Joint work with Bernd Siebert (2019).

History of mirror symmetry.

1989–1990: Candelas et al, Greene–Plesser

Calabi–Yau 3-folds tend to come in pairs $X, \check{X}$

(Calabi–Yau: compact complex manifold with a nowhere vanishing top-dim'l holomorphic form.)

$$h^{1,1}(X) = h^{1,2}(\check{X})$$

$$h^{1,2}(X) = h^{1,1}(\check{X}).$$

Example: The quintic 3-fold and its mirror.

$$X = Z(x_0^5 + \cdots + x_4^5) \subseteq \mathbb{CP}^4$$

(the quintic 3-fold)

$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 101.$

Greene-Plesser construction:

$\mathbb{Z}_5^5 \circlearrowright \mathbb{CP}^4 \qquad (a_0, \ldots, a_4) \in \mathbb{Z}_5^5$

acts on $\mathbb{CP}^4$ by

$$(x_0, \ldots, x_4) \longmapsto (\xi^{a_0} x_0, \ldots, \xi^{a_4} x_4)$$

where $\xi = e^{2\pi i/5}$.

$G = \mathbb{Z}_5^4 \subseteq \mathbb{Z}_5^5$

$G = \{(a_0, \ldots, a_4) \in \mathbb{Z}_5^5 \mid \sum a_i = 0\}.$

G acts on X, and then X/G has a Calabi-Yau resolution $\check{X} \to X/G$.

$h^{1,1}(\check{X}) = 101, \quad h^{1,2}(\check{X}) = 1.$

Candelas, de la Ossa, Green and Parkes

One can calculate the numbers of rational curves on X by performing certain period integrals on $\check{X}$ $\left(\int_\alpha \Omega_{\check{X}}, \ \alpha \in H_3(\check{X}, \mathbb{Z}) \right.$ $\Omega_{\check{X}}$ is a nowhere vanishing holomorphic 3-form. $\left.\right)$

A rational curve should be thought, to first approximation, as a map $f: \mathbb{C}\mathbb{P}^1 \longrightarrow X$ representing some homology class $\beta \in H_2(X, \mathbb{Z})$, i.e. $f_*[\mathbb{C}\mathbb{P}^1] = \beta$. (Work modulo reparametrization)

For example, the number of lines in X is the # of holomorphic maps $f: \mathbb{CP}^1 \to X$ representing the class of a line in $H_2(X, \mathbb{Z})$

(Number = 2875.)

conics = 609250 (S. Katz)

<u>Constructions of mirror pairs:</u>

<u>Batyrev:</u> Mirror pairs can be constructed as hypersurfaces in toric varieties. (1992)

<u>Question:</u> Is there a general mirror construction?

<u>Yes!</u>

Our context:

We fix a log Calabi-Yau manifold, i.e., a pair (X, D) where X is a non-singular variety and $D \subseteq X$ is a normal crossings divisor with $K_X + D = 0$. In other words, $\exists$ a nowhere holomorphic form on $X \setminus D$ with simple poles along D.

In general, we will either consider the case that X is compact **or** the case that there is a family

$$\begin{array}{ccc} X_0 & \subseteq & X \\ \downarrow & & \downarrow \pi \\ 0 & \in & S \end{array} \qquad D = X_0$$

S = one-dimensional germ of a curve

Here the fibres of π are CY manifolds

and π should be proper.

Let $D = D_1 + \cdots + D_s$

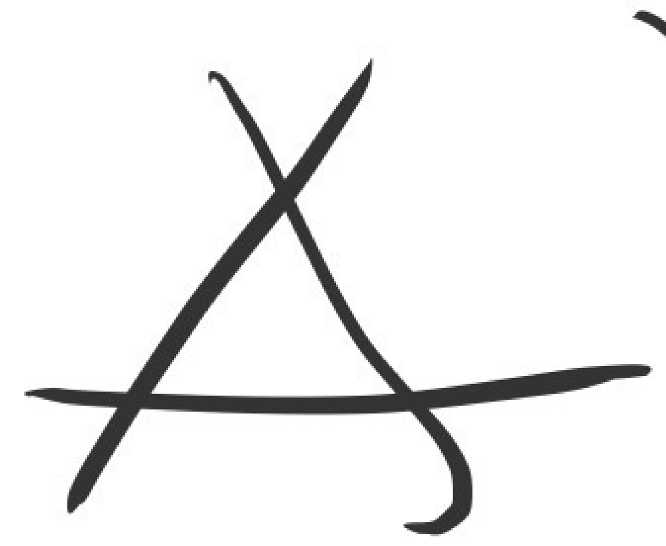
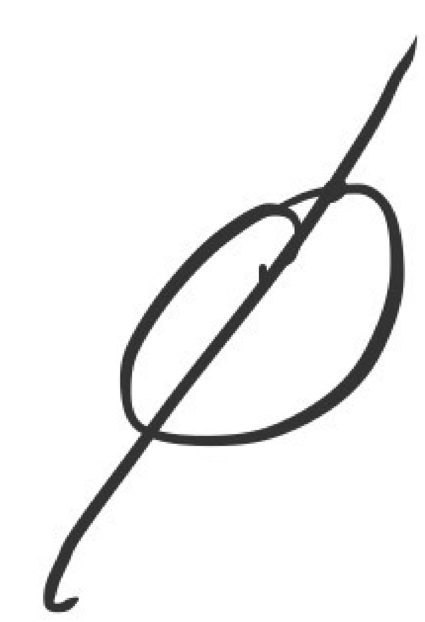
be the irreducible decomposition of D.

