Geometry from Donaldson-Thomas invariants

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1. Introduction
2. Stability conditions and DT invariants
3. Detour: Stokes data
4. DT Riemann-Hilbert problem
5. Joyce structures

Topological strings and the derived category

Given a Calabi-Yau threefold X we are interested in relations between:

- String theory compactified on X,
- The derived category of coherent sheaves $D(X) = D^b \operatorname{Coh}(X)$.

For example:

- Dualities in string theory induce equivalences $D(X_1) \cong D(X_2)$,
- Π-stability for D-branes corresponds to stability in $D(X)$,
- BPS invariants = DT invariants,
- Partition function = ~~generating function for rank 1 PT invariants~~.

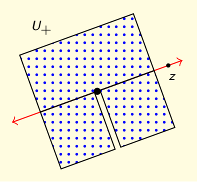
We are only interested in 'natural' constructions with $D(X)$. This means that everything should be manifestly covariant under equivalences $D(X_1) \cong D(X_2)$.

This leads us to consider non-perturbative aspects of topological strings.

A Riemann-Hilbert problem

Fix $z \in \mathbb{C}^*$ and set $U_\pm = \mathbb{C}^* \setminus \mathbb{R}_{>0} \cdot (\pm iz)$. Seek holomorphic functions

$$y_\pm \colon U_\pm \to \mathbb{C}^*$$



satisfying the conditions

(i) $y_\pm(\epsilon) \to 1$ as $\epsilon \to 0$,

(ii) $|\epsilon|^{-k} < |y_\pm(\epsilon)| < |\epsilon|^k$ as $|\epsilon| \to \infty$,

(iii) on $U_- \cap U_+$ there is a relation

$$y_-(\epsilon) = \begin{cases} y_+(\epsilon)(1 - e^{-z/\epsilon}) & \text{if } \mathrm{Re}(\epsilon/z) > 0 \\ y_+(\epsilon)(1 - e^{+z/\epsilon}) & \text{if } \mathrm{Re}(\epsilon/z) < 0 \end{cases}.$$

There is a unique solution

$$y_\pm(\epsilon) = \Lambda\left(\frac{\pm z}{2\pi i \epsilon}\right)^{\pm 1}, \qquad \Lambda(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}}.$$

Non-perturbative partition function of $X = T^*S^3$

Define $\tau_\pm(z,\epsilon)$ by the condition

$$2\pi i \cdot \frac{\partial \log \tau_\pm(z,\epsilon)}{\partial z} = \frac{\partial \log y_\pm(z,\epsilon)}{\partial \epsilon}.$$

Then using the Barnes G-function we can write

$$\tau_\pm(z,\epsilon) = \Upsilon\Big(\frac{\pm z}{2\pi i\epsilon}\Big), \qquad \Upsilon(w) = \frac{e^{-\zeta'(-1)}\, e^{\frac{3}{4}w^2}\, G(w+1)}{(2\pi)^{\frac{w}{2}}\, w^{\frac{w^2}{2}}}.$$

There are asymptotic expansions as $\epsilon \to 0$

$$\log \tau_\pm(z,\epsilon) \sim -\frac{1}{12}\log\Big(\frac{\pm 2\pi i\epsilon}{z}\Big) + \sum_{g\geq 2} \frac{B_{2g}}{2g(2g-2)}\Big(\frac{2\pi i\epsilon}{z}\Big)^{2g-2}.$$

- The RH problem came from the DT theory of the Fukaya category of X.
- The sum is the genus expansion of the B-model partition function for X.

