# Geometry and topology of wild character varieties

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# Outline

- Based on joint works with with **Arinkin and Toën** , with **Diaconescu and Donagi**, and with **Chuang, Diaconescu, Donagi, and Nawata**.
- Review of wild character varieties.
- Geometry - resolution of singularities, Poisson structures, symplectic leaves.
- Topology - weighted Poincaré polynomials, Calabi-Yau enumerative geometry, and refined Chern-Simons theory of torus knots.

# Wild character varieties

# Character varieties (i)

**Character varieties** = special moduli spaces of matrices.

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Parametrize monodromy data for meromorphic systems of linear ODE on compact Riemann surfaces. Historically come in three flavors

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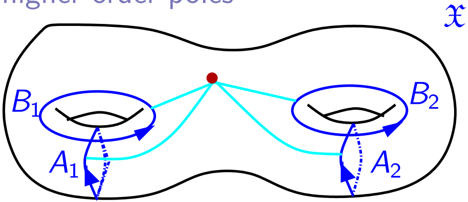
- tame, $\leftarrow$ logarithmic poles
- wild. $\leftarrow$ higher order poles

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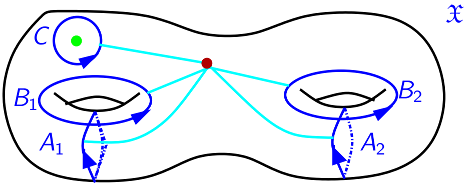


$$\prod_{i=1}^{g} [A_i, B_i] \cdot C_1 \cdots C_k = \mathbf{1}$$

# Character varieties (ii)

Depend on:

**Background data:**
- $\mathfrak{X}$ - a smoooth projective curve/$\mathbb{C}$.
- $D = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ - distinct points.
- $G$ - a complex reductive group, $\mathfrak{g} = \mathrm{Lie}(G)$.

**Invariants:**
- $\mathbf{Q} = \{Q_1, \ldots, Q_k\}$ - irregular type at $D$.
- Fixed conjugacy classes of formal monodromy.

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Study monodromies of $(V, \nabla)$ with $V$ a principal $G$-bundle and $\nabla$ meromorphic flat connection with poles at most at $D$.

**Invariants:**

- $\mathbf{Q} = \{Q_1, \ldots, Q_k\}$ - irregular type at $D$.
- Fixed conjugacy classes of formal monodromy.

Study monodromies of $(V, \nabla)$ when $\nabla$ has polar parts $Q_i$ at $x_i$ and formal monodromy in the given conjugacy classes.

# Irregular type (i)

**Definition:** A $G$-valued **irregular type** is a Laurent polynomial

$$Q = \frac{A_{n-1}}{z^{n-1}} + \cdots + \frac{A_1}{z}, \ \ A_i \in \mathfrak{t},$$

where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra.

$(V, \nabla)$ has **irregular type** $Q$ **at** $x_i \in D$ if locally near $x_i$

$$\nabla = d + dQ \mod z^{-1}\mathfrak{g}\{z\}dz$$

in an analytic local trivialization of $V$ near $x_i$ and for some finite root $z$ of a local coordinate on $\mathfrak{X}$ centered at $x_i$

**Note:** Up to a Puiseux gauge transformation the $\leqslant -2$ polar part of $\nabla$ always has coefficients in $\mathfrak{t} \subset \mathfrak{g}$.

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Consider $\widehat{\mathfrak{X}}$ - the **real oriented blow up** of $\mathfrak{X}$ at $D$

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- each $x_i$ is replaced by a circle $\partial_i$ parametrizing real oriented directions emanating from $x_i$;
- $\widehat{\mathfrak{X}}$ is a surface with boundary $\partial\widehat{\mathfrak{X}} = \partial_1 \sqcup \ldots \sqcup \partial_k$.