$\rightsquigarrow$ dual complex of D.

<u>Assume:</u> for $I \subseteq \{1, \dots, s\}$,

$$D_I := \bigcap_{i \in I} D_i \quad \text{is connected.}$$

e.g.

$(\mathbb{P}^2,$  $)$ $\qquad$ $(\mathbb{P}^2,$  $)$

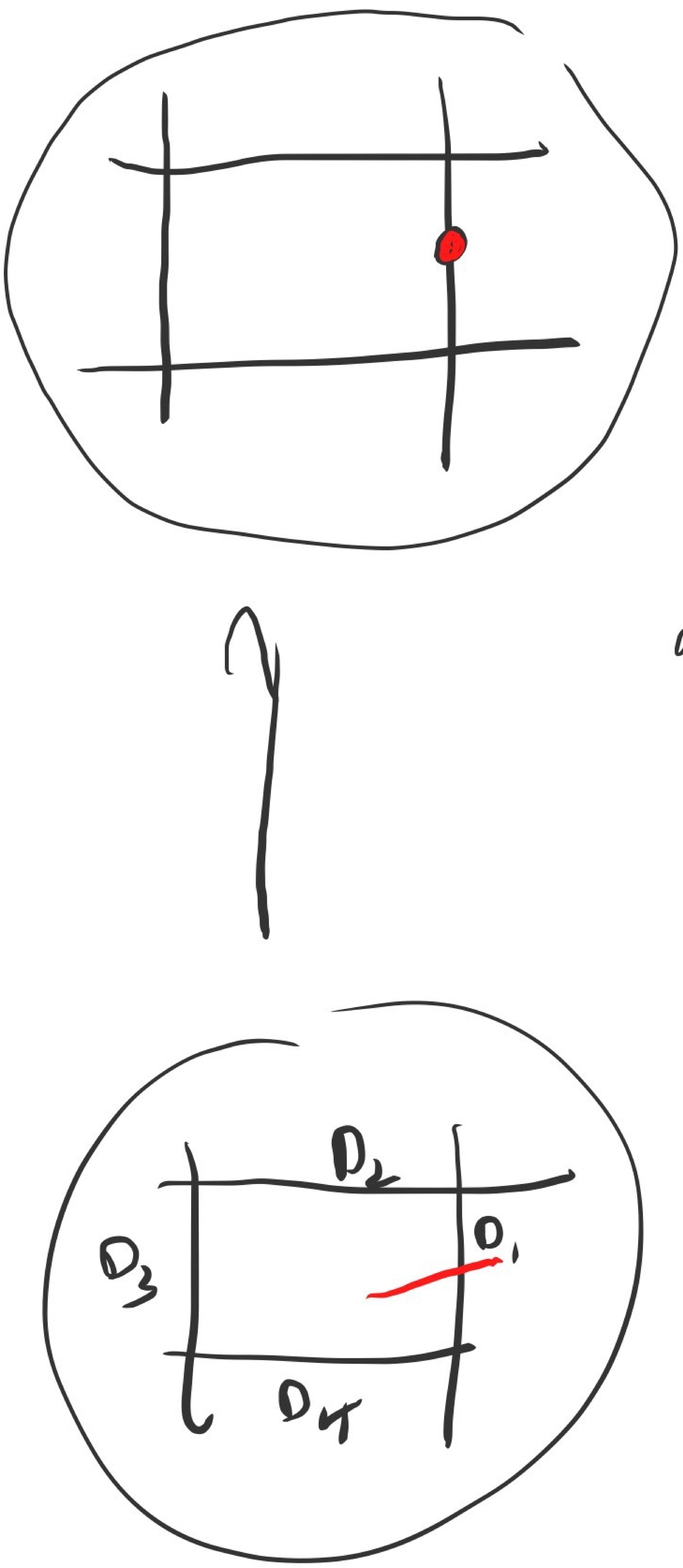
We will build the dual complex of D as a cone complex in the $\mathbb{R}$-vector space with basis $D_1, \dots, D_s$.

