How is the hypersimplex like the amplituhedron?

Lauren K. Williams, Harvard

Slides at `http://people.math.harvard.edu/~williams/Williams.pdf`

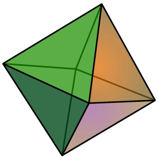
Based on:

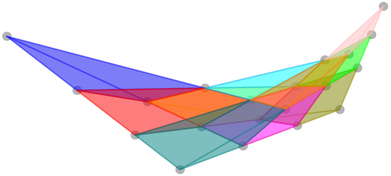
- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Tomasz Lukowski and Matteo Parisi, arXiv:2002.06164
- "The positive Dressian equals the positive tropical Grassmannian," with David Speyer, arXiv:2003.10231

Overview of the talk

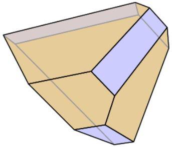
I. Hypersimplex and moment map '87
Gelfand-Goresky-MacPherson-Serganova
matroids, torus orbits on $Gr_{k,n}$

II. Amplituhedron '13
Arkani-Hamed–Trnka
$\mathcal{N} = 4$ SYM





III. Positive tropical Grassmannian '05
Speyer–W.
associahedron, cluster algebras



Program

- Geometry of Grassmannian and the moment map. GGMS '87. Hypersimplex, matroid stratification, matroid polytopes.
- Add *positivity* to the previous picture. Postnikov '06. Positroid stratification of $(Gr_{kn})_{\geq 0}$, positroid polytopes, plabic graphs.
- Simultaneous generalization of $(Gr_{k,n})_{\geq 0}$ and polygons: amplituhedron. Arkani-Hamed and Trnka '13.
- Triangulations of the amplituhedron and triangulations of the hypersimplex.
- Connection with positive tropical Grassmannian

The Grassmannian and the moment map

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

Have torus action of $H = (\mathbb{C}^*)^n$ on $Gr_{k,n}$ which rescales columns of each $A \in Gr_{k,n}$. Gives moment map ...

Let $\{e_1, \dots, e_n\}$ be basis of $\mathbb{R}^n$; for $I \subset [n]$, let $e_I := \sum_{i \in I} e_i$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

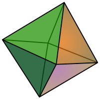
The Grassmannian and the moment map

Let $\{e_1, \dots, e_n\}$ be basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is $\quad \mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$

Convexity Theorem (Atiyah, Guillemin-Sternberg)

$\mu(\overline{H.A}) = \mathrm{Conv}\{\mu(Q) \mid Q \text{ a torus fixed point of } \overline{H.A}\}$



Let $\Delta_{k,n} := \mathrm{Conv}\{e_I \ : \ |I| = k\} \subset \mathbb{R}^n$ be the *hypersimplex.*
Is moment map image of generic point of $Gr_{k,n}$.

The *matroid* associated to $A \in Gr_{k,n}$ is $\mathcal{M}(A) := \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0.\}$

Gelfand-Goresky-MacPherson-Serganova showed that polytope $\mu(\overline{H.A})$ equals $\mathrm{Conv}\{e_I \mid I \in \mathcal{M}(A).\}$ Called *matroid polytope*.

Matroid polytopes and the matroid stratification

Recall: *matroid* assoc to $A \in Gr_{k,n}$ is $\mathcal{M}(A) := \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0.\}$
Let $\mathcal{M} = \mathcal{M}(A)$. The *matroid polytope* is $\Gamma_{\mathcal{M}} = \mathrm{Conv}\{e_I \mid I \in \mathcal{M}.\}$

Gelfand-Goresky-MacPherson-Serganova '87: gave beautiful characterization of matroid polytopes. In particular, every edge of a matroid polytope is parallel to $e_i - e_j$ for some i, j.

GGMS introduced the *matroid stratification* of $Gr_{k,n}$.
Given $\mathcal{M} \subset \binom{[n]}{k}$, let $S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid p_I(A) \neq 0 \text{ iff } I \in \mathcal{M}\}$.
Called *matroid stratum*.
Have the matroid stratification

$$Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}.$$

Unfortunately, the topology of matroid strata is terrible – Mnev's *universality theorem* (1987): "The topology of the matroid stratum $S_{\mathcal{M}}$ can be as complicated as that of any algebraic variety."

What if we add the adjective "positive" to the whole story?

Background: Lusztig's theory of total positivity for G/P 1994, Rietsch 1997, Postnikov's 2006 preprint on the *totally non-negative* (TNN) or "positive" Grassmannian.

Let $(Gr_{k,n})_{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

Inspired by matroid stratification, one can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

In contrast to terrible topology of matroid strata ...

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$(Gr_{k,n})_{\geq 0} = \sqcup S_{\mathcal{M}}^{tnn}.$$

What if we add the adjective "positive" to the whole story?