# Irregular type (ii)

Consider $\widehat{\mathfrak{X}}$ - the **real oriented blow up** of $\mathfrak{X}$ at $D$

**Boalch:** An irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$ decorates $\widehat{\mathfrak{X}}$ with:

- A connected reductive group $L_i \subset G$ , the centralizer of $Q_i$ in $G$.
- A finite set $\mathrm{A}_i \subset \partial_i$ of singular/anti-Stokes directions for all $i = 1, \ldots, k$.
- For each $d \in \mathrm{A}_i$ a unipotent group $\mathrm{Sto}_d \subset G$, normalized by $L_i$.

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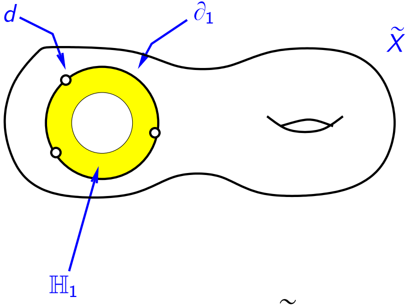
- For each $d \in \mathrm{A}_i$ a unipotent group $\mathrm{Sto}_d \subset G$, normalized by $L_i$.

**Note:** These give rise to a new punctured surface surface $\widetilde{X}$ defined by puncturing $X = \mathfrak{X} - D$ once near each singular direction..

# Irregular type (iii)

## Concretely:

- Glue an annulus $\mathbb{H}_i = (0,1] \times \partial_i$ to each boundary circle $\partial_i \subset \widehat{\mathfrak{X}}$ to obtain a new surface $X^+$.
- Puncture $X^+$ at each singular direction $d \in \mathsf{A}_i \subset \partial_i$ to obtain $\widetilde{X}$:




The surface $\widetilde{X}$

# Stokes local systems

The analytic covariantly constant local sections in a flat meromorphic $G$-bundle of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$ form a locally constant sheaf on $\widetilde{X}$ of a special type:

**Definition:** A **Stokes $G$-local system** of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$ is a locally constant principal $G$ bundle on $\widetilde{X}$ with a locally constant reduction of the structure group to $L_i$ on a finite etale cover of $\mathbb{H}_i$, so that the local monodromy around each $d \in \mathrm{A}_i \subset \partial_i$ is in $\mathrm{Sto}_d$ for any choice of a base point in $\mathbb{H}_i$.

# Riemann-Hilbert correspondence

The **irregular Riemann-Hilbert correspondence (Stokes, Malgrange-Sibuya, Deligne-Malgrange, Boalch)** can now be formulated as the following

> **Theorem:** The category of flat meromorphic $G$-bundles of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$ is equivalent to the category of Stokes $G$-local systems of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$.

This leads to our main object of interest: the **wild character varieties**.

# Wild character varieties

Fix $(\mathfrak{X}, D)$, a group $G$, an irrequalr type $\mathbf{Q}$, and a collection $\mathbf{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_k\}$, where $\mathbf{C}_i \subset L_i$ is a conjugacy class in $L_i$.

**Definition:** The **wild character variety** of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$ is the coarse moduli space

$$M_G(\mathfrak{X}, D; \mathbf{Q}) = M_G(\mathbf{Q})$$

of Stokes $G$-local systems of irregular type $\mathbf{Q}$ on $(\mathfrak{X}, D)$.

The **$\mathbf{C}$-restricted wild character variety** is the subvariety $M_G(\mathbf{Q}, \mathbf{C}) \subset M_G(\mathbf{Q})$ of Stokes local systems whose formal monodromy around $\partial_i$ is in $\mathbf{C}_i \subset L_i$.

# Geometry

# Poisson and symplectic structures

Fix $(\mathfrak{X}, D)$, a group $G$, an irrequalr type $\mathbf{Q}$, and a collection $\mathbf{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_k\}$, where $\mathbf{C}_i \subset L_i$ is a conjugacy class in $L_i$.

**Classical story:** Boalch, Boalch-Yamakawa

- $M_G^{\mathrm{sm}}(\mathbf{Q})$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(\mathbf{Q})$ are the $\mathbf{C}$-restricted moduli spaces $M_G^{\mathrm{sm}}(\mathbf{Q}, \mathbf{C})$.
- There is a **quasi-Hamiltonian** description of symplectic leaves.