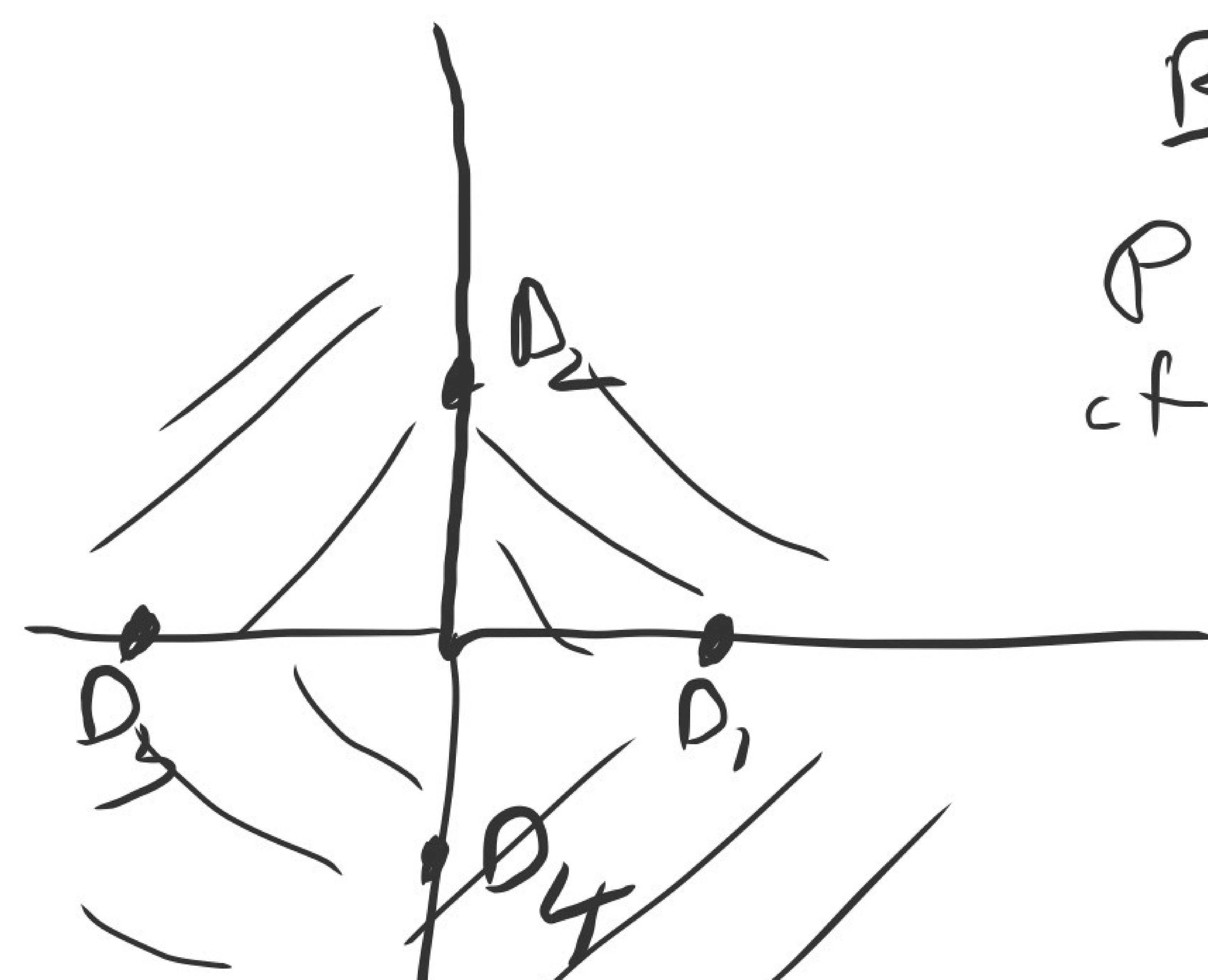
$$\mathcal{P} := \left\{ \sum_{i \in I} \mathbb{R}_{\geq 0} D_i \;\middle|\; I \subseteq \{1, \dots, s\},\ D_I \neq \emptyset \right\}$$

Let $B = \bigcup_{\sigma \in \mathcal{P}} \sigma$.

