$\dim = 3g-3+N$

X - smooth projective curve $/\mathbb{C}$
with distinct points $t_1, .., t_N$
$\mathrm{Bun}^s_m(X)$ - moduli space of stable
(quasi) parabolic PGL_2 - bundles
of degree m on X, i.e. rank 2 bundles
modulo tensoring with line bundles

Parabolic slope: for a rank 2
bundle:

$$s(E) = \frac{1}{2}\deg E + \frac{N}{4}$$

for a line subbundle: $L \subset E$

$$s(L) = \deg L + \frac{N_L}{2}$$

N_L = # of parabolic lines in L.

Stable: $s(L) < s(E)$ $\forall L \subset E$.

$$\mathrm{Bun}^s(X) = \mathrm{Bun}^s_0(X) \sqcup \mathrm{Bun}^s_1(X).$$

$\mathcal{H}_i = L^2(\mathrm{Bun}^s_i(X))$ - square integrable Half-densities.

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = L^2(\mathrm{Bun}^s(X))$$

$x \in X$, $x \neq t_j$

Hecke modification: $E \in Bun_i^s$

$s \in \mathbb{P}E_x$

$\mathcal{H}_{x,s}(E)$ – bundle whose sections are meromorphic sections of E with $\leq$ first order pole at x, with residue in s.

This is not necessarily stable, but generically it is.

Hecke operator:

$H_x : \mathcal{H} \to \mathcal{H}$ (densely defined initially).

$$(H_x \psi)(E) = \int_{s \in \mathbb{P}E_x} \psi(\mathcal{H}_{x,s}(E)) \|ds\|$$

One can show that this makes sense canonically.

Conjecture 1 These operators are compact, self-adjoint, and pairwise commuting. (for different x).

Theorem 2 The operators H_x commute with the quantum Hitchin hamitonians on $Bun^s(X)$.

If conjecture 1 holds, By the spectral theorem for compact self-adjoint operators, $\{H_x\}$ admit a basis of eigenfunctions ψ_Λ with

naturally $\frac{1}{2}$ density

$$H_x \psi_\Lambda = \beta_\Lambda(x)\, \psi_\Lambda.$$

What are eigenvalues $\beta_\Lambda(x)$?

Theorem. $\forall \Lambda$ $\exists$ a real SL_2-oper L_Λ on X

$$L_\Lambda : \underline{K^{-1/2}} \longrightarrow \underline{K^{3/2}}$$

with monodromy in $SL_2(\mathbb{R})$ $= SU(1,1)$, with $L_\Lambda \beta_\Lambda(x) = 0$. $\overline{L}_\Lambda \beta_\Lambda = 0$

In this case $\exists !$ up to scale real-valued analytic solution $\beta = f_1 \overline{f}_2 + f_2 \overline{f}_1$ where f_1, f_2 is a basis of solutions of $L_\Lambda \beta = 0$.

Recall that according to Beilinson–Drinfeld, if $\mathcal{A}$ is the quantum

Hitchin algebra then
Spec $\mathcal{A}$ = Op, the
space of opers.
So $\forall L \in Op$ have character
$$\chi_L : \mathcal{A} \to \mathbb{C}.$$

Prop. $\forall D \in \mathcal{A}$
$$D\,\psi_\Lambda = \chi_{L_\Lambda}(D)\,\psi_\Lambda.$$

Conjecture The spectrum
is simple, labled by
all real opers.

(Almost) all these
conjectures are now
proved in genus 0.

Also this problem

makes sense over $\mathbb{R}$ (and other local fields).

Some (not all!) historical references.

Braverman-Kazhdan 2006

Kontsevich 2007

Langlands 2018

Teschner 2017

Gaiotto - Witten 2021

Beukers 2007 for $\mathbb{P}^1$ with 4 pts

Nekrasov & collaborators 2021 and earlier

Kapustin - Witten

. . . . .

Quantum Hitchin:

Usual Hitchin system:

$T^*\mathrm{Bun} \longrightarrow \mathcal{B}$ - The Hitchin

$\mathcal{O}(\mathcal{B}) \hookrightarrow \mathcal{O}(T^*\mathrm{Bun})$ $\quad$ $\underset{\text{base}}{H^0(X, K_X^2)}$

Quantum:

Comm. algebra of same size in $\mathcal{D}(\mathrm{Bun})$

$$\mathcal{O}(\tilde{\mathcal{B}}) \longleftrightarrow \mathcal{D}(\mathrm{Bun})$$

space of opers.

Feigin Frenkel Casimirs

$$\mathrm{Bun} = G(X \setminus pt) \backslash G((t)) / G[[t]]$$