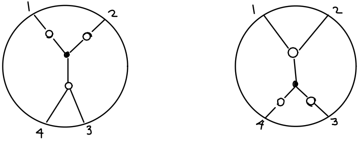
- Recall: *matroid* assoc to $A \in Gr_{k,n}$ is $\mathcal{M}(A) := \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0.\}$
- And the *matroid polytope* is $\Gamma_{\mathcal{M}} = \text{Conv}\{e_I \mid I \in \mathcal{M}.\}$
- If $A \in (Gr_{k,n})_{\geq 0}$, call $\mathcal{M}(A)$ a *positroid* and $\Gamma_{\mathcal{M}}$ a *positroid polytope*.
- Can restrict moment map from $Gr_{k,n}$ to $(Gr_{k,n})_{\geq 0}$: each positroid polytope is moment map image of positroid cell. (Tsukerman-W.)
- Positroid polytopes are precisely the matroid polytopes whose facets all have the form $x_i + x_{i+1} + \cdots + x_j = c$. (Ardila-Rincon-W.)

Theorem (Postnikov)

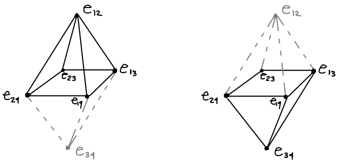
The positroid cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with *decorated permutations* π on $[n]$ with k antiexcedances. Also in bijection with equivalence classes of *planar bicolored (plabic) graphs*, or *on-shell graphs*.

How to read off a positroid (polytope) from a plabic graph

- Positroid cells $\leftrightarrow$ *plabic graphs*, planar bicolored graphs embedded in disk with boundary vertices labeled $1, 2, \ldots, n$ and internal vertices colored black or white.



- WLOG we assume graph G is bipartite and that every boundary vertex is incident to a white vertex.
- Let $\mathcal{M}(G) := \{\partial(P) \mid P \text{ is a perfect matching of } G\}$.
- $\mathcal{M}(G)$ a positroid, and all positroids obtained this way (Postnikov).



Background and Motivation for the amplituhedron

- Introduced by Arkani-Hamed and Trnka in 2013.
- The amplituhedron is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$:

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}(Z)$ depends on Z but its combin. properties appear not to.
- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" is supposed to compute scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory;
  the *BCFW recurrence* for scattering amplitudes can be reformulated as giving a triangulation of the $m = 4$ amplituhedron.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.

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Special cases:

- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$. *Do $m = 2$ example*

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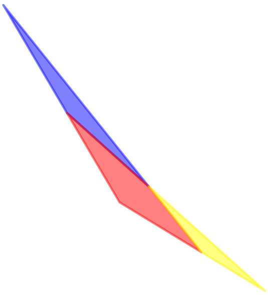
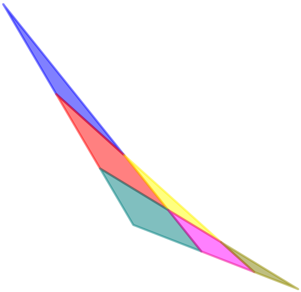
Special cases:

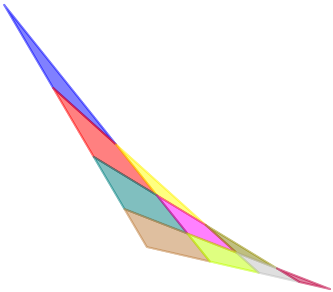
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)



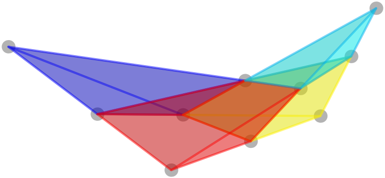
$\mathcal{A}_{4,2,1}$



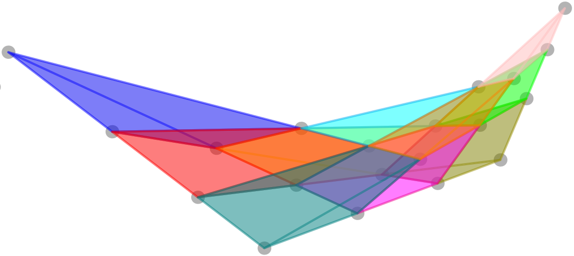
$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{5,3,1}$



$\mathcal{A}_{6,3,1}$

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$:
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

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Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)
- $m = 4$: case of main physical interest.
- $m = 2$: toy model for $m = 4$ – and connected to hypersimplex! ...

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^+_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Physicists want to understand "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim (Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Call this a **(positroid) triangulation**.
- BCFW recurrence conjecturally gives triangulation of $\mathcal{A}_{n,k,4}$.

