Gauged Gromov-Witten theory and infinite-dimensional GIT

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For whole talk, C Riemann surface

Consider algebraic stack $Bun_n(C) = Map(C, BGL_n)$

$$\left(T \xrightarrow{\text{maps}} Bun_n(C)\right) = \left(\begin{matrix}\text{rank } n \text{ vector bundles} \\ \text{on } C \times T\end{matrix}\right)$$

"functor of points"

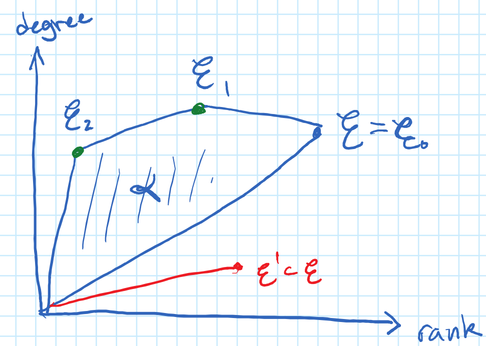
Solution to classification problem = special stratification of this moduli problem

<u>Semistable locus</u>: $\mathcal{E}$ is semistable if $\nexists$ subbundle of larger slope

$\rightsquigarrow$ defines open substack $Bun_n^{ss} \subset Bun_n(C)$ that admits a projective good moduli space.

<u>Unstable locus</u>:

<u>Thm</u>: (Harder-Narasimhan) Every unstable bundle $\mathcal{E}$ admits unique <u>convex</u> filtration whose graded pieses are semistable.



$$Bun_n(C) = Bun_n^{ss} \cup \bigcup_\alpha \left\{\begin{matrix}\text{bundles w/ HN filtration} \\ \text{of shape } \alpha\end{matrix}\right\} \leftarrow \text{stratification by locally closed substacks}$$

$$\left\{\begin{matrix}\text{bundles w/ HN filtration} \\ \text{of shape } \alpha\end{matrix}\right\} \underset{\oplus}{\overset{gr}{\rightleftarrows}} Z_\alpha^{ss} = \underbrace{Bun_{r_1,d_1}^{ss} \times \cdots \times Bun_{r_p,d_p}^{ss}}_{\text{has projective good moduli space}}$$

<u>Goal</u>: Discover this structure on other moduli stacks.

$\boxed{\text{Moduli of gauged maps}}$

Let G be reductive group, X a linear representation

$$M_C^G(X) = \mathrm{Map}(C, X/G) = \left\{ \begin{array}{l} G\text{-bundles } P \to C \text{ and} \\ \text{a section of } P \times^G X \to C \end{array} \right\}$$

Ex: 1) vector bundle + endomorphism. $\rightsquigarrow$ Higgs bundles

2) vector bundle + section of $\Lambda^2 \mathcal{E}$

Motivation: There's a family of stability conditions depending on $\delta \geq 0$. For $\delta \gg 0$, integrals of tautological classes on $M_C^G(X)^{\delta\text{-ss}}$ is related by wall-crossing to GW of X^{ss}/G.

K-theoretic gauged GW-invariants

$$\chi(M_C^G(X)^{\delta\text{-ss}}, \text{taut.}) = \sum (-1)^i \dim \mathbb{H}^i(\text{taut.})$$

Strategy for computing them:

1) $p : M_C^G(X) \longrightarrow \mathrm{Bun}_G$ forgets section

$p_*(\mathcal{O}) \sim \mathrm{Sym}(P^\circ)$ explicit computation in K-theory

$$\chi(M_C^G(X), p^*(\text{taut.})) = \underbrace{\chi(\mathrm{Bun}_G, \mathrm{Sym}(P^\circ) \otimes (\text{taut.}))}_{\text{Teleman-Woodward gives explicit formulas}}$$

2) virtual non-abelian localization

$$\chi(\underbrace{M_C^G(X)}^{\delta\text{-ss}}, p^*(\text{taut.})) = \chi(\mathrm{Bun}_G, \mathrm{Sym}(P^\circ) \otimes (\text{taut.}))$$
$$+ \left(\begin{array}{c}\text{correction terms from} \\ \text{unstable strata}\end{array}\right)$$

applies to a stack with a Θ-stratification (generalized HN stratification of Bun_G)

The main application (in progress) for tautological classes on Bun_G, $\chi(M_C^G(X)^{\delta\text{-ss}}, p^*(\text{taut.}) \otimes \mathcal{L})$ is given by Teleman-Woodward formula if $\mathcal{L}$ has sufficiently positive level.