# Goal

**Remark:**

- [Boalch,Boalch-Yamakawa] For a generic choice of irregular type and formal monodromies ensure that $M_G^{\text{sm}}(\mathbf{Q}) = M_G(\mathbf{Q})$ and $M_G^{\text{sm}}(\mathbf{Q}, \mathbf{C}) = M_G(\mathbf{Q}, \mathbf{C})$.
- In contrast with the unramified+tame case here is no obvious cohomological reason for non-degeneracy of the symplectic form on $M_G^{\text{sm}}(\mathbf{Q}, \mathbf{C})$.

# Goal

- Construct Poisson structures everywhere $M_G(\mathbf{Q})$ including the singular points.
- Describe their symplectic leaves.
- Find **cohomological** and **quasi-Hamiltonian** descriptions of the symplectic form at the singular points.
- Extend the whole story to higher dimensional smooth varieties $X = \mathfrak{X} - D$.

# Setup

**Natural approach:** Resolve the singularities of $M_G(\mathbf{Q})$ and $M_G(\mathbf{Q}, \mathbf{C})$ in a minimal way so that the Poisson and symplectic structures extend to the resolution.

**Lucky break:** $M_G(\mathbf{Q})$ and $M_G(\mathbf{Q}, \mathbf{C})$ admit natural resolutions which are again moduli spaces.

**Note:** These resolutions/refinements of $M_G$ are not schemes but rather are derived algebraic stacks whicih are locally finitely presentable and in particular have perfect tangent complexes.

# Refinements (i)

## Unramified+tame cases

The moduli $M_G(X)$ can be refined to the derived stack

$$Loc_G(X) = \mathrm{Map}_{\mathrm{dSt}}(X, BG)$$

parametrizing $G$-local systems on $X = \mathfrak{X} - D$.

### Key point:

- Any non-degenerate $\kappa \in (\mathrm{Sym}^2\, \mathfrak{g}^\vee)^G$ corresponds to a 2-shifted symplectic structure $\omega_\kappa$ on the Artin stack $BG$
- $\omega_\kappa$ induces 0-shifted symplectic or Poisson structures on $Loc_G(X)$ in the unramified or tame case respectively.

# Refinements (ii)

**Wild case:**

The moduli $M_G(\mathbf{Q})$ can be refined to the derived stack

$$Loc_G(\mathbf{Q}) = \Gamma\left(X^+, \mathrm{DMS}_{G,\mathbf{Q}}\right)$$

parametrizing Stokes $G$-local systems of irregular type $\mathbf{Q}$.

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**Note:** $\mathrm{DMS}_{G,\mathbf{Q}}$ denotes the Deligne-Malgrange-Stokes sheaf of Artin stacks on $X$ classifying Stokes $G$-local systems of irregular type $\mathbf{Q}$.

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**Key point:** $\mathrm{DMS}_{G,\mathbf{Q}}$ is equipped with a natural 0-shifted relative symplectic structure which induces a Poisson structure on $Loc_G(\mathbf{Q})$.

# Deligne-Malgrange-Stokes sheaves

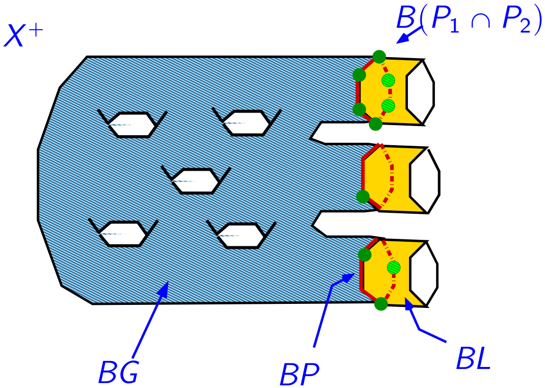


Figure: The sheaf $\mathrm{DMS}_{G,\mathbf{Q}}$

# Symplectic structures

**Theorem: [Arinkin-P-Toën]**

**(1)** The 2-shifted symplectic form on $BG$ extends canonically to a relative $K_{X^+}$-valued 0-shifted symplectic form on $\mathrm{DMS}_{G,\mathbf{Q}}$

**(2)** The relative form $\omega$ induces a 0-shifted Poisson structure on the stack $Loc_G(\mathbf{Q}) = \Gamma\left(X^+, \mathrm{DMS}_{G,\mathbf{Q}}\right)$ of Stokes $G$-local systems if irregular type $\mathbf{Q}$.