Example: $(\mathbb{P}^1 \times \mathbb{P}^1, \ \overline{D} = (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1)$
$\cup \ (\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}))$

Blow-up a point in $\{0\} \times \mathbb{P}^1$ and let X be the blow-up and D the strict transform $\overline{D}$.





B

P collections of cones.

Note this is a <u>piecewise</u> linear object.

$$B(\mathbb{Z}) = \left\{ \sum a_i D_i \in B \;\middle|\; a_i \in \mathbb{Z} \right\}.$$

Point of $B(\mathbb{Z})$ record tangency orders for holomorphic maps of curves

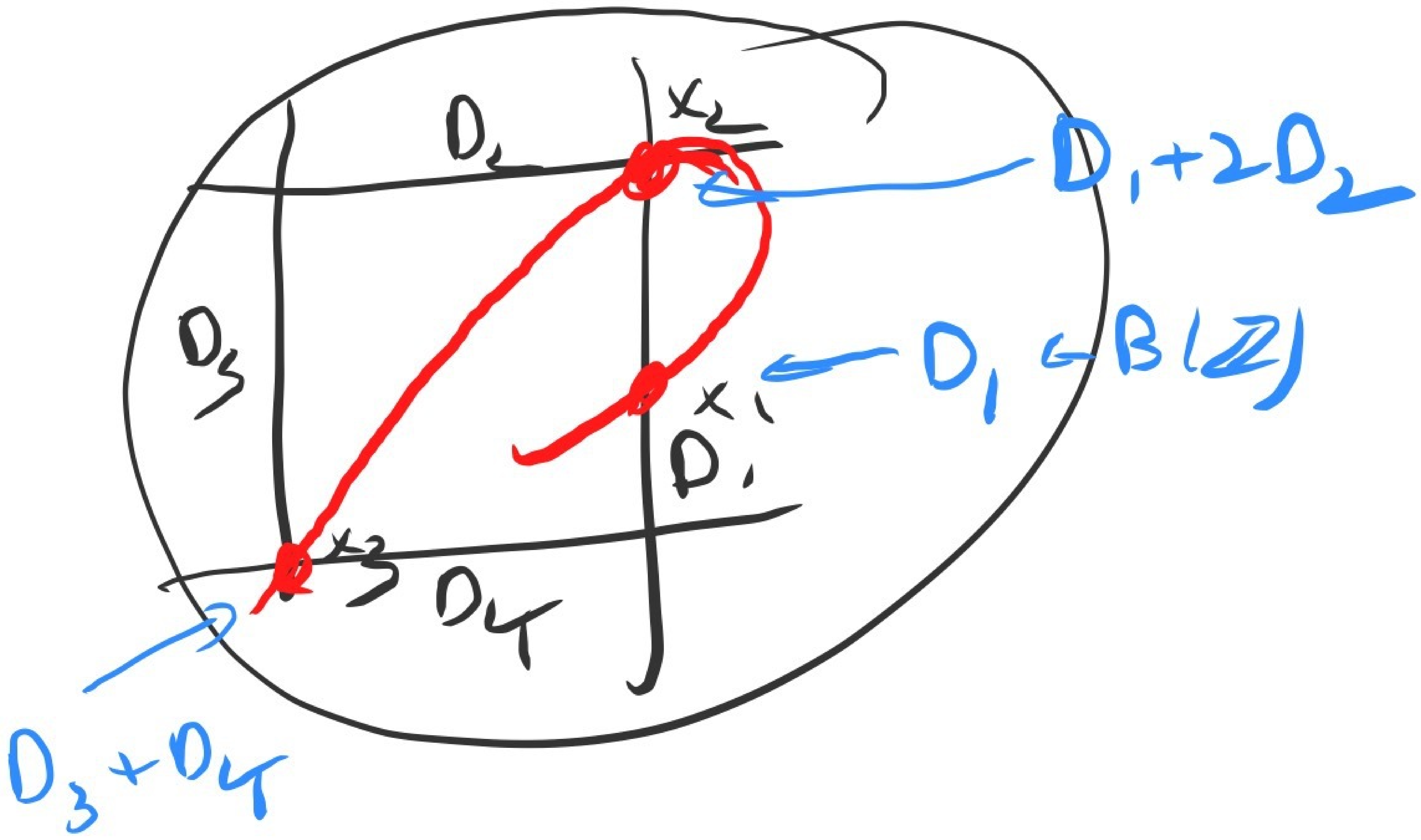
$f: C \longrightarrow X$.

