





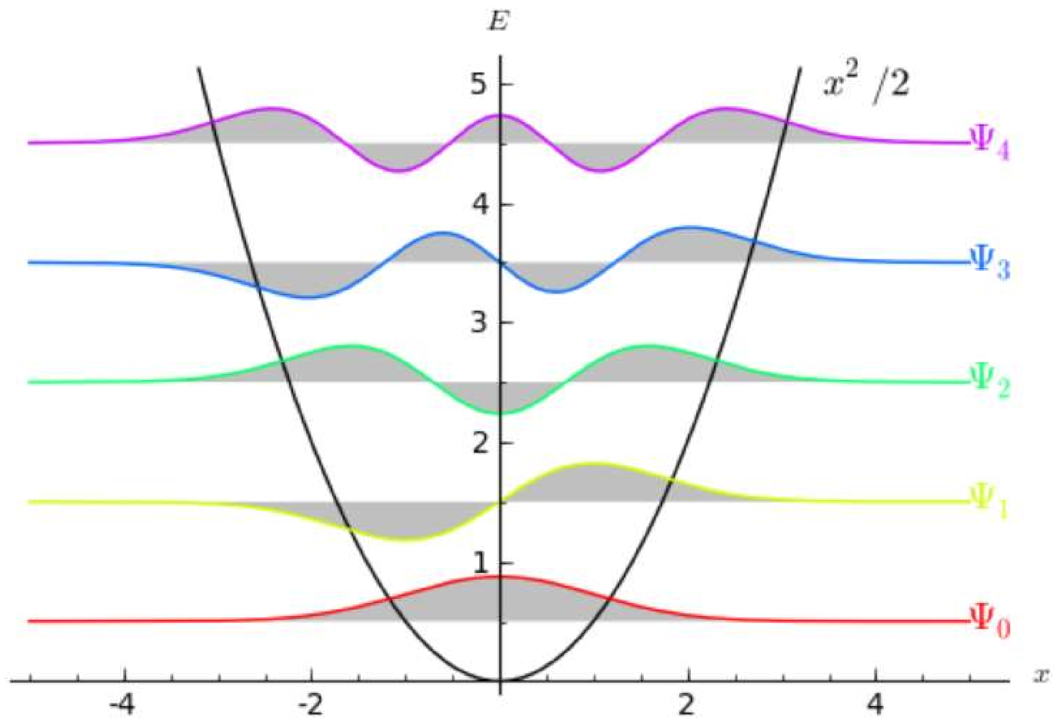


Based on:

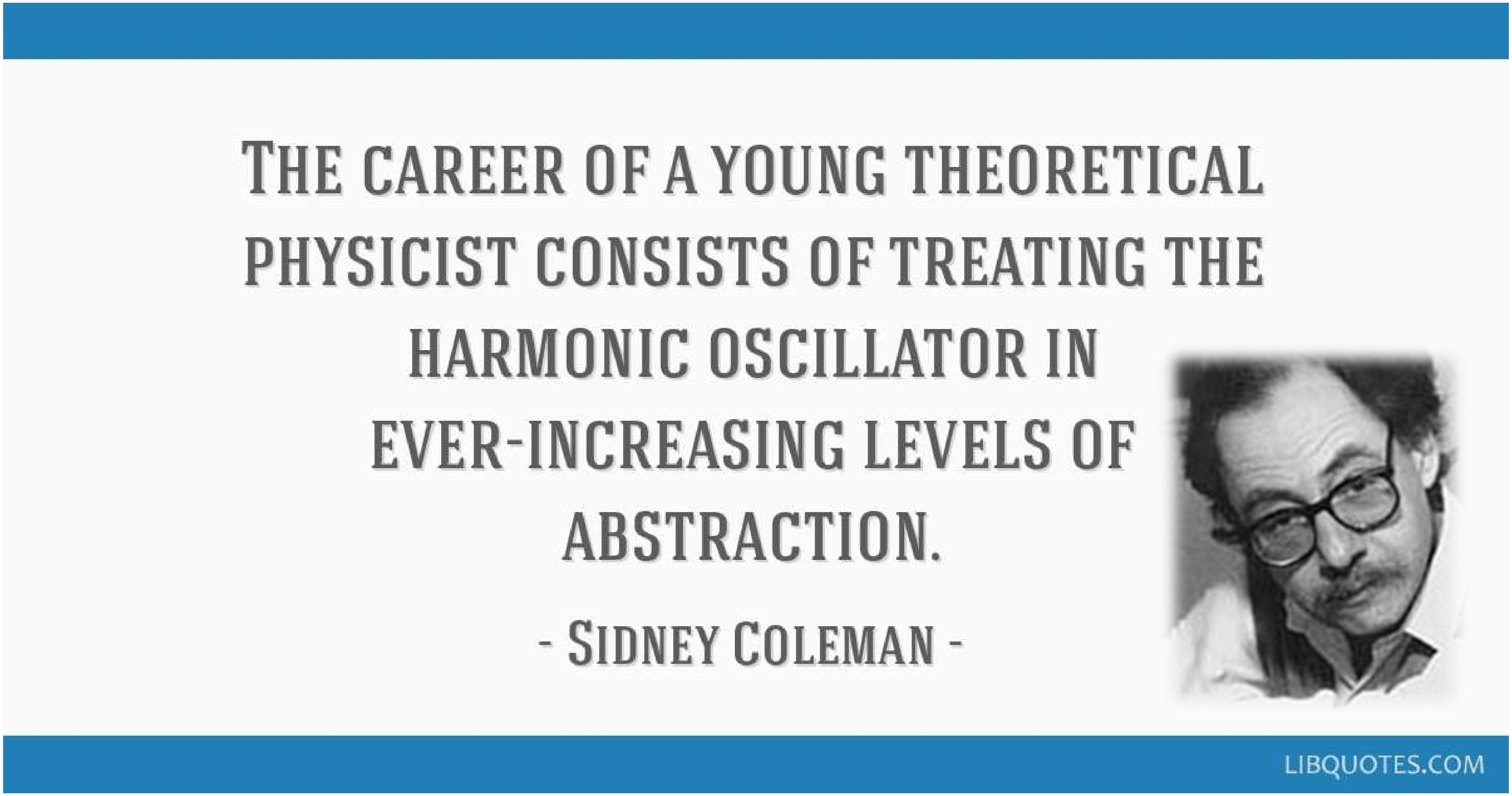
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko   arXiv:2005.05347