**(3)** The moduli $Loc_G(\mathbf{Q}, \mathbf{C})$ of $\mathbf{C}$-restricted Stokes local systems is a symplectic leaf for this Posson structure.

**Note:** **(2)** and **(3)** follow from a general pushforward theorem for symplectic structures - a generalization of symplectic reduction.

# Topology

# Prelude: counting $G$-covers (i)

$G$ - a finite group, $X$ a smoooth compact surface of genus $g$.

**Problem:** **[Riemann,Hurwitz, . . . ]** Count all unramified $G$-Galois covers of $X$ up to isomorphism.

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**Reformulation:** Find

$$\left(\begin{array}{l}\text{orbifold Euler character-}\\ \text{istic of the moduli stack}\\ M_G(X) \text{ of flat } G \text{ bundles}\\ \text{on } X\end{array}\right) = \left(\begin{array}{l}\text{orbifold partition function}\\ \text{of a } 2d \text{ Dijkgraaf-Witten}\\ \text{theory with gauge group}\\ G\end{array}\right).$$

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**Reformulation:** $M_G(X) \cong [\mathrm{Hom}(\pi_1(X), G)/G]$, so find

$$\chi^{\mathrm{orb}}(M_G(X)) = \sum_{[\rho]\in \mathrm{Hom}(\pi_1(X),G)/G} \frac{1}{|\,\mathrm{Aut}(\rho)|} = \frac{|\,\mathrm{Hom}(\pi_1(X), G)|}{|G|}.$$

# Prelude: counting $G$-covers (ii)

## Formulas:

Frobenius'1896 If $g = 1$, then

$$\chi^{\mathrm{orb}}(M_G(X)) = \frac{|\operatorname{Hom}(\pi_1(X), G)|}{|G|} = |\operatorname{Irrep}(G)|$$

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**Goal:** FInd analogous formulas for $\chi(M_G(\mathbf{Q}, \mathrm{C}))$.

# Steps in the physics approach

Wild character varieties

↕

Moduli of irregular parabolic Higgs bundles

↕

Sheaves on a CY3

↕

Stable pairs on a CY3

↕

Torus links

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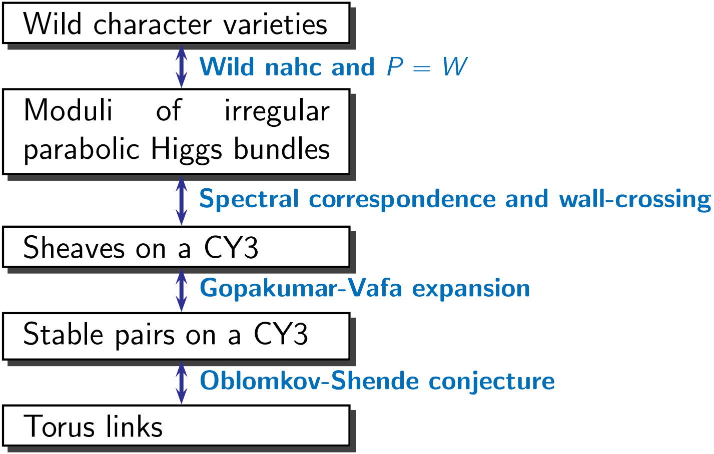

# Steps in the physics approach

| | |
|---|---|
| Wild character varieties | Mixed Poincaré polynomial |
| ↕ **Wild nahc and $P = W$** | |
| Moduli of irregular parabolic Higgs bundles | Perverse Poincaré polynomial |
| ↕ **Spectral correspondence and wall-crossing** | |
| Sheaves on a CY3 | Refined DT invariants |
| ↕ **Gopakumar-Vafa expansion** | |
| Stable pairs on a CY3 | Refined PT invariants |
| ↕ **Oblomkov-Shende conjecture** | |
| Torus links | Refined colored link invariants |