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Charge lattice notation

Consider a $\mathbb{C}$-linear triangulated category $\mathcal{D}$ with the CY_3 property:

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(B, A[3])^*.$$

Fix a group homomorphism

$$\mathrm{ch}\colon K_0(\mathcal{D}) \to \Gamma = \mathbb{Z}^{\oplus n}.$$

Assume given a skew form $\langle -, - \rangle\colon \Gamma \times \Gamma \to \mathbb{Z}$ such that

$$\langle \mathrm{ch}([E]), \mathrm{ch}([F]) \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(E, F[i]).$$

Define the Poisson algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n, \qquad \mathbb{C}[\mathbb{T}] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma,$$

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_{\alpha+\beta}.$$

Stability conditions

A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where:

- $\mathcal{P} = \cup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$ is a full subcategory "the semistable objects",
- $Z \colon \Gamma \to \mathbb{C}$ is a homomorphism of abelian groups "the central charge",

satisfying a short list of axioms.

The set $\mathrm{Stab}(\mathcal{D})$ is a complex manifold with an action of $\mathrm{Aut}(\mathcal{D})$.

The forgetful map $\mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) = \mathbb{C}^n$ is a local homeomorphism.

We shall assume the existence of DT invariants

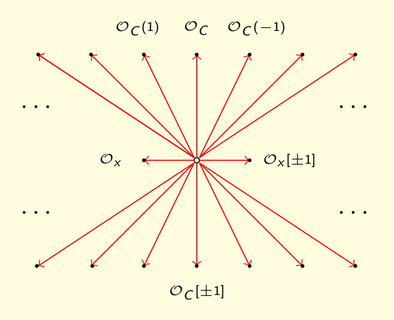
$$\Omega_\sigma(\gamma) \in \mathbb{Q}, \qquad \sigma \in \mathrm{Stab}(\mathcal{D}), \qquad \gamma \in \Gamma.$$

Equivalent data: $\mathrm{DT}_\sigma(\gamma) = \sum_{\gamma = n\alpha} \Omega_\sigma(\alpha)/n^2$.

Ray diagram

A stability condition $\sigma = (Z, \mathcal{P})$ has an associated ray diagram in $\mathbb{C}$.

It is the set of rays $\mathbb{R}_{>0} \cdot Z(E) \subset \mathbb{C}^*$ for all $E \in \mathcal{D}$ semistable.



This example is for $D(X)$ with $X = \operatorname{Tot} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ the resolved conifold.

DT automorphisms

Try to define, for each ray $\ell \subset \mathbb{C}^*$, a Poisson automorphism of $\mathbb{T}$:

$$\mathbb{S}_\sigma(\ell)^*(x_\beta) = \exp\Big\{ \sum_{Z(\gamma)\in\ell} \mathrm{DT}_\sigma(\gamma)\cdot x_\gamma, - \Big\}(x_\beta) = x_\beta \cdot \prod_{Z(\gamma)\in\ell} (1 - x_\gamma)^{\Omega_\sigma(\gamma)\cdot\langle\beta,\gamma\rangle}$$

(We should replace $\mathbb{T}$ by a torsor over it which introduces some signs.)

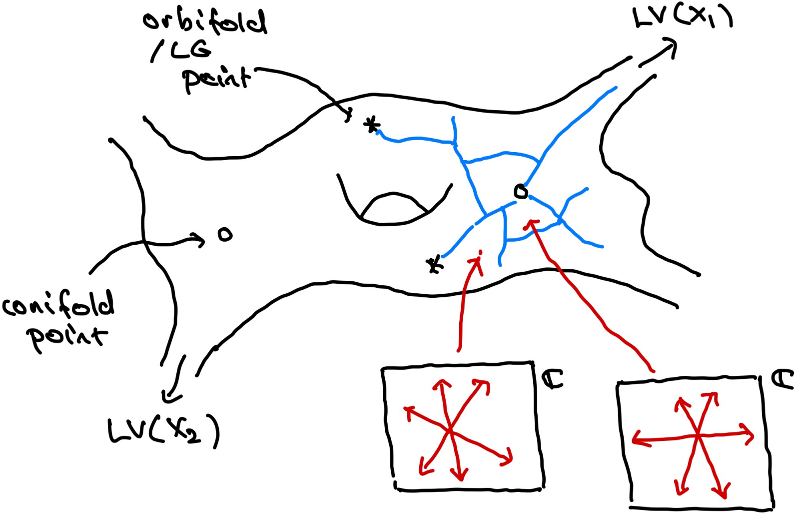
Work is required to make rigorous sense of this. Possible approaches:

- formal series: replace $\operatorname{Aut}\mathbb{C}[x_i^{\pm 1}]$ with $\operatorname{Aut}\mathbb{C}[[x_i]]$,
- assume that for some $R > 0$

$$\sum_{\gamma\in\Gamma} |\Omega_\sigma(\gamma)| \cdot e^{-R|Z(\gamma)|} < \infty,$$

  and use automorphisms defined on analytic open subsets of $\mathbb{T}$.