S.G., S.Park, P.Putrov   arXiv: 2009.11874

+ previous work & work in progress



$$\sum_{n=0}^{\infty} q^{E_n} = q^{1/2}(1 + q + q^2 + q^3 + q^4 + \dots)$$
$$= \frac{q^{1/2}}{1-q}$$



$$\sum_{n=0}^{\infty} \pm q^{E_n} = q^{-\frac{1}{2}}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \ldots)$$

't Hooft fluxes $v \in H^2(M_4, \mathcal{Z})$

$$Z_{VW}^{(v)}(M_4, q) \;=\; q^{h_v}\left(a_0^{(v)} + a_1^{(v)} q + a_2^{(v)} q^2 + \ldots\right)$$

$$= \text{ VOA character}$$

BPS: counting curves, solitons, solutions to PDEs, ...

$$\sum_{n=0}^{\infty} \pm q^{E_n} = q^{-\frac{1}{2}}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \ldots)$$

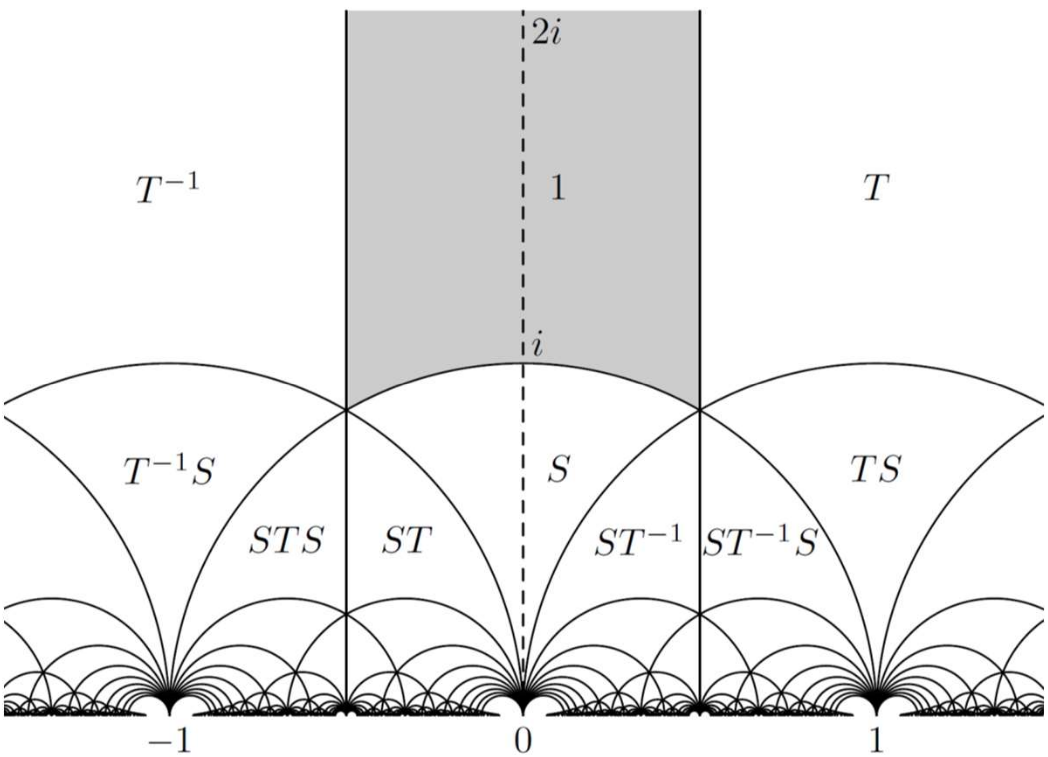
Today: 3d analogue of Vafa-Witten theory

