Topological Strings

Twistors

and

Skyrmions

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Background On Twistors

- $\mathbb{PT} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{CP}^1$
- $\mathbb{PT} \longrightarrow \mathbb{R}^4$ twistor fibration: fibres are $\mathbb{CP}^1$
- $\bar{\partial}$ equation on $\mathbb{PT} \longleftrightarrow$ ordinary PDE on $\mathbb{R}^4$
- e.g. $H^1(\mathbb{PT}, \mathcal{O}(-2)) = \{\varphi \in C^\infty(\mathbb{R}^4), \Delta\varphi = 0\}$

More generally,

Holomorphic field theory on $\mathbb{PT}$

$\Longrightarrow$ Ordinary field theory on $\mathbb{R}^4$

Examples:

– $A \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2)) = \Omega^{0,1}(\mathbb{PT}, K^{1/2})$

gauge field, $S(A) = \int_{\mathbb{PT}} A \bar{\partial} A$

$\Longrightarrow$ Free massless scalar $\int_{\mathbb{R}^4} \varphi \Delta \varphi$

Many more examples:

- Holomorphic BF theory
  $\longleftrightarrow$ Self-Dual Yang Mills theory
  (Penrose-Ward, Movshev, Mason et al.)

- Holomorphic Chern-Simons on Super-Twistor space $\longleftrightarrow$ Self dual limit of $N = 4$ Yang-Mills
  (Witten, Berkovits, Mason et al., ...)

All these theories

are free or almost free (one loop exact)

e.g. Full $N=4$ SYM requires adding non-local terms to the Lagrangian on twistor space

(D1-instantons: Witten, Berkovits, Boels, Mason, Skinner)

Goal Today:

Study a new twistor string theory

- Local field theory on twistor space (holomorphic Chern-Simons)
- Corresponding field theory on $\mathbb{R}^4$: σ-model with target $SO(8)$

No supersymmetry

Holomorphic Chern-Simons

X Calabi-Yau 3-fold

Ω holomorphic volume form

Gauge field $A \in \Omega^{0,1}(X, \mathfrak{g})$

$$S(A) = \int_X CS(A) \wedge \Omega$$

hCS on Twistor Space

$\mathbb{PT}$ is not Calabi-Yau!

$K_{\mathbb{PT}} = \mathcal{O}(-4)$

Take Ω to have second order poles at $z=0$ and $z=\infty$

Locally $\Omega = dv_1 dv_2 dz / z^2$

z: coordinate on $\mathbb{P}^1$, v_i on $\mathcal{O}(-1)^2$ fibres

$A \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$ hCS gauge field

$$\int_{\mathbb{PT}} CS(A)\,\Omega$$ not gauge invariant

because Ω has poles

Fix:

- $z=0$, $z=\infty$ set $A=0$

(and gauge transformations $=1$)

Dirichlet boundary conditions

Field theory on $\mathbb{R}^4$

$x \in \mathbb{R}^4$, $\quad \mathbb{P}^1_x \subseteq \mathbb{PT}$ twistor fibre

$A\big|_{\mathbb{P}^1_x}$ is a holomorphic G-bundle trivialized at $z=0$, $z=\infty$

$\Longleftrightarrow$ an element $\sigma(x) \in G$

(compare trivializations)

$4d$ field: $\sigma : \mathbb{R}^4 \longrightarrow G$

σ-model Lagrangian

A short calculation shows hCS Lagrangian on $\mathbb{PT}$ gives

$$\int_{\mathbb{R}^4} \mathrm{Tr}(J \wedge * J) - \frac{1}{3}\int_{\mathbb{R}^4 \times \mathbb{R}_{\geq 0}} \mathrm{Tr}(\hat{J} \wedge \hat{J} \wedge \hat{J})\,\omega$$

$J = \sigma^{-1} d\sigma$

$\hat{\sigma}$ extension to $\mathbb{R}^4 \times \mathbb{R}_{\geq 0}$, $\hat{J} = \hat{\sigma}^{-1} d\hat{\sigma}$