# Steps in the physics approach

**Wild character varieties** — Mixed Poincaré polynomial

↕ **Wild nahc and $P = W$**

**Moduli of irregular parabolic Higgs bundles** — Perverse Poincaré polynomial

↕ **Spectral correspondence and wall-crossing**

**Sheaves on a CY3** — SUSY $D2$-$D0$ branes

↕ **Gopakumar-Vafa expansion**

**Stable pairs on a CY3** — $D6$-$D2$-$D0$ bound states

↕ **Large $N$ duality near a conifold point**

**Torus links** — Refined CS invariants

# Assumptions:

- $G = GL(r, \mathbb{C})$, $D = p$ consists of a single point.
- The centralizer $H_Q = \prod_{i=1}^{\ell} GL(m_i, \mathbb{C}) \subset GL(r, \mathbb{C})$ of $A_1, \ldots, A_{k-1}$ equals the centralizer of $A_{k-1}$.
- The formal monodromy at $p$ is in the conjugacy class $\mathbf{C} \subset H_Q$ of a block diagonal matrix M of the form

$$\mathrm{M} = \begin{pmatrix} \tau_1 \mathbf{1}_{m_1} & 0 & \cdots & 0 \\ 0 & \tau_2 \mathbf{1}_{m_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \tau_\ell \mathbf{1}_{m_\ell} \end{pmatrix}$$

where $\tau_1, \ldots, \tau_\ell$ are pairwise distinct complex numbers.

# Topology of wild character varieties

**Goal:** Understand the topology of $M_G(Q, \mathbf{C})$, e.g. compute its mixed Poincaré polynomial.

**Mixed Poincaré polynomial:**

$$WP(M_G(Q, \mathbf{C}); u, v) = \sum_{k,j} \dim Gr_k^W H^j(M_G(Q, \mathbf{C}))u^{k/2}v^j$$

where $W_k H^j$ is the weight filtration for the mixed Hodge structure on $M_G(Q, \mathbf{C})$.

# Non-abelian Hodge and $P = W$

- **Wild non-abelian Hodge correspondence** [Biquard-Boalch,Witten,Sabbah,Mochizuki]

  (Irregular flat connections) $\leftrightarrow$ (Irregular Higgs bundles)

  and the associated moduli spaces are related by hyper-Kähler rotation.

- $P = W$ **correspondence** [de Cataldo, Hausel, Migliorini]

$$W_{2k} H^j \begin{pmatrix} \text{character} \\ \text{variety} \end{pmatrix} = P_k H^j \begin{pmatrix} \text{Higgs bundle} \\ \text{moduli space} \end{pmatrix}$$

  where $P_k H^j$ is the perverse Leray filtration arising from the Hitchin map.

# The Hausel-Mereb-Wong conjecture (i)

**Definition:** The **HMW partition function** is the function

$$Z_{HMW}(z,w) = \sum_{\lambda} \Omega_{\lambda}^{g,n}(z,w)\widetilde{H}_{\lambda}(\mathbf{x}; z^2, w^2).$$

Where:

- The sum is over all Young diagrams $\lambda$;
- $$\Omega_{\lambda}^{g,n}(z,w) = \prod_{\Box\in\lambda} \frac{\left(-z^{2a(\Box)}w^{2l(\Box)}\right)^r \left(z^{2a(\Box)+1} - w^{2l(\Box)+1}\right)^{2g}}{\left(z^{2a(\Box)+2} - w^{2l(\Box)}\right)\left(z^{2a(\Box)} - w^{2l(\Box)+2}\right)};$$

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- $\mathbf{x} = (x_1, x_2, \ldots)$ is an infinite sequence of formal variables;
- $\widetilde{H}_{\lambda}(\mathbf{x}; z, w)$ is the modified Macdonald polynomial.