Cartoon of Stab $D(X)/\operatorname{Aut} D(X)$



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A meromorphic connection

Consider an equation for $Y\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{d\epsilon} Y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon} \right) Y(\epsilon)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$, such that

(i) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\mathrm{reg}}$ is diagonal with $u_i \neq u_j$,

(ii) V has zeroes on the diagonal.

The Stokes rays of the equation at $\epsilon = 0$ are defined to be the rays

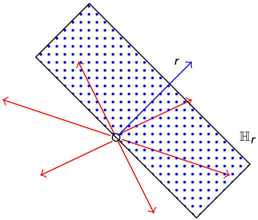
$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha).$$

Canonical solutions in half-planes

Theorem

For any half-plane $\mathbb{H}_r \subset \mathbb{C}^*$ centered on a non-Stokes ray $r \subset \mathbb{C}^*$, there is a unique solution $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(\epsilon) \cdot \exp(U/\epsilon) \to 1 \text{ as } \epsilon \to 0.$$

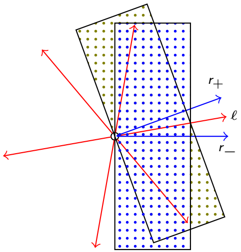


Stokes factors

The Stokes factor $\mathbb{S}(\ell)$ associated to a Stokes ray ℓ is defined by

$$Y_{r_-}(\epsilon) = Y_{r_+}(\epsilon) \cdot \mathbb{S}(\ell), \qquad \mathbb{S}(\ell) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G,$$

where $r_\pm$ are small perturbations of ℓ.



Riemann-Hilbert problem: $G = \mathrm{GL}_n(\mathbb{C})$ setting

Suppose given $U \in \mathfrak{h}^{\mathrm{reg}}$ and the Stokes factors $\mathbb{S}(\ell)$.

To reconstruct V we first construct the half-plane solutions $Y_r(t)$.

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(\epsilon) \cdot \exp(U/\epsilon) \to 1 \text{ as } \epsilon \to 0 \text{ in } \mathbb{H}_r,$$

$$|\epsilon|^{-k} < \| Y_r(\epsilon) \| < |\epsilon|^k \text{ as } \epsilon \to \infty \text{ in } \mathbb{H}_r,$$

and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

$$Y_{r_-}(\epsilon) = Y_{r_+}(\epsilon) \cdot \overset{\curvearrowright}{\prod_{\ell \in \Delta}} \mathbb{S}(\ell) \text{ for } \epsilon \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

Donaldson-Thomas Riemann-Hilbert problem

Fix data $(\Gamma, \langle -,- \rangle, Z, \Omega)$ as above, and a point $\xi = \exp(\theta) \in \mathbb{T}$.

Take $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$ and compose $Y \colon \mathbb{C}^* \to G$ with $\mathrm{eval}_\xi \colon G \to \mathbb{T}$.