$\omega \in \Omega^2(\mathbb{R}^4)$ Kähler form

$$\int_{\mathbb{R}^4} J \wedge * J$$ usual σ-model Lagrangian

$$\int_{\mathbb{R}^4 \times \mathbb{R}_{\geq 0}} \mathrm{Tr}(\hat{J} \wedge \hat{J} \wedge \hat{J})\,\omega$$ analog of WZ term

Lorentz invariance broken

Interpretation of WZ term

4d σ-model with target G has a topological $U(1)$ symmetry

Current is $\mathrm{Tr}(J \wedge J \wedge J)$

WZ term: couple to background $U(1)$ gauge field for this symmetry:

$\int A \,\mathrm{Tr}\, J^3$ $\quad F(A) = \omega$, Kähler form

Skyrme Model

Low energy EFT for QCD

$G = SU(N_F)$ $\quad \sigma : \mathbb{R}^4 \longrightarrow G,$

$\sigma = \exp(q\bar{q})$ encodes mesons

Topological $U(1)$ charge

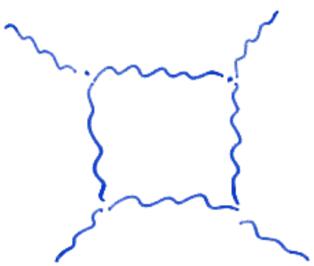
= Baryon number

Anomalies for holomorphic CS

So far: hCS on twistor space $\rightsquigarrow$ σ-model with target G, classically.

At the quantum level, hCS has a one-loop anomaly: (KC, Si Li)

Gauge variation of diagram



is $\int \mathrm{Tr}_{\mathfrak{g}}\left(c(\partial A)^3\right)$

(Gauge variation is $\delta A = \bar{\partial} c + [c, A]$)

Green-Schwarz mechanism

Introduce closed string fields

$$\alpha \in \Omega^{2,1}(\mathbb{PT}, \log D), \quad \partial\alpha = 0$$

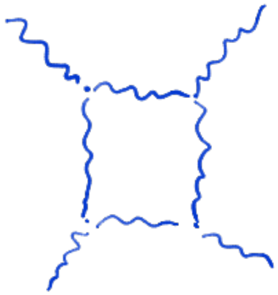
(Type I Kodaira-Spencer theory)

Lagrangian

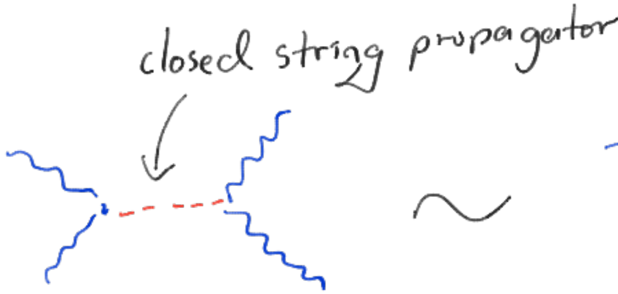
$$\int_{\mathbb{PT}} \mathrm{Tr}_{\mathfrak{g}} (\alpha A \partial A) + \int_{\mathbb{PT}} \bar{\partial}\alpha \, \bar{\partial}^{-1}\alpha + \ldots$$

These cancel the anomaly of hCS for

$$G = SO(8) \qquad (\mathbb{C}, S_i L_i)$$



$$\sim \mathrm{Tr}_{\mathfrak{g}}\left(c(\partial A)^3\right)$$



$$\sim \mathrm{Tr}_{\mathfrak{g}}(c\,\partial A)\,\mathrm{Tr}_{\mathfrak{g}}\left((\partial A)^2\right)$$

These cancel for a Lie algebra $\mathfrak{g}$ where

$$\mathrm{Tr}_{\mathfrak{g}}(X^4) \propto \left(\mathrm{Tr}_{\mathfrak{g}}\, X^2\right)^2$$

