Universally counting curves in Calabi--Yau threefolds

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There are lots of ways of counting curves!

Most come from moduli spaces with virtual fundamental cycle lying over space of 1-cycles

$$
\begin{tikzcd}
\text{(stable maps)}\ \mathcal{M}'(X) \arrow[dr] & & P(X)\ \text{(stable pairs)} \arrow[dl] \\
& Z(X)\ \text{(1-cycles)} &
\end{tikzcd}
$$

Functorial under open embeddings

Most are invariant under deformation

$$Z(X/B) = \bigcup_{b \in B} Z(X_b)$$

Universal invariant with these three properties!

$$\bigoplus_{X \in CY3/\Delta^1} H^0_c(Z(X/\Delta^1)) \xrightarrow{ev_0 - ev_1} \bigoplus_{X \in CY3} H^0_c(Z(X)) \longrightarrow H_0(CY3, H^0_c(Z)) \longrightarrow 0$$

(no compactness/properness assumption!)

(inspired by work of Ionel--Parker on Gopakumar--Vafa integrality conjecture)

$$GW : H_0(CY3, H^0_c(Z)) \longrightarrow \mathbb{Q}((u))$$

$$PT : \text{------ '' ------} \longrightarrow \mathbb{Z}((q))$$

(ring homomorphisms)

Recover usual invts by applying to $(X, 1_\beta \in H^0_c(Z(X)))$ for X projective

Two such pairs could be called "enumeratively equivalent" if they represent the same class in the group above.

Can also include "higher deformation invariance" wrt families over any simplex

$$C_*\left(CY3, C_c^*(Z)\right) = \bigoplus_{X \in CY3/\Delta^k} C_c^{*+k}\left(Z(X/\Delta^k)\right)$$

$$\Big\downarrow \text{homology}$$

$$H_c^*(Z/CY3)$$

multiplication: disjoint union of cycles

co-multiplication: sum of cycles

$$W \xrightarrow{\pi} B \xrightarrow{\alpha} *$$

$$H_c^*(W/B) := H_c^*(W, \pi^! \omega_B) = \alpha_! \pi_! \pi^* \alpha^! \mathbb{Z}$$

homology of B with coefficients in fiberwise compactly supported cohomology

$$H_c^*\left(Z(Univ/Cpx_3)/Cpx_3\right)$$

Theorem: This homology group (for complex CY3's) is supported in cohomological degree <=0, and

in degree 0 it is the free polynomial algebra on "equivariant local curve elements" x_{g,m}.

Corollary: GW and PT are related by MNOP transformation on CY3's iff they are so on x_{g,m}.

Prop: Eval on x_{g,m} coincides with localized equivariant count on local curve of genus g in class m.

Bryan--Pandharipande compute equivariant GW of local curves

Okounkov--Pandharipande compute equivariant DT of local curves

Conclude: MNOP correspondence on all CY3's

Generation statements is essentially a *transversality* assertion.

Almost complex geometry: transversality wrt generic almost complex structures

--> compute $H_0\left(ACpx_3, H_c^0(Z^{CY})\right)$

Complex geometry: transversality in total space after enlarging base, *locally* on cycle space

--> compute $H_c^*\left(Z^{CY}/Cpx_3\right)$

Curve $C \subseteq X$ is <u>regular</u> when def problem of $\tilde{C} \to X$ w/ its given jet/incidence constraints is unobstructed.

Cycle $z = \sum_i m_i C_i$ is <u>semi-regular</u> when $\bigcup_i C_i \subseteq X$ is regular. $Z_{\text{semi-reg}} \subseteq Z$ constructible

$$H_c^*((Z^{CY})^{\circ}_{\text{semi-reg}}/\mathrm{Cpx}_3) \longrightarrow H_c^*(Z^{CY}/\mathrm{Cpx}_3)$$

$$\dim Z^{CY}(X/B)_{\text{semi-reg}} = \dim B$$

Lemma: $H_c^*((Z^{CY})^{\circ}_{\text{semi-reg}}/\mathrm{Cpx}_3)$ vanishes in degrees <0 and in degree 0 is freely generated by monomials in "geometric local curve elements" $y_{g,m}$ = P.D. of smooth point of $Z^{CY}(X/\Delta^r)^{\circ}_{\text{semi-reg}}$ of genus g + multiplicity m

Proposition: (Enough Divisors) Let X-->B be a family of threefolds, and let $K \subseteq Z(X/B)$ be compact analytic set whose projection map K-->B is injective. After removing from X a closed set disjoint from the support of K, there exist

disjoint! relative divisors $D_i \subseteq X \times_B U_i$ ($U_i \subseteq B$ open) which together "control" all cycles z in K.

(A cycle $z = \sum_i m_i C_i$ is *controlled* by a divisor when said divisor intersects all C_i) and no C_i contained in divisor

Proof: Induct on dimension of base B. Choosing divisors generically reduces to base of two real dimensions less. QED

Proposition: Comparison map $H_c^*((Z^{CY})^{\circ}_{\text{semi-reg}}/\mathrm{Cpx}_3) \longrightarrow H_c^*(Z^{CY}/\mathrm{Cpx}_3))$ is an isomorphism.

Proof: Use enough divisors and "generic transversality". QED

Generic transversality: Given a smooth divisor $D \subseteq X$ we can deform X by any subspace of $H^0(D, TX(-\infty D)|_D)$ and in the resulting family every connected curve intersecting D can be made regular using a suitable subspace.

Equivariant local curve elements $X_{g,m}$

E = rank two vector bundle over curve C of genus g

Fix weight r_i maps $\lambda_i : Z(E,m) \longrightarrow \mathbb{C}$ with compact joint zero set. $\lambda_1, \ldots, \lambda_N$

$$X_{g,m} = \left(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}, 1_m \cdot \prod_i \frac{\lambda_i^* c_1(\mathcal{L}^{\otimes r_i})}{r_i} \in H_c^{2n}\left(Z\left(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} / \mathbb{CP}^N\right)\right)\right)$$

(independent of choice of λ_i's)

Proposition: Monomials in equivariant and geometric local curve elements coincide modulo cycles of smaller covering multiplicity.

Question: $X_{g,m} = Y_{g,m}$?

Remark: Can use an algebraic trick to show that if x_{g,m} generate then they necessarily freely generate. (analyze possible kernels and show they must be trivial)

Question: How to keep track of multiple covers in this framework?

Conjecture: For any complex CY3, the element $(X, t^{[\cdot]}) \in H_c^0(Z^{CY}/C_{p \times_3})[[t^{H_2(X)}]]$ has the form

$$\prod_{\beta \neq 0} \prod_{g \geq 0} \left(\sum_{m \geq 0} x_{g,m} t^{m\beta} \right)^{e_{\beta,g}(X)}$$

for integers $e_{g,\beta}(X)$ (compare Ionel--Parker).