# The Hausel-Mereb-Wong conjecture (ii)

Define $\mathbb{H}_{\mu,n}(z,w)$ by the expansion

$$\log Z_{HMW}(z,w) = \sum_{k\geqslant 1}\sum_{\mu} \frac{(-1)^{n|\mu|} w^{kd_{\mu,n}} \mathbb{H}_{\mu,n}(z^k,w^k)}{(1-z^{2k})(w^{2k}-1)} m_\mu(\mathrm{x}^k)$$

where $m_\mu(\boldsymbol{x})$ are the monomial symmetric functions and $\boldsymbol{x}^k = (x_1^k, x_2^k, \ldots)$.

**Conjecture: [HMW]**

$$WP(M_G(Q,\mathbf{C}); u, v) = \mathbb{H}_{(1^r),n}(u^{1/2}, -u^{-1/2}v^{-1})$$

for any $r, n \geqslant 1$, any $Q$, and any generic **regular** $\mathbf{C}$.

# The Shende-Treumann-Zaslow conjecture (i)

Suppose $\Sigma \subset \mathbb{A}^2$ is a reduced rational plane curve with a single singular point $\nu$ and $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ projection onto one of the coordinate axes.

Then $\Sigma \simeq$ affine part of a spectral curve for a meromorphic Hitchin system on $\mathbb{P}^1$ with a pole at $\infty$.

The wild non-abelian Hodge correspondence identifies the symplectic leaf containing the compactified Jacobian of the projective completion of $\Sigma$ with a wild character variety $\mathcal{S}_\Sigma$ on $\mathbb{P}^1$ parametrizing Stokes data at $\infty$.

# The Shende-Treumann-Zaslow conjecture (ii)

- $L \subset S^3$ - the link of the singular point $\nu \in \Sigma$.
- $P_L(a,u) \in \mathbb{Q}(u^{1/2})[a,a^{-1}]$ - the HOMFLY polynomial of $L$.
- $P_L^{(0)}(u)$ - the $a^0$ term in $(au^{-1/2})^{m(\nu)} P_L(a,u)$, where $m(\nu)$ is the Milnor number of $\nu \in \Sigma$.

**Conjecture: [STZ]**

$$P_L^{(0)}(u) = (1-u)^{b(\nu)} WP(\mathcal{S}_\Sigma; u, -1)$$

where $b(\nu) =$ number of branches of $\Sigma$ at $\nu$.

# The Shende-Treumann-Zaslow conjecture (iii)

**Note:** We need a refined colored generalization of the STZ conjecture proposed by [Oblomkov, Rassmusen,Shende] and [Diaconescu,Hua,Soibelman] and based the stable pair theory of the conifold.

# Irregular parabolic Higgs bundles (i)

- $(C, p)$ curve with marked point,
- $D = np$, $n \geqslant 1$,
- $M = K_C(D)$,
- $\underline{\xi} = (\xi_1, \ldots, \xi_\ell)$, $\ell \geqslant 1$, sections of $K_C(D)|_D$,

**Definition:** An **irregular $\underline{\xi}$-parabolic Higgs bundle** $(E, \Phi, E_D^\bullet, \underline{\alpha})$ consists of

$$
\begin{array}{l}
E : \text{vector bundle on } C, \quad E_D = E \otimes \mathcal{O}_D \\
\Phi : E \longrightarrow E \otimes M, \quad \Phi_D = \Phi|_D \\
0 \subset E_D^1 \subset \cdots \subset E_D^\ell = E_D \quad E_D^i / E_D^{i-1} \text{ locally free } \mathcal{O}_D - \text{modules} \\
\Phi_D(E_D^i) \subseteq E_D^i, \quad \mathrm{gr}^i(\Phi_D) = \xi_i \otimes \mathbf{1} \\
\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \quad \text{real parabolic weights} \\
0 < \alpha_\ell < \cdots < \alpha_1 < 1.
\end{array}
$$

# Irregular parabolic Higgs bundles (ii)

Imposing the parabolic stability conditions we get a moduli stack

$$\mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$$

of semistable parabolic Higgs bundles, where:

- $\underline{m} = (m_1, \ldots, m_\ell)$, $m_i = \text{length}_{\mathcal{O}_D}(E^i_D / E^{i-1}_D)$,
- $d = \deg(E)$, $\quad \sum_{i=1}^{\ell} m_i = \text{rk}(E)$.