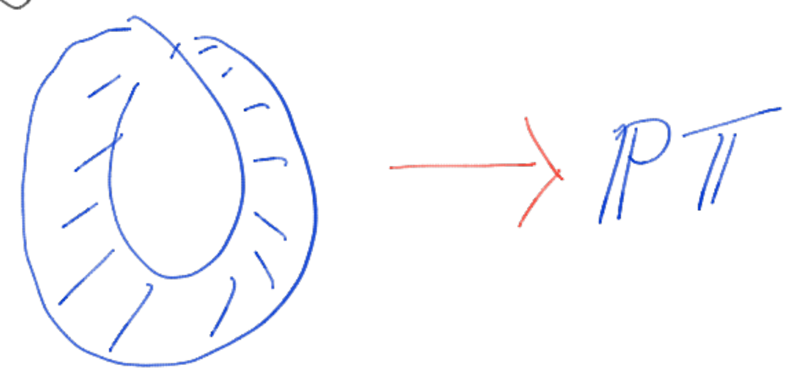
(trace in adjoint representation)

This happens for $\mathfrak{g} = so(8)$ (and E_8 ...)

$\mathfrak{g} = so(8)$: All higher-loop anomalies for hCS + CS cancel, and counterterms fixed uniquely (KC, S_i L_i)

Worldsheet Perspective:

Unoriented topological B-model with target $\mathbb{PT}$



Type I top' string.

Compare to Witten, Berkovits: ordinary B-model on super-twistor space $\mathbb{CP}^{3|4}$

$\rightsquigarrow$ $N=4$ SYM on $\mathbb{R}^4$

Here: Type I B-model on ordinary twistor space $\mathbb{PT}$

$\rightsquigarrow$ 4d σ-model with target $SO(8)$

Implications of the twistorial origin

- Control over the RG flow and counterterms
- Integrable properties and a Lax matrix
- Good analytic behaviour of correlation functions

RG flow

$\mathbb{R}_{>0}$ acts on $\mathbb{R}^4$ by scaling.

On $\mathbb{PT}$ comes from scaling $\mathcal{O}(-1)^2$ fibres of projection $\mathbb{PT} \to \mathbb{CP}^1$

Extends to an action of $\mathbb{C}^\times$

Corollary For any theory of twistorial origin the RG flow is periodic with period i

If λ = log scale, $T(\lambda)$ theory at scale λ, then periodicity says

$$T(\lambda+i) = T(\lambda)$$

Periodicity takes place in the world of analytically continued theories. Strongly constrains divergences in Feynman diagrams: no divergences like $\log \varepsilon$ or $\varepsilon^{-k} \log \varepsilon$

Very unusual behaviour!

- Interacting scalar field theory can not come from twistor space (because of log divergences)
- 4d YM can not come from twistor space

How can σ-model with SO(8) target have periodic RG trajectory?

Answer 4d version of Green-Schwarz mechanism that cancels log divergence instead of anomalies

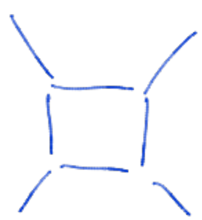
$\sigma : \mathbb{R}^4 \to G$ $\qquad\qquad$ $\varphi = \log \sigma : \mathbb{R}^4 \to \mathfrak{g}$

$$\int \omega J \wedge * J - \frac{1}{3}\int \omega J^3$$

$$\rightsquigarrow \int \mathrm{Tr}_{\mathfrak{g}}\, \varphi \Delta \varphi + \int \mathrm{Tr}_{\mathfrak{g}} \left(\varphi [\partial\varphi, \bar\partial\varphi]\right)\omega \quad (*)$$

$+$ higher order terms

One loop log divergence



$$\sim (\log \varepsilon)\, \mathrm{Tr}_{\mathfrak{g}} \left(\partial\varphi \bar\partial\varphi \partial\varphi \bar\partial\varphi\right)$$

(*) Lagrangian studied by Goncharov, "Hodge Field Theory"