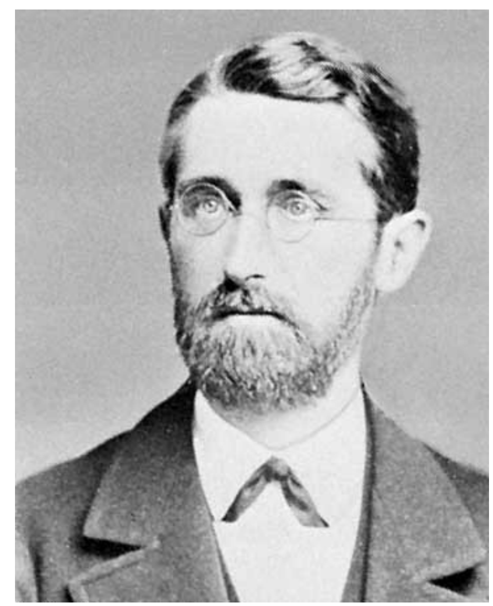
- quantum groups at generic q
- BPS "partition vector"
- also has labels $b \in \mathrm{Spin}^c(M_3) \cong H_1(M_3;\mathbb{Z})$
- also intriguing connections to chiral algebras
- very computable, ready-to-use

$$\sum_{n=0}^{\infty} \pm q^{E_n} = q^{-\frac{1}{2}}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \ldots)$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

$$= q^{\frac{1}{24}}\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots\right)$$

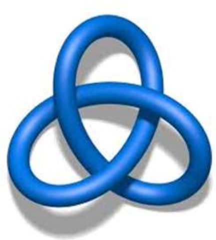


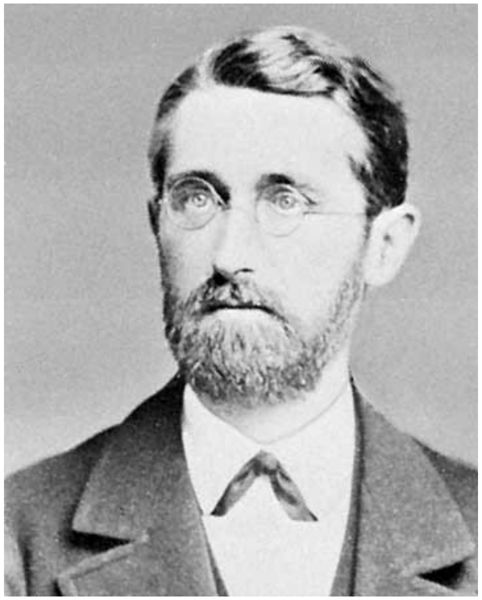


Richard Dedekind
(1877)

$$\eta(q) \;=\; q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \;=\; \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}}$$

$$\epsilon_m \;=\; \begin{cases} +1, & m = 1, 11 \pmod{12} \\ -1, & m = 5, 7 \pmod{12} \\ 0, & \text{otherwise} \end{cases}$$



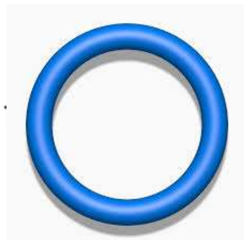
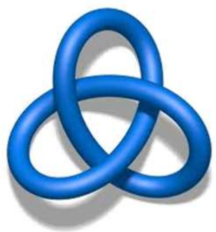


Richard Dedekind
(1877)

$$\Delta\left(\quad\right) - \Delta\left(\quad\right) = \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\Delta\left(\quad\right)$$

James Waddell Alexander II
(1923)

$$\Delta\left(\quad\right) = 1$$

knot	Alexander polynomial
	$\Delta_{\text{unknot}}(x) = 1$
	$\Delta_{\mathbf{3_1}}(x) = x^{-1} - 1 + x$
	$\Delta_{\mathbf{4_1}}(x) = -x^{-1} + 3 - x$
	$\Delta_{\mathbf{5_2}}(x) = 2x^{-1} - 3 + 2x$