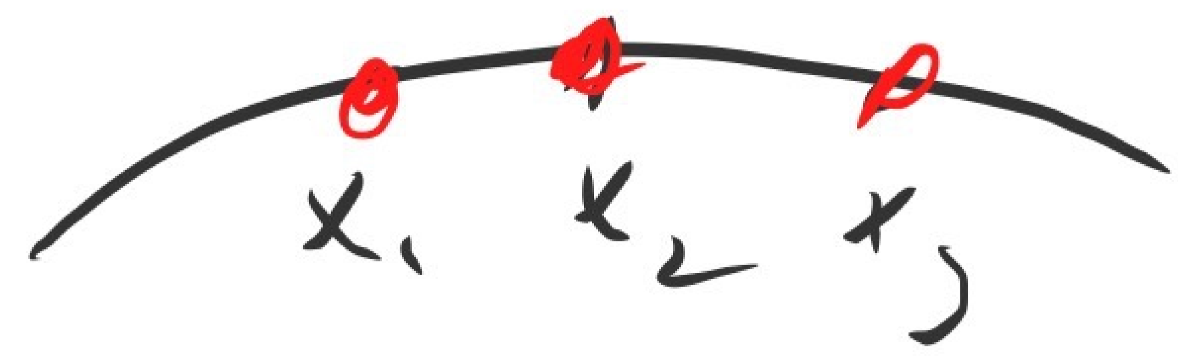
Typical question: consider maps from punted curves

$$f: (C, x_1, \ldots, x_n) \longrightarrow X$$

$x_1, \ldots, x_n \in C$ distinct points,

and we wish to impose tangency of f at a point x_i with divisor D_j.





f Being tangent to D_j at x_i to order a means: take a local defining equation for D_j near $f(x_i)$, say $t_j = 0$. Then $t_j \circ f$ is a function on C in neighborhood of x_i. It has an order of vanishing, which should be a.

A point $\sum a_j D_j \in B(\mathbb{Z})$ encodes such a tangency condition: i.e., we should be tangent to D_j to order a_j.

<u>Assume</u>: $\dim_{\mathbb{R}} B = \dim_{\mathbb{C}} X$

(In the case $X \to S$ with CY fibres, this is the maximally unipotent condition.)

We will construct a ring R which is either the affine or projective coordinate ring (in the log and relative cases respectively) of the mirror.

Fix $P \subseteq H_2(X, \mathbb{Z})$ with the property that
- P contains the classes of every effective curve,
- P closed under addition
- $0 \in P$
- $P \cap (-P) = H_2(X, \mathbb{Z})_{tors}$

$A = k[P]$ = monoid ring of P

$$= \bigoplus_{p \in P} k \cdot t^p,$$

$$t^p \cdot t^{p'} = t^{p+p'}.$$

Fix also a monomial ideal $I \subseteq A$ such that $A_I := A/I$ is Artinian.

Will construct an A_I-algebra R_I, giving a family

$$\begin{array}{ccc} \operatorname{Spec} R_I & & \operatorname{Proj} R_I \\ \downarrow & \text{or} & \downarrow \\ \operatorname{Spec} A_I & & \operatorname{Spec} A_I \end{array}$$

$\uparrow$

Infinitesimal nbhd of large Kähler limit of X.

(Taking limit over all I will give mirror construction in a formal neighborhood of the large Kähler limit.)

Construction of R_I:

"Theta functions"

$$R_I := \bigoplus_{p \in B(\mathbb{Z})} A_I \cdot \theta_p$$

a free A_I-module.

Need to define

$$\theta_p \cdot \theta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \theta_r$$

where $\alpha_{pqr} \in A_I = k[P]/I$

$$\alpha_{pqr} = \sum_{\beta \in P \setminus I} N_{pqr}^{\beta} \cdot t^{\beta}$$

where $N_{pqr}^{\beta} \in \mathbb{Q} \subseteq k$

Def'n of N^β_{pqr}.

$r \in B(\mathbb{Z})$, $r = \sum_{i \in I} a_i D_i$ with $a_i > 0$.