Log divergence is cancelled by closed string fields. On $\mathbb{PT}$, have $\alpha \in \Omega^{2,1}(\mathbb{PT}, \log D)$

$\partial\alpha = 0$, $\int \bar{\partial}\alpha\, \partial^{-1}\alpha$

On $\mathbb{R}^4$, $\alpha \rightsquigarrow B \in \Omega^2_-$, γ a scalar,

$dB + *d\gamma = 0$, two point function

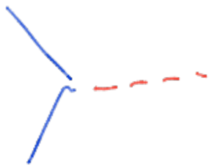
$\langle B(0)\, B(x)\rangle = \|x\|^{-4}$, $\langle \gamma, \gamma\rangle = 0$, $\langle \gamma, B\rangle = \ldots$

B, γ are coupled by varying Kähler form

$$\omega \rightsquigarrow \omega + B + \omega\cdot\gamma$$

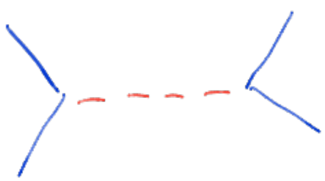
(B, γ are gravitational fields)

In terms of $\varphi = \log \sigma$, fields couple by

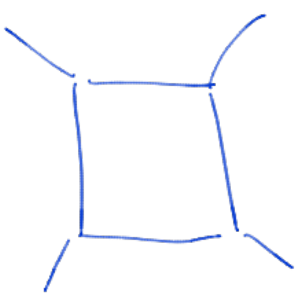


$$\int \mathrm{Tr}(\partial\varphi\,\bar\partial\varphi)\, B$$

Since $\langle B(0), B(x)\rangle = \|x\|^{-4}$



$$\sim \log\varepsilon\; \mathrm{Tr}_g(\partial\varphi\,\bar\partial\varphi)^2$$



$$\sim \log\varepsilon\; \mathrm{Tr}_g(\partial\varphi\,\bar\partial\varphi\,\partial\varphi\,\bar\partial\varphi)$$

Cancel for $G = SO(8)$!

Fixing higher loop counter-terms

- σ-model is non-renormalizable
- 4d perspective: ∞ many possible choices of counter-terms
- $\{$4d local counter-terms$\} \supseteq \{$6d local counterterms$\}$

  6d: many fewer possibilities
- (KC, Si Li) All 6d counter-terms fixed by anomaly cancellation

Towards integrability of the $SO(8)$ σ-model

There are many 2-dimensional integrable field theories.

- classically characterized by a Lax matrix $\mathcal{L}(z)$, a 1-form so that

$$d\mathcal{L}(z)+\frac{1}{2}[\mathcal{L}(z),\mathcal{L}(z)]=0 \Leftrightarrow$$

Equations of motion

- Quantum level:

$P\,Exp\int\mathcal{L}(z)$ is a conserved quantity

(Better: a topological line defect)

<u>4d:</u> It is believed Lorenz invariant theories cannot be integrable.

SO(8) σ-model has a Lax matrix $\mathcal{L}(z)$ $z \in \mathbb{C}^\times$ so that $\mathcal{L}(z)$ is a (0,1) form on $\mathbb{R}^4$ in complex structure z:

$$\mathcal{L}(z) \in \Omega_z^{0,1}(\mathbb{R}^4) \otimes so(8)$$

Satisfying the Lax equation

$$\bar{\partial}_z \mathcal{L}(z) + \frac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = 0$$

$$\in \Omega_z^{0,2}(\mathbb{R}^4) \otimes so(8)$$

At the classical level this is very simple:

$\Pi_z :=$ projection onto $(0,1)$ forms in

complex structure z

$$\mathcal{L}(z) = \Pi_z \Pi_0 J$$

Lax equation follows from the equations of motion: $u_i, \bar{w}_i$ holomorphic coordinates in complex structure $z=0$

EOM are $\partial_{u_i} J_{\bar{w}_i} = 0$

$$A \in \Omega^{0,1}(\mathbb{PT}) \otimes so(8), \quad \mathcal{L}(z) = \langle A(z) \rangle$$

A CS gauge field

At the quantum level:

$\mathcal{L}(z)$ can couple to chiral fields living on a surface

$$S \subseteq \mathbb{R}^4$$

which is holomorphic in complex structure z. This gives a surface defect on S which depends holomorphically (in complex structure z) on the position of S.