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $X_r \colon \mathbb{H}_r \to \mathbb{T}$ such that

$$X_r(\epsilon) \cdot \exp(Z/\epsilon) \to \xi \in \mathbb{T} \text{ as } \epsilon \to 0 \text{ in } \mathbb{H}_r,$$

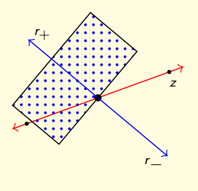
$$|\epsilon|^{-k} < \|X_r(\epsilon)\| < |\epsilon|^k \text{ as } \epsilon \to \infty \text{ in } \mathbb{H}_r,$$

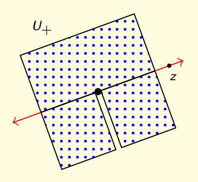
and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

$$X_{r_-}(\epsilon) = X_{r_+}(\epsilon) \cdot \overset{\curvearrowright}{\prod_{\ell \in \Delta}} \mathbb{S}(\ell) \text{ for } \epsilon \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

Example from the introduction

- $\Gamma = \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m$ with $\langle \gamma_m, \gamma_e \rangle = 1$,
- $Z(\gamma_e) = z \in \mathbb{C}^*$ and $Z(\gamma_m) = 0$,
- $\Omega(\pm\gamma_e) = 1$ with all other BPS invariants 0,
- Take constant term $\theta(\gamma_e) = \theta(\gamma_m) = 0$,
- Jumping: $x_e \mapsto x_e$, $x_m \mapsto x_m(1 - x_e^{\pm 1})^{\pm 1}$,
- Then $Y_{r,\gamma_e}(\epsilon) = e^{-z/\epsilon}$.
- Write $y_\pm(\epsilon) = Y_{r_\pm,\gamma_m}(\epsilon)$ for $r_\pm \in \pm\,\mathrm{Im}(\epsilon/z)$.
- Get the RH problem from the introduction.



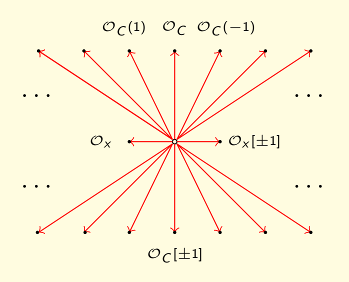


Another non-perturbative partition function

Let X be the resolved conifold.

Take $\mathcal{D} = D^b_c \operatorname{Coh}(X)$.

Set $Z(\mathcal{O}_C) = v$ and $Z(\mathcal{O}_x) = w$.



$$\log \tau(v, w, \epsilon) = \int_{\mathbb{R}+i\delta} \frac{e^{vs}-1}{e^{ws}-1} \cdot \frac{-e^{\epsilon s}}{(e^{\epsilon s}-1)^2} \cdot \frac{ds}{s}$$

$$+\Big(\frac{w}{2\pi i\epsilon}\Big)^2 \big(\operatorname{Li}_3(e^{2\pi i v/w}) - \zeta(3)\big) + \frac{\pi i v}{12 w}.$$

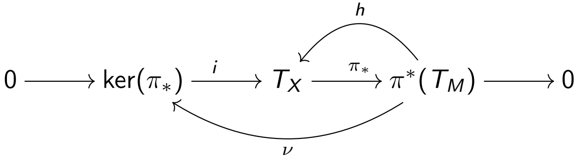
As $\epsilon \to 0$ there is an asymptotic expansion

$$\log \tau(v, w, \epsilon) \sim (\cdots) + \sum_{g\geq 2} \frac{B_{2g}}{2g\,(2g-2)!} \Big(\operatorname{Li}_{3-2g}(e^{2\pi i v/w}) + \frac{B_{2g-2}}{2g-2}\Big) \Big(\frac{2\pi i \epsilon}{w}\Big)^{2g-2}.$$

The $g \geq 2$ terms match the GW generating function.

Joyce (≈ non-linear Frobenius) structure

Let M be a complex manifold with tangent bundle $\pi\colon X = T_M \to M$.



There is a canonical isomorphism $\nu\colon \pi^*(T_M) \to \ker(\pi_*)$. Set $v = i \circ \nu$.

A non-linear connection on π is a splitting $h\colon \pi^*(T_M) \to T_X$.

Definition

A pre-Joyce structure on M consists of

(i) a holomorphic symplectic form ω,

(ii) a non-linear connection h on $\pi\colon X = T_M \to M$,

such that for each $\epsilon \in \mathbb{C}^*$ the connection $h_\epsilon = h + \epsilon^{-1} v$ is flat and symplectic.