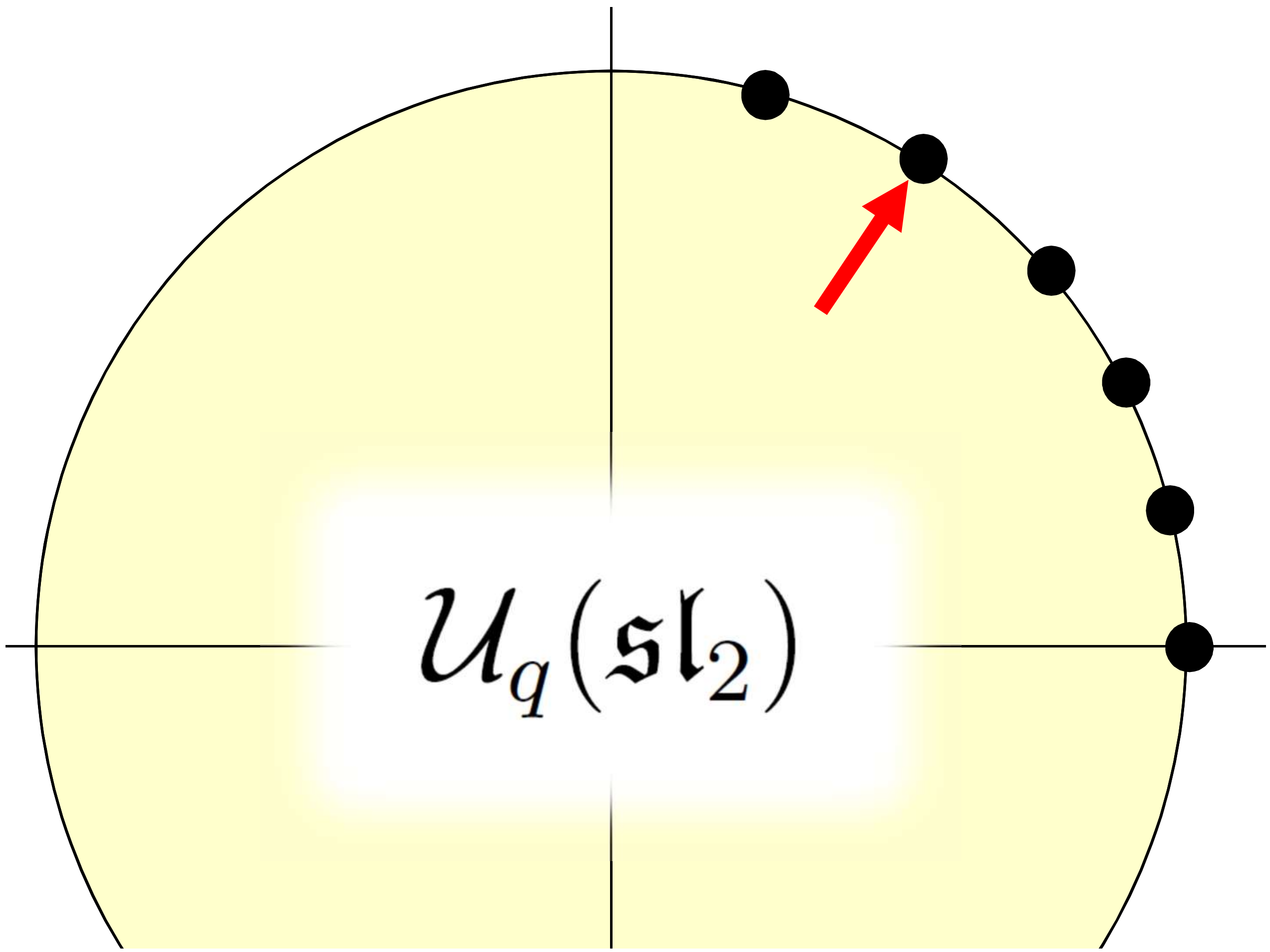
Dedekind

Alexander

$$\eta(q) = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}}$$

$$\frac{x^{-1} - x}{\Delta_{\mathbf{3_1}}(x^2)} = \sum_{m=1}^{\infty} \epsilon_m \left(x^m - x^{-m}\right)$$

$$F_{\mathbf{3_1}}(x, q) = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}} \left(x^m - x^{-m}\right)$$



$$KEK^{-1} = q^2 E\,, \qquad KFK^{-1} = q^{-2} F\,, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$$

$$KEK^{-1} = q^2 E\,, \quad KFK^{-1} = q^{-2} F$$

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$q = e^{\frac{i\pi}{p}}$$

2p-th root of unity

"restricted quantum group"

$$E^p = 0 = F^p \qquad , \qquad K^{2p} = \mathbf{1}$$

$$\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$$

$$KEK^{-1} = q^2 E\,, \quad KFK^{-1} = q^{-2} F$$

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2p-th root of unity

"restricted quantum group"

$$E^p = 0 = F^p \qquad , \qquad K^{2p} = \mathbf{1}$$

$$\Rightarrow \dim \overline{\mathcal{U}}_q(\mathfrak{sl}_2) = 2p^3$$

$\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$

2p-th root of unity
"restricted quantum group"

$$\dim \overline{\mathcal{U}}_q(\mathfrak{sl}_2) = 2p^3$$

$$\dim \mathfrak{Z} = 3p - 1$$

SL(2,ℤ)

S and T matrices of a log-VOA

$\mathcal{U}_q(\mathfrak{sl}_2) \xrightarrow[\text{root of 1}]{q = e^{\frac{i\pi}{p}}}$ (3p-1)-dimn'l SL(2,$\mathbb{Z}$) rep $\xrightarrow[\text{semisimpl.}]{}$ (p-1)-dimn'l SL(2,$\mathbb{Z}$) rep

Triplet (1,p) model

WZW $SU(2)_{p-2}$

[B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin]
[T.Creutzig, A.Gainutdinov, I.Runkel]
[C.Negron]
[P.Etingof, V.Ostrik]
[M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison]

	$\mathcal{U}_q^{\text{small}}(\mathfrak{sl}_2)$ with k odd	$\overline{U}_q(\mathfrak{sl}_2)$ with $k=2p$ even
dimension	k^3	$2p^3$
$\dim(\mathfrak{Z})$	$(3k-1)/2$	$3p-1$
block decomposition of $\mathfrak{Z}$	$\underbrace{\mathbf{3}\oplus\cdots\oplus\mathbf{3}}_{\frac{k-1}{2}\text{ times}}\oplus\mathbf{1}$	$\underbrace{\mathbf{3}\oplus\cdots\oplus\mathbf{3}}_{p-1\text{ times}}\oplus\mathbf{1}\oplus\mathbf{1}$
$SL(2,\mathbb{Z})$ rep.	$W_{\frac{k+1}{2}}\oplus\mathbb{C}^2\otimes\mathrm{Ver}_{\frac{k-1}{2}}$	$W_{p+1}\oplus\mathbb{C}^2\otimes\mathrm{Ver}_{p-1}$
log-VOA	not known	$(1,p)$ triplet

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LOGARITHMIC KNOT INVARIANTS ARISING FROM RESTRICTED QUANTUM GROUPS

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We construct knot invariants from the radical part of projective modules of the restricted quantum group $\overline{\mathcal{U}}_q(sl_2)$ at $q=\exp(\pi\sqrt{-1}/p)$, and we also show a relation between these invariants and the colored Alexander invariants. These projective modules are related to logarithmic conformal field theories.