Infinitely many conserved charges

Take $4d$ spacetime to be

$$T^3 \times \mathbb{R}$$

For countably many complex structures z this is $E \times \mathbb{C}^\times$, E an elliptic curve, coordinates u, v

Let $T(z,v)$ be surface defect on $E \times v$. Then $\frac{\partial}{\partial \bar{v}} T(z,v) = 0$

As in a 2d CFT a holomorphic operator gives infinitely many conserved charges:

$$\oint_{|v|=1} v^k \, T(z,v)$$

On $\mathbb{PT}$, surface defects come from D1 branes in the fibre over $z \in \mathbb{CP}^1$

D1's at different values of z are disjoint in $\mathbb{P}\mathbb{T}$

$$\Longrightarrow \left[T(z,v), T(z',v') \right] = 0$$

$$\Longrightarrow \left[\oint_{|v|=1} T(z,v), \oint T(z',v') \right] = 0$$

Infinitely many conserved (non-local) charges!

(Those z for which $T^3 \times \mathbb{R}$ contains elliptic curve E includes $z = e^{2\pi i (n/m)}$)

So far:

4d $SO(8)$ σ-model, with WZW term and additional fields, has

- Periodic RG trajectory, period i
- A well-defined quantization, despite being non-renormalizable by power-counting
- Infinitely many commuting non-local charges

"Standard" twistor string theory:

Scattering amplitudes $\longleftrightarrow$ Geometry of algebraic curves in $\mathbb{PT}$

Here: Correlation functions of special local operators are related to algebraic curves in $\mathbb{PT}$

Vertical D1 branes: wrap $\mathbb{CP}^1$ in

$$\mathbb{PT} = \mathcal{O}(-1)^2 \to \mathbb{CP}^1$$

Theory on D1 brane is symplectic bosons on moduli space $\mathcal{M}$ of charge 1 instantons for $so(8)$

$z = 0, \infty$: need to choose boundary conditions:

Lagrangians $L_0, L_\infty \subseteq \mathcal{M}$

Vertical D1 brane $\Rightarrow$ local operator

Assume $L_0 \cap L_\infty$ = a point $p \in \mathcal{M}$

$SO(8)$ acts on $\mathcal{M}$ by Hamiltonian functions H_a

If $\varphi = \log \sigma$, then the local operator is

$$\exp\left(\varphi^a H_a(p)\right) + \cdots$$

These local operators have special analytic properties: consider correlation functions

$$\langle \mathcal{O}(x_1), \ldots, \mathcal{O}(x_n) \rangle \qquad x_i \in \mathbb{R}^4$$

$$G(2,4) = \{ \mathbb{CP}^1 \subseteq \mathbb{CP}^3 \}$$

$$G(2,4) \supseteq \mathbb{C}^4 = \{ \mathbb{CP}^1 \subseteq \mathbb{PT} \}$$

Correlation functions analytically extend to $x_i \in \mathbb{C}^4$

As, on $\mathbb{PT}$ the $D1$ branes are defined on $\mathbb{CP}^1 \subseteq \mathbb{PT}$

$\langle \mathcal{O}(x_1), \dots, \mathcal{O}(x_n) \rangle$ has poles when

$\mathbb{CP}^1_{x_i} \cap \mathbb{CP}^1_{x_j} \Longleftrightarrow x_i, x_j \in \mathbb{C}^4$ are such that

$x_i - x_j$ is null

This implies that there is a well-behaved OPE:

$$\mathcal{O}_1(0)\cdot\mathcal{O}_2(x) \sim \frac{1}{\|x\|^{2D}} \sum x^{I}$$

(I is a multi-index)

Suggests a bootstrap-like way to understand correlation functions