Joyce structures from the DT RH problem

The RH problem depends on $\sigma \in M = \mathrm{Stab}(\mathcal{D})$ and $\xi \in \mathbb{T}$.

Recall that $T_\sigma M = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*)$.

If the RH problem had unique solutions we would get maps

$$Y_r(\epsilon) \colon X = T_M \to \mathbb{T}.$$

Then we could define $\mathrm{im}(h_\epsilon) = \ker(dY_r(\epsilon)) \subset T_X$.

A Joyce structure has extra compatibilities with

(i) integral affine structure $T_M^{\mathbb{Z}} \subset T_M$

(ii) $\mathbb{C}^*$-action on M,

(iii) $-1\colon T_M \to T_M$

This is the expected geometric structure on $M = \mathrm{Stab}(\mathcal{D})$.

Given (slightly murky) extra data we can define an appropriate $\tau\colon X \to \mathbb{C}$.

Categories of class $S[A_1]$

Theorem (Ivan Smith, TB)

Fix $g \geq 0$, and $m = (m_1, \cdots, m_d)$, with $d \geq 1$ and $m_i \geq 3$. Then there is a CY_3 triangulated category $\mathcal{D} = \mathcal{D}(g, m)$ such that

$$\operatorname{Stab} \mathcal{D}(g, m) / \operatorname{Aut} \mathcal{D}(g, m) \cong \operatorname{Quad}(g, m),$$

where $\operatorname{Quad}(g, m)$ parameterizes pairs (C, q), with C a compact Riemann surface of genus g, and $q = q(x)dx^{\otimes 2}$ a quadratic differential on C having d poles of order m_i, and simple zeroes.

The category $\mathcal{D}(g, m)$ embeds in the derived Fukaya category of a non-compact CY threefold $\pi \colon X \to C$ defined locally by

$$X = \{u^2 + v^2 + w^2 = q(x)\} \subset \mathbb{C}^3 \times C.$$

The theorem was extended to the holomorphic case $d = 0$ by Haiden.

Joyce structures for categories of class $S[A_1]$

Can construct a natural Joyce structure on $\mathrm{Quad}(g, m)$.

Holomorphic case on arxiv, meromorphic extension forthcoming [Menelaos Zikidis].

Geometric construction: moduli of bundles with connections and Higgs fields.

Involves isomonodromic deformations for connections of form $\nabla - \epsilon^{-1}\Phi$.

Like a complexification of the Hitchin system, but much simpler (e.g. algebraic).

In simple cases, e.g. $g = 0, m = (7)$, we have explicit formulae.

The τ-function is an isomonodromy τ-function (e.g. Painlevé I).

Cluster construction of twistor space in ADE type

Geometric approach to the RH problem joint with Helge Ruddat.

Quotient $X = T_M$ by the integrable distribution $\mathrm{im}(h_\epsilon) \subset T_X$.

Varying $\epsilon \in \mathbb{P}^1$ we get a twistor space $q\colon Z \to \mathbb{P}^1$.

The central fibre Z_0 is $M = \mathrm{Stab}(\mathcal{D})$.

For a quiver Q of ADE type we give a cluster-theoretic construction of Z.

The generic fibre is an étale cover of the cluster Poisson variety.

Glue together cells $\{(w_i) \in \mathbb{C}^n : \mathrm{Im}(w_i/\epsilon) \geq 0\}$ by maps

$$w_j \mapsto \begin{cases} w_j + \epsilon \cdot \log\left(1 + e^{+w_i/\epsilon}\right)^{v_{ij}} & \text{if } i \neq j \text{ and } v_{ij} \geq 0 \\ w_j + \epsilon \cdot \log\left(1 + e^{-w_i/\epsilon}\right)^{v_{ij}} & \text{if } i \neq j \text{ and } v_{ij} \leq 0 \\ -w_i & \text{if } i = j. \end{cases}$$

The solutions to the RH problem are then twistor lines!