$\mapsto \quad x = -\text{cross ratio}(\vec{p})$

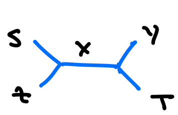
• Equations
cut out image

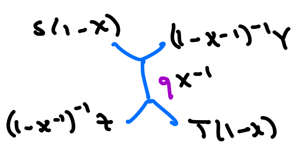


$1 - x_{e_1} + x_{e_1}x_{e_2} - \cdots = 0$

$x_{e_1} \cdots x_{e_n} = (-1)^n$

3 • Mutation
([F-G])

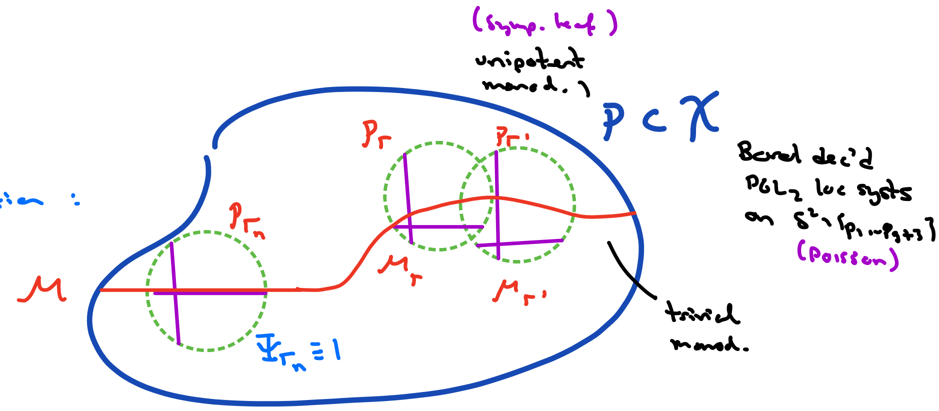




• Quantization $\quad x_e x_{e'} = q^{\langle e,e'\rangle} x_{e'} x_e$ from effective CS thy $x_e = e^{u_e}$

• Globalization: Equations consistent eg. $1 - S + SX - SXY \rightsquigarrow$
$1 - S(1-X) + S(1-X)(1-X^{-1})^{-1}Y$ ✓

• Interpretation :



• Conjugation $\quad x_e' = \mathrm{Ad}_{\Phi(x_k)}\, x_e$

$\quad \Rightarrow \quad \Psi' = \Phi(x_k)\Psi$

Problem: x_k's don't commute
Re-express in coords of
phase / frame.

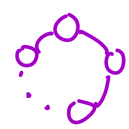
Change of frame : easy
" " phase : F.T. $\Rightarrow$

• Analytification $\quad \Phi \rightsquigarrow \varphi \quad$ NC q-dilog

4. **RESULTS**

• Conj. $\quad \Psi_r = \prod_{s,d} \varphi(q^s x^d)^{n_{d,s}}$ ($n \in \mathbb{Z}$) $\quad$ & $\quad \log \Psi_r = OGW(L_r)$

• Compute : bootstrap $\quad \Psi_{\text{necklace}_g} \equiv 1$



check
(exact)

• Framing Duality

cliff

$g \times g$ symm

$$\Psi^{\text{std phase, framing } A}_{\text{canoe}_g} = DT(Q^A)$$

([K-S])

Ex: (g=1, Luo-Zhu) $\qquad Q^A =$ 

A = 1

LHS = $\Phi = \prod_{k \geq 0} \frac{1}{1-q^k x}$ $\qquad$ RHS: $\mathrm{Rep}_d = \mathbb{C}^{d^2}/GL_d$

H^{2n} =
symm polys deg n
in $x_1 \dots x_d$ $\leftrightarrow$
partitions of n in d parts

$$\Rightarrow DT = \sum p_d(n) q^n x^d = \prod_{k \geq 0} \frac{1}{1-q^k x} \quad ✓$$