Triangulating the amplituhedron

(Positroid) triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2$, $m = 4$, $k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

(Positroid) triangulations of $\mathcal{A}_{n,1,2}$

- Recall that $\mathcal{A}_{n,1,2}$ is a polygon (n-gon) in projective space.

- Positroid triangulations of $\mathcal{A}_{n,1,2}$ are ordinary triangulations of the n-gon
- Each triangulation consists of $n-2$ triangles, each of dimension 2
- The total number of triangulations of $\mathcal{A}_{n,1,2}$ is the *Catalan number* $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.
- When $n=2$, have two triangulations of $\mathcal{A}_{n,1,2}$ (quadrilateral).

Triangulating the hypersimplex

In analogy with positroid triangulations of the amplituhedron, we define positroid triangulations of the *hypersimplex.*
Also known as *fine positroidal subdivisions.*
Recall the **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

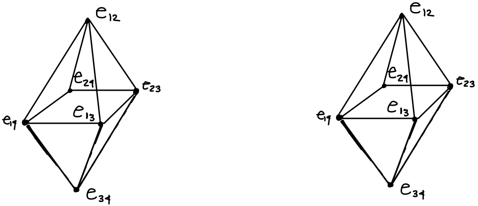
(Positroid) triangulations of the hypersimplex

Have $\mu : (Gr_{k,n})_{\geq 0} \to \mathbb{R}^n$; we have $\dim \mu(Gr_{k,n})_{\geq 0} = \dim \Delta_{k,n} = n-1$.
A **(positroidal) triangulation** of $\Delta_{k,n}$ is a collection of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images are disjoint and cover $\Delta_{k,n}$.

(Positroid) triangulations of $\Delta_{2,n}$

- Each positroid triangulation consists of $n-2$ positroid cells ("triangles"), each of full dimension $n-1$;
- The total number of positroid triangulations of $\Delta_{2,n}$ is the *Catalan number* $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ (Speyer-W.)

Example: $\mu : (Gr_{2,4})_{\geq 0} \to \Delta_{2,4} \subset \mathbb{R}^4$



Comparison with $\mathcal{A}_{n,1,2}$.

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \text{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

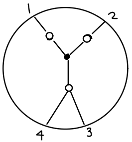
- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

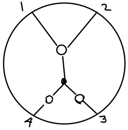
For $k = 1$, it seems that:
positroidal triangulations of $\mathcal{A}_{n,k,2} \leftrightarrow$ positroidal triangulations of $\Delta_{k+1,n}$.
Claim: this holds for general k!

Indexing of positroid cells by permutations

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

- A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
- Cells S_π of $(Gr_{k,n})_{\geq 0}$ $\leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
- One can read off description of cell S_π from π.
- Given (reduced) plabic graph representing positroid cell, can read off permutation π by following "rules of road":
  right at black, left at white.





T-duality map on positroid cells

Conjecture (Lukowski-Parisi-W.)

Positroid triangulations of the amplituhedron $\mathcal{A}_{n,k,2}$ are in bijection with positroid triangulations of the hypersimplex $\Delta_{k+1,n}$. Bijection TBD ...

- Triangulations of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$, while triangs of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

Recall that cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ decorated permutations π on $[n]$ with k antiexcedances.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \dots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \dots, a_{n-1}),$$

where any fixed points declared to be loops. Call it *T-duality map.*

Lemma (Lukowski–Parisi–W.)

The T-duality map $S_\pi \leftrightarrow S_{\hat{\pi}}$ is a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

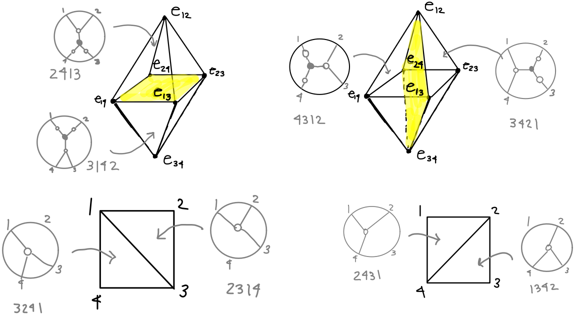
Remark: the T-duality map is $m = 2$ analogue of an $m = 4$ map used by Arkani-Hamed-Bourjaily-Cachazo-Goncharov-Postnikov-Trnka to relate twistor space to momentum-twistor space.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

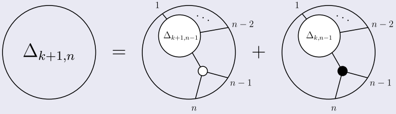
A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.



Conjecture true for many triangulations

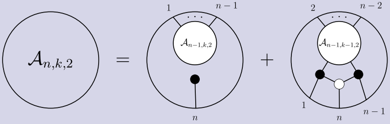
Theorem (Lukowski–Parisi–W.)