Keywords: Invariants of knots and links; restricted quantum group.

Mathematics Subject Classification 2000: 57M27, 17B37

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ELSEVIER

Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to $\mathfrak{sl}_2$

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ABSTRACT

We study the tensor structure of the category of finite-dimensional modules of the restricted quantum enveloping algebra associated to $\mathfrak{sl}_2$. Tensor product decomposition rules for all indecomposable modules are explicitly given. As a by-product, it is also shown that the category of finite-dimensional modules of the restricted quantum enveloping algebra associated to $\mathfrak{sl}_2$ is *not* a braided tensor category.

$\mathcal{U}_q^{\mathrm{DK}}(\mathfrak{sl}_2)$ De Concini-Kac form

$\mathcal{U}_q^{\mathrm{L}}(\mathfrak{sl}_2)$ Lusztig (divided powers)

$\mathcal{U}_q^{\mathrm{small}}(\mathfrak{sl}_2)$ $\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$

$\overline{\mathcal{U}}_q^{\mathrm{KS}}(\mathfrak{sl}_2)$ without $K^{\frac{1}{2}}$ (e.g. Kondo-Saito)

$\overline{\mathcal{U}}_q^{\mathrm{MN}}(\mathfrak{sl}_2)$ with $K^{\frac{1}{2}}$ (e.g. Murakami-Nagatomo)

$\overline{\mathcal{U}}_q^{(\Phi)}(\mathfrak{sl}_2)$ quasi-Hopf modification

$\overline{\mathcal{U}}_q^{H}(\mathfrak{sl}_2)$ unrolled restricted

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INVARIANTS OF COLORED LINKS

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Let us discuss connection of the new colored link invariants to the multivariable Alexander polynomial. It was shown by J. Murakami [15] that a colored link invariant which corresponds to $\hat{\Phi}(L, \alpha)$ for the $N=2$ case is a version of the multivariable Alexander polynomial (the Conway potential function). Therefore the new colored link invariants $\hat{\Phi}(L, \alpha)$ for $N = 3, 4, \ldots,$ are generalizations of the multivariable Alexander polynomial.

Osaka J. Math.
45 (2008), 541–564

COLORED ALEXANDER INVARIANTS AND CONE-MANIFOLDS

JUN MURAKAMI

1. Introduction

New link invariants are introduced in [1] for colored links. They are defined for each positive integer N and considered as a generalization of the multivariable Alexander polynomial [12], which corresponds to the case $N = 2$. Here we redefine these invariants by using the universal R-matrix of $\mathcal{U}_q(sl_2)$.

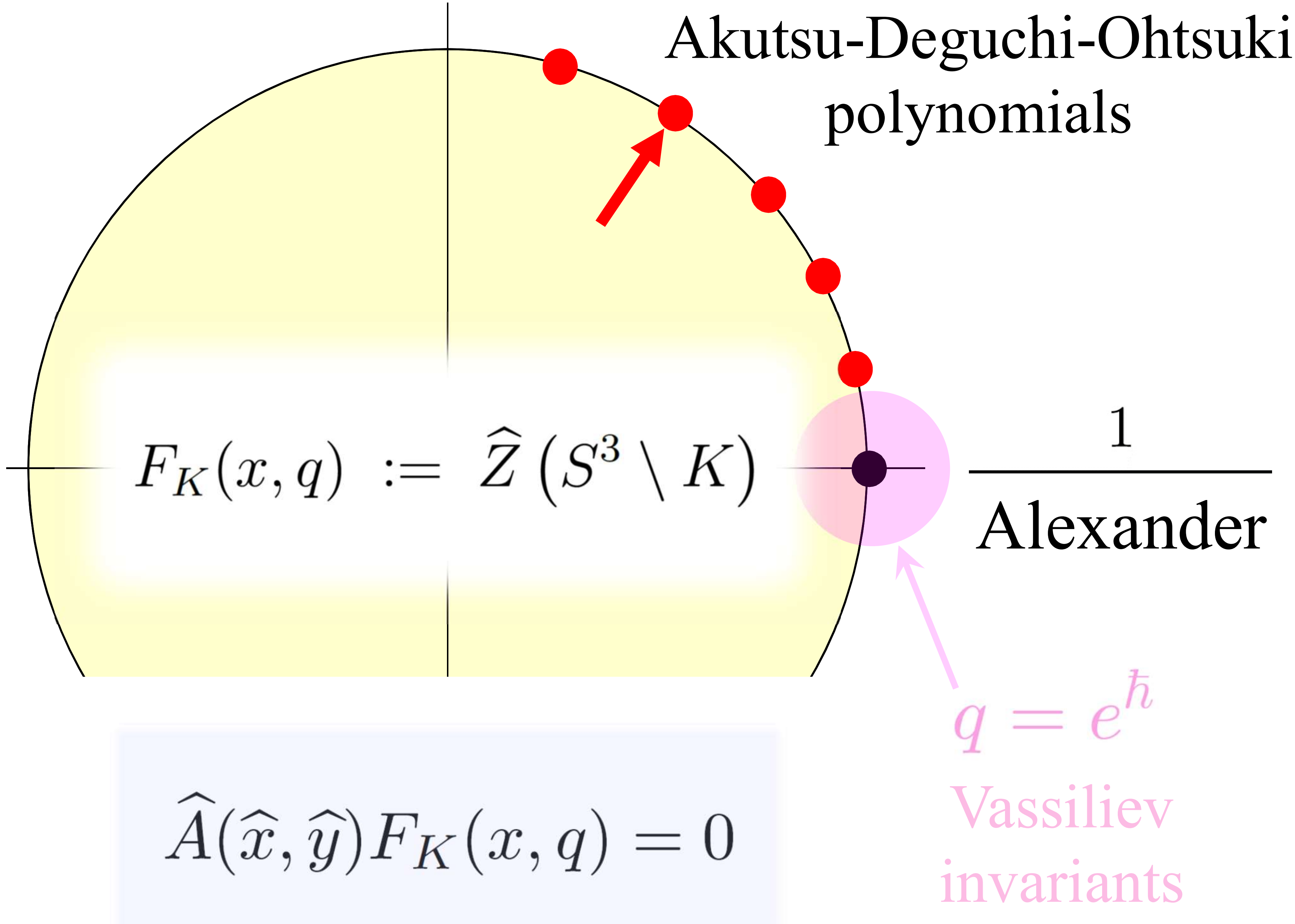
Let $q = \exp(\pi\sqrt{-1}/N)$ be a $2N$-th root of unity. Let $\mathcal{U}_q(sl_2)$ be the quantum enveloping algebra corresponding to the Lie algebra sl_2 defined by the following generators and relations:

$$\mathcal{U}_q(sl_2) = \left\langle K, K^{-1}, E, F \;\middle|\; KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\right.$$
$$\left. [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle.$$

1	1
2	$-x-1-x^{-1}$
3	$\zeta_3 x^2+\zeta_3 x+(\zeta_3-\zeta_3^{-1})+\zeta_3 x^{-1}+\zeta_3 x^{-2}$
4	$ix^3+ix^2+(1+i)x+(1+2i)+(1+i)x^{-1}+ix^{-2}+ix^{-3}$

1	1
2	$x+3+x^{-1}$
3	$x^2+3x+5+3x^{-1}+x^{-2}$
4	$x^3+3x^2+6x+7+6x^{-1}+3x^{-2}+x^{-3}$
5	$x^4+3x^3+(6+\zeta_5+\zeta_5^{-1})x^2+(9+\zeta_5+\zeta_5^{-1})x+10+\dots$

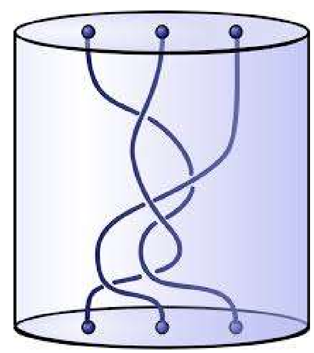
Enter physics: 2020



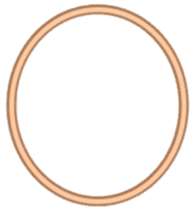
Akutsu-Deguchi-Ohtsuki
polynomials
$F_K(x,q) := \widehat{Z}\left(S^3 \setminus K\right)$
$\frac{1}{\text{Alexander}}$
$q = e^{\hbar}$
Vassiliev
invariants
$\widehat{A}(\widehat{x},\widehat{y})F_K(x,q) = 0$

LARGE COLOR R-MATRIX FOR KNOT COMPLEMENTS AND STRANGE IDENTITIES

SUNGHYUK PARK

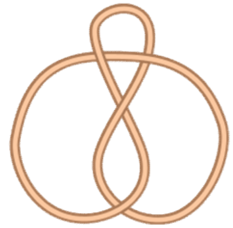


$$KEK^{-1} = q^2 E\,, \qquad KFK^{-1} = q^{-2} F\,, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

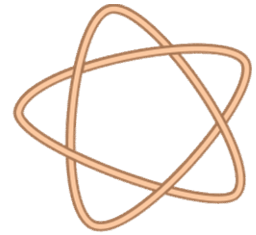


Unknot
[GPV'16]

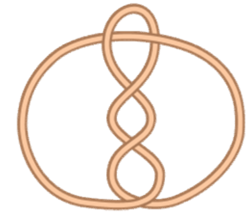
3_1
[GM'19]



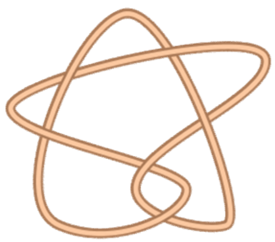
4_1
[GM'19]



5_1
[GM'19]



5_2
[Park'20]

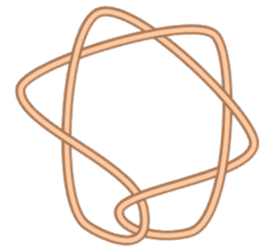


6_1
[Park'20]

6_2
?

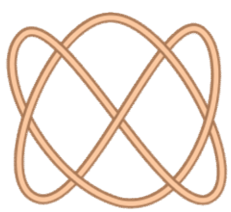
6_3
?

7_1
[GM'19]



7_2
[Park'20]

7_3
[Park'20]



7_4
[Park'20]

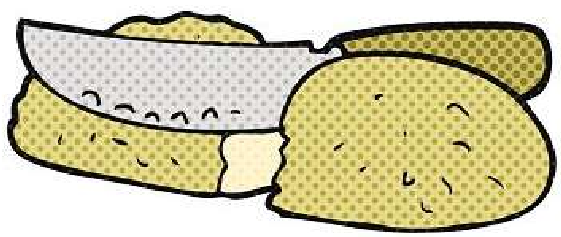
7_5
[Park'20]

7_6
?