Θ-stratifications

$\Theta = \mathbb{A}^1/\mathbb{G}_m$ a filtration in a general stack $\mathcal{X}$ is a map $f : \Theta \to \mathcal{X}$ $\rightsquigarrow$ $f(1)$ underlying object
$f(0)$ associated graded point

In our example: $G = GL_n$

$$\left(\text{maps } f : \Theta \to \mathcal{M}_C^G(X)\right) = \left(\begin{array}{l}\mathbb{Z}\text{-weighted filtered vector bundle} \\ \cdots \subset \mathcal{E}_{w+1} \subset \mathcal{E}_w \subset \cdots \text{ that is compatible} \\ \text{with section of associated bundle for} \\ \mathcal{E} = \mathcal{E}_w \text{ for } w \ll 0\end{array}\right)$$

What is Θ-stratification?

For any stack $\mathcal{X}$, $\mathrm{Filt}(\mathcal{X}) = \mathrm{Map}(\Theta, \mathcal{X})$ $\rightsquigarrow$ also an algebraic stack

A Θ-stratification is an open substack $\mathcal{S} \subset \mathrm{Map}(\Theta, \mathcal{X})$ with components $\mathcal{S}_\alpha$ such that $ev_1 : \mathcal{S}_\alpha \to \mathcal{X}$ is a locally closed immersion (+ condition on how closures of strata meet)

$\rightsquigarrow$ Key point: need to find canonical filtrations $(f : \Theta \to \mathcal{M}_C^G(X))$ for unstable points

Canonical filtrations determined by a numerical invariant:
Assigns $\mu_f(\Theta \xrightarrow{f} \mathcal{M}_C^G(X)) \in \mathbb{R}$

$$\| $$

$$\frac{1}{\sqrt{b}}\left(\ell_{\mathrm{Bun}_G} + \delta \ell_X\right)$$

degree/rank of $\mathcal{E}$

- $\ell_{\mathrm{Bun}_G} = \sum_{w \in \mathbb{Z}} w\left(\deg(\mathrm{gr}_w \mathcal{E}_\bullet) - \frac{D}{R}\,\mathrm{rank}(\mathrm{gr}_w \mathcal{E}_\bullet)\right)$
- $\ell_X =$ Hilbert-Mumford weight in X/G at generic point of C
- $b = \sum w^2\, \mathrm{rank}(\mathrm{gr}_w \mathcal{E}_\bullet)$

(Motivation: in GIT, $\mu(f : \Theta \to \mathcal{X}) = \dfrac{wt(f^*\mathcal{L})}{\|f\|}$)

Def: • A point $(\mathcal{E},s)$ is unstable if $\exists\, f: \Theta \to \mathcal{M}_C^G(X)$ with $f(1)=(\mathcal{E},s)$ and $\mu(f)>0$

• A HN-filtration is an f that maximizes $\mu(f)$ subject to $f(1)=(\mathcal{E},s)$

Thm: This numerical invariant defines a Θ-stratification of $\mathcal{M}_C^G(X)$.

Comments on infinite dimensional GIT:

$$\mathcal{M}_C^G(X) \text{ ``}=\text{'' } Gr^{BD} \big/ Map_{rat}(C,G)$$

informal

line bundles used for numerator of μ are ample on affine Grassmannian Gr^{BD}

definition Gr^{BD}

$$\begin{tikzcd} Gr^{BD} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_C^G(X) \arrow[d, "\text{relative ind-projective ind-scheme}"] \\ T \arrow[r] & \underbrace{Map_{rat}}_{\text{behaves like classifying stack of reductive group.}}(C, X/G) \text{ ``}=\text{'' } B\, Map_{rat}(C,G) \end{tikzcd}$$