The following recursion constructs triangulations of $\Delta_{k+1,n}$ in terms of triangulations of $\Delta_{k+1,n-1}$ and $\Delta_{k,n-1}$:



Theorem (Bao-He)

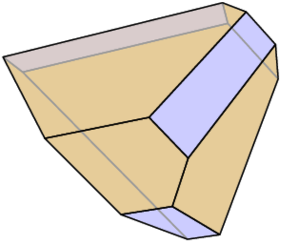
The following recursion constructs triangulations of $\mathcal{A}_{n,k,2}$ in terms of triangulations of $\mathcal{A}_{n-1,k,2}$ and $\mathcal{A}_{n-1,k-1,2}$:



Theorem (L-P-W): These recursions are in bijection via T-duality.

How did we guess connection between $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$?

- – Studying "good" triangulations of the amplituhedron
- and noticing a numerology connection to the *positive tropical Grassmannian*



Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_π, the image of a cell S_π in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Data on good triangulations of the amplituhedron

$\#$ of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
$\#$ of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
$\#$ of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
$\#$ of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Remark: $\text{Trop}^+ Gr_{k,n}$ is closely related to the cluster algebra structure on $\mathbb{C}[Gr_{k,n}]$ – in finite type, it is a coarsening of the corresponding generalized associahedron.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of our work.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\text{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\text{Trop}^+(f)$ for $f \in I$.

- So we can define $\text{Trop}^+ Gr_{k,n} = \cap \text{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.
  (I'll explain positive tropical hypersurface on next slide)
- It turns out that the three-term Plücker relations suffice!

The positive tropical Grassmannian

Recall $\text{Trop}^+ Gr_{k,n} = \cap \text{Trop}^+(f)$ where f in Plücker ideal.

Fix k, n. Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$.
Call P a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

This is the **pos trop hypersurface** associated to the 3-term Plücker relation.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors**.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to regular matroidal subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- Look for positive analogue of this.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

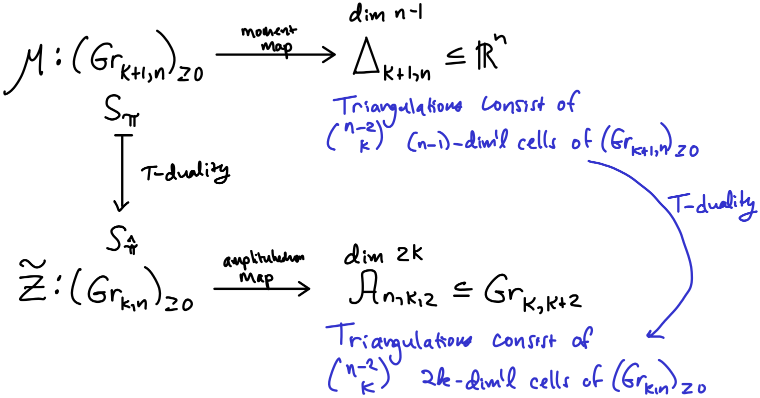
- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte and appeared shortly after our paper in work of Arkani-Hamed–Lam–Spradlin.

Corollary (Lukowski–Parisi–W)

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

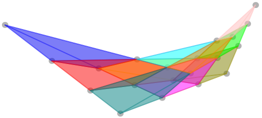
Summary and questions



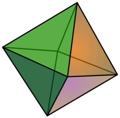
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?

Thank you for listening!

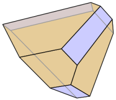
I. Amplituhedron '13



II. Hypersimplex and moment map '87



III. Positive tropical Grassmannian '05



- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Lukowski and Parisi, arXiv:2002.06164
- "The positive Dressian equals the positive tropical Grassmannian," with Speyer, arXiv:2003.10231.

How positroid cells are encoded by decorated permutations

Given a $k \times n$ matrix $C = (c_1, \ldots, c_n)$ (representing a point of $(Gr_{k,n})_{\geq 0}$) written as a list of its columns, we associate a decorated permutation π as follows.

- Given $i, j \in [n]$, let $r[i,j]$ denote the rank of $\langle c_i, c_{i+1}, \ldots, c_j \rangle$, where we list the columns in cyclic order, going from c_n to c_1 if $i > j$.
- We set $\pi(i) := j$ to be the label of the first column j such that $c_i \in \text{span}\{c_{i+1}, c_{i+2}, \ldots, c_j\}$.
- If c_i is the all-zero vector, we call i a loop or black fixed point, and if c_i is not in the span of the other column vectors, we call i a coloop or white fixed point.

We define S_π^{tnn} to be the set of all elements $C \in (Gr_{k,n})_{\geq 0}$ which give rise to this π.

What is the positive tropical Grassmannian?

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n} :=$ Trop $V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.