7_7
?

Theorem [Lickorish, Wallace, Kirby]:

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link $K \hookrightarrow S^3$ (*i.e.* a surgery along a framed link)



Theorem [Lickorish, Wallace, Kirby]:

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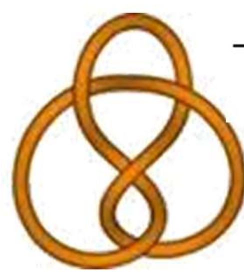


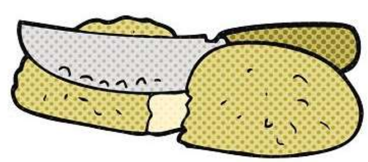
$+\frac{4}{17}$

$M_3 = S^3_{-1}($  $) = S^3_{+1}($  $)$:

$$\widehat{Z}(q) = q^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1};q)_n}$$

$M_3 = S^3_{-1/2}($  $)$:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \dots$$
$$\dots - 15040q^{500} + \dots)$$



dependence on M_3

(Topology)

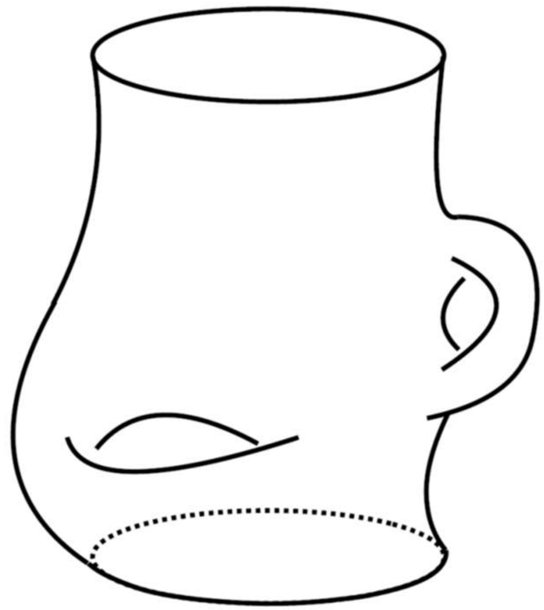
dependence on q

See a talk by Miranda Cheng in this series

$$\widehat{Z}_b(M_3, q)$$

dependence on

$$b \in \mathrm{Spin}^c(M_3) \cong H_1(M_3; \mathbb{Z})$$



d	0	1	2	3	4	5	6	7	8	9	…
Ω_d^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	…

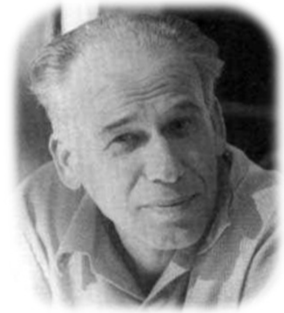
A.Kapustin, R.Thorngren, A.Turzillo, Z.Wang (2014)
D.Freed, M.Hopkins (2016)

d	0	1	2	3	4	5	6	7	8	9	$\dots$
Ω_d^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\dots$

$$\mu(M_3, s) := \sigma(M_4) \mod 16$$

d	0	1	2	3	4	5	6	7	8	9	…
Ω_d^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	…

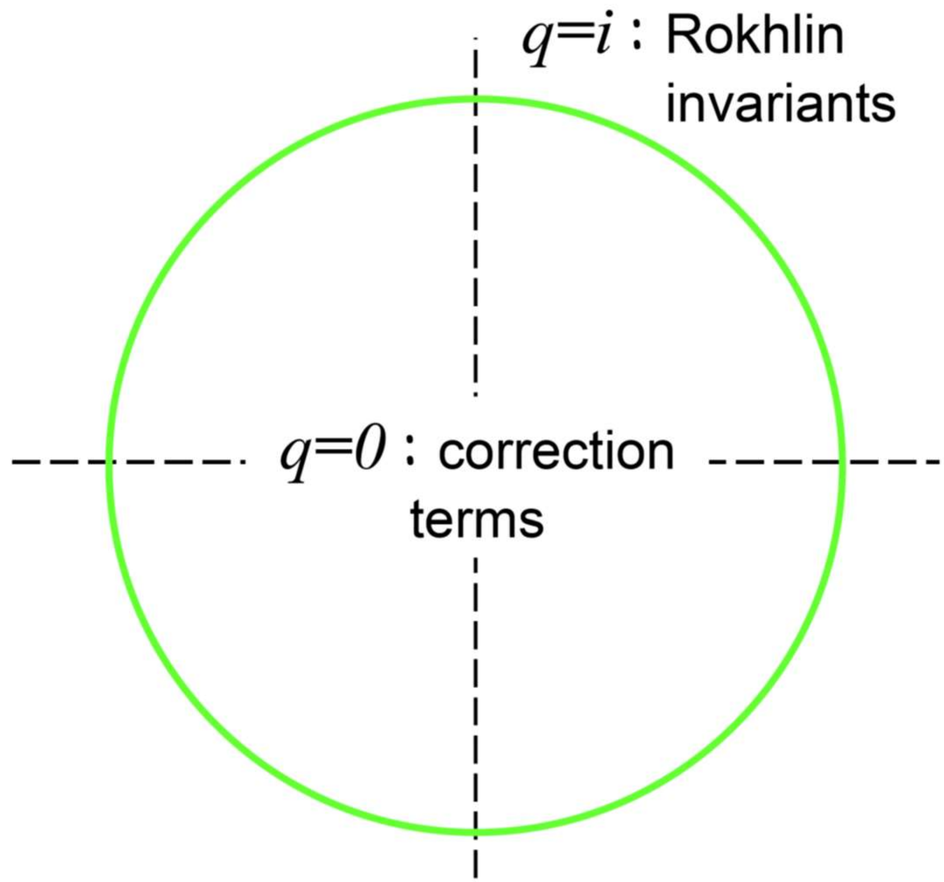
Rokhlin invariant (shows $|\Theta^3_{\mathbb{Z}}| \geq 2$)