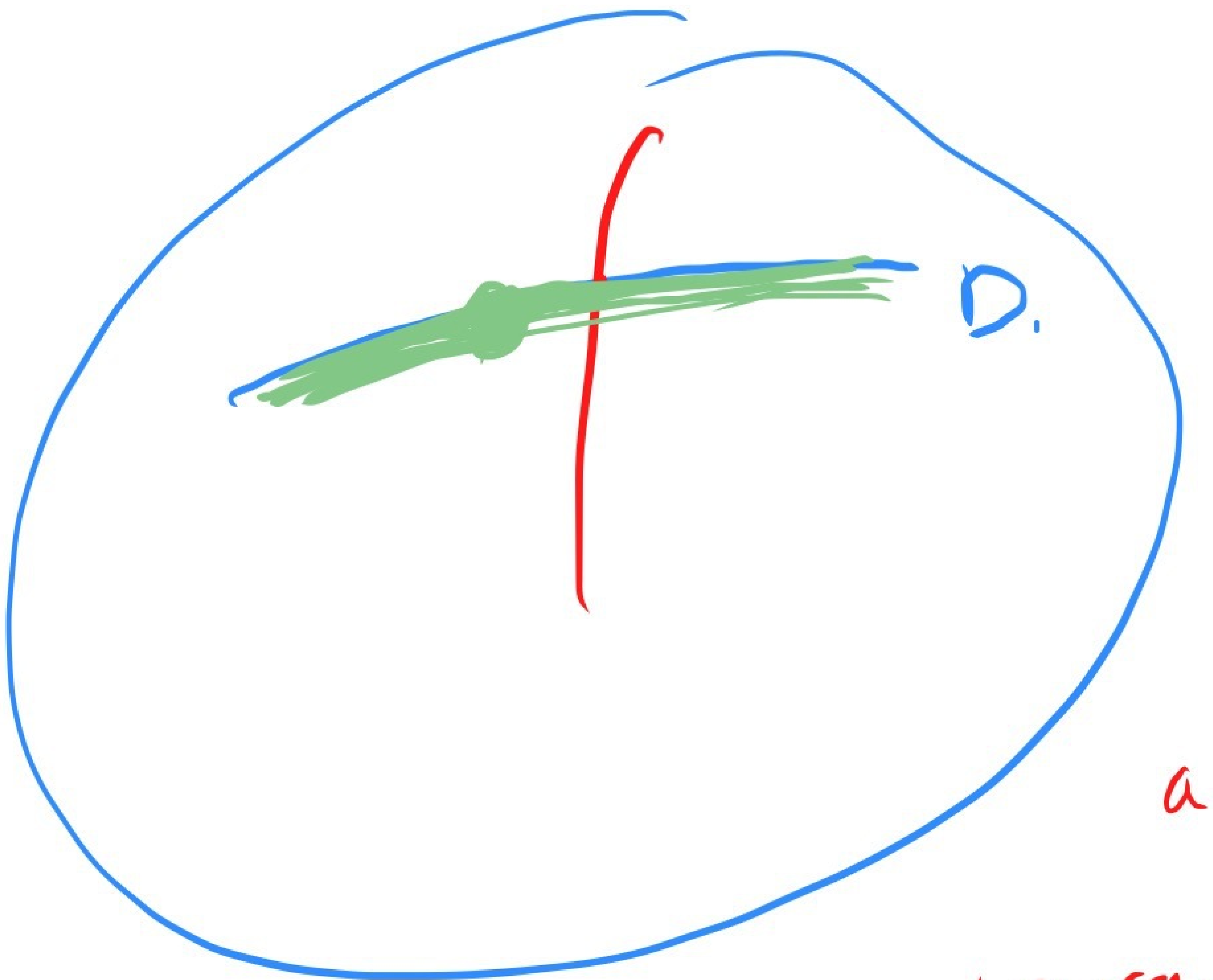
$\rightsquigarrow D_I$

Fix a point $z \in D_I$. (a point of the variety.)

Let $N^\beta_{pqr} = \#$ of maps

$$f : (C, x_1, x_2, x_{out}) \longrightarrow X$$

with
- C rational (genus 0)
- $f_*[C] = \beta$
- The tangency condition at x_1 is given by p.
- The tangency condition at x_2 is given by q.
- The tangency condition at x_{out} is given by $-r$ and $f(x_{out}) = z$.



To define negative orders of tangency, one uses logarithmic geometry and defines **punctured invariants** (Abramovich, Chen, –, Siebert)

Theorem (G–Siebert) The numbers N^{β}_{pqr} can be defined rigourously, and with this product R_I is an associative, commutative algebra with $1 = \theta_0$.