Froyshov invariant
(shows $\Theta^3_{\mathbb{Z}}$ contains a $\mathbb{Z}$-summand)

invariants α, β, γ introduced by Manolescu


“correction terms”
of Ozsvath and Szabo

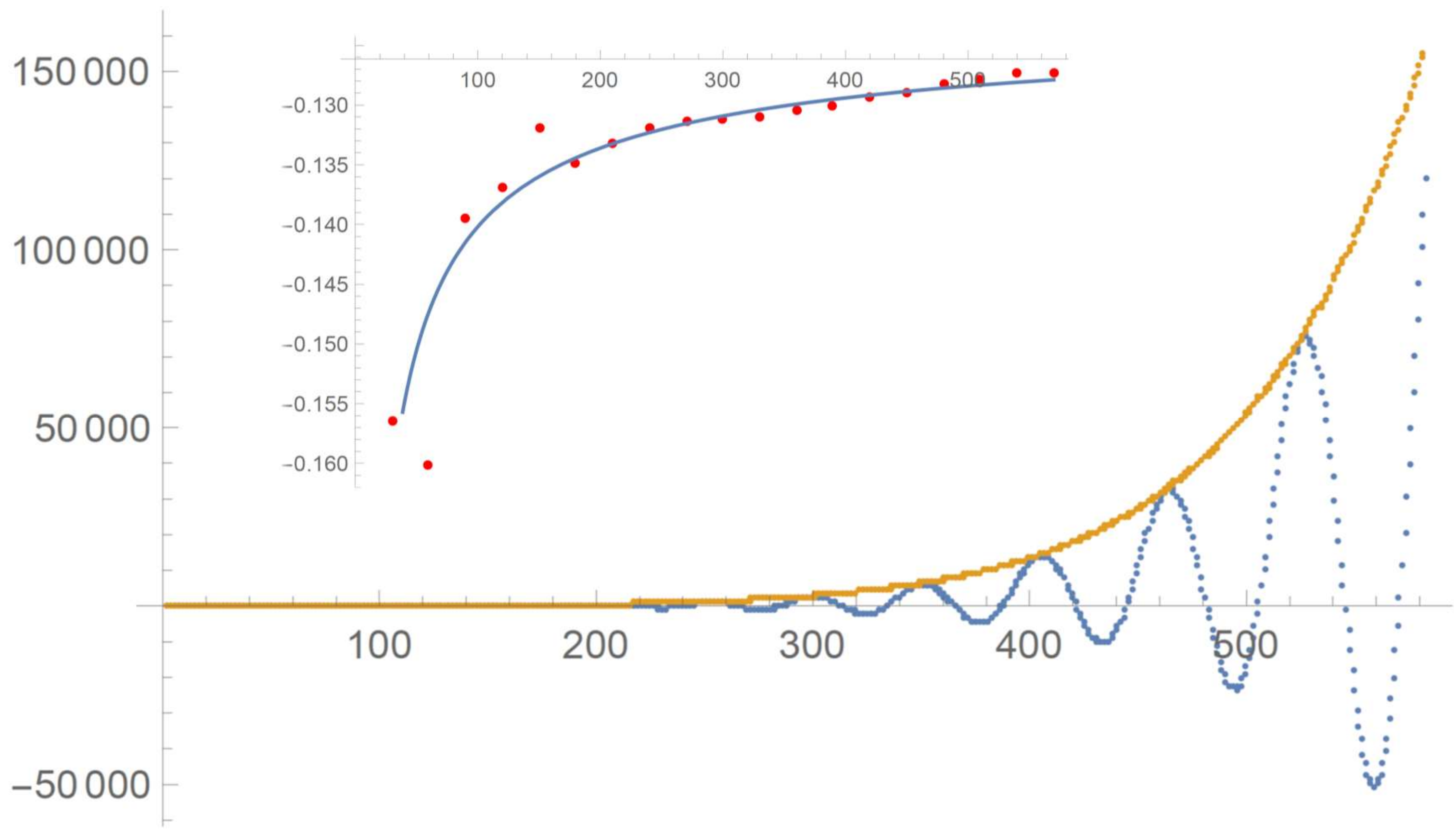


correction term

$$\widehat{Z} = q^{\Delta}\left(a_0 + a_1 q + a_2 q^2 + \dots\right) \quad \in \; q^{\Delta}\mathbb{Z}[[q]]$$

$M_3 = S^3_{-1/2}($$)$:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}\big(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} + \ldots$$
$$\ldots - 15040q^{500} + \ldots\big)$$



Thanks for listening.

Questions?