**Note:** For generic weights $\mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ is a $\mathbb{G}_m$-gerbe over its coarse moduli space.

# Spectral correspondence (i)

**Goal:** construct a holomorphic symplectic surface $S_{\underline{\xi}}$ so that

$$(*) \qquad \mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d) \cong \left( \begin{array}{l} \text{Moduli stack Bridgeland} \\ \text{stable pure dimension one} \\ \text{sheaves on } S_{\underline{\xi}} \end{array} \right)$$

**Note:**

- Kontsevich and Soibelman proposed a construction of $S_{\underline{\xi}}$ and proved that the Hitchin base for $\mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ is isomorphic to a linear system on $S_{\underline{\xi}}$.
- S.Szabo proved the isomorphism $(*)$ for certain open subsets.

# Spectral correspondence (ii)

**Construction:** [Kontsevich-Soibelman]

- Let $T_{\underline{\xi}}$ = blow-up of the images of the sections $\xi_1, \ldots, \xi_\ell : D \to M_D$ on $M$ (assumed pairwise distinct and nonzero).
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$$f + \sum_{i=1}^{\ell} \sum_{a=1}^{n-1} \Xi_{a,i}$$

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- For any $\underline{m} = (m_1, \ldots, m_\ell)$, partition of $r$, the linear system $\Sigma_{\underline{m}}$ on $S_{\underline{\xi}}$ given by
$$\Sigma_{\underline{m}} = r\Sigma_0 - \sum_{i=1}^{\ell} m_i \left(\sum_{a=1}^{n} a\Xi_{a,i}\right)$$
is non-empty and consists of compact divisors which are finite covers of $C$.

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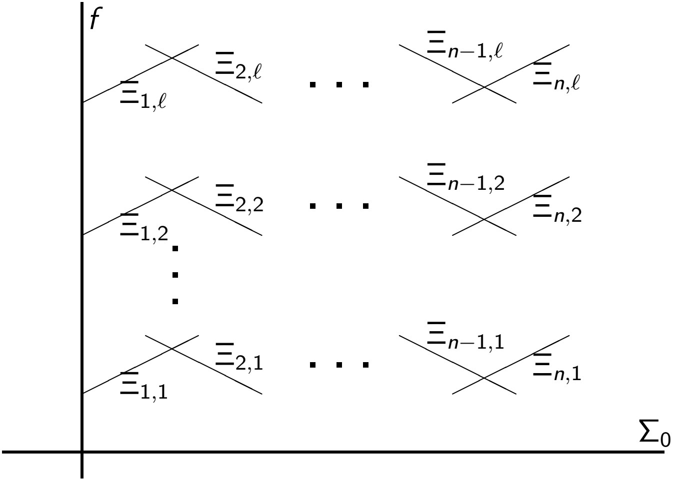
proper transform of the zero section of $M$

# Spectral correspondence (iii)



# Spectral correspondence (iv)

## Stability:

- For any collection of real numbers $\beta_1, \ldots, \beta_\ell$, there is a compactly supported $B$-field $\beta \in H^2_c(S_{\underline{\xi}}, \mathbb{R})$whose only non-zero periods are

$$\beta(\Xi_{n,i}) = -\beta_i, \;\; i = 1, \ldots, \ell.$$

- The $B$-field defines a slope function $\mu_\beta$ for compacly supported pure dimension one sheaves on $S_{\underline{\xi}}$:

$$\mu_\beta(F) = \frac{\chi(F) + \beta(\mathrm{ch}_1(F))}{r_F},$$

where $r_F$ is the rank of the sheaf $\pi_* F$, for the natural projection $\pi_S : S_{\underline{\xi}} \to C$.

# Spectral correspondence (v)

**Definition:** The Bridgeland stability given by $\mu_\beta$ will be called **$\beta$-stability**. The moduli stack of $\beta$-semistable compactly supported pure dimension one sheaves on $S_{\underline{\xi}}$ will be denoted by $\mathfrak{M}^{ss}_\beta(S_{\underline{\xi}}; \underline{m}, c)$.

**Theorem:** [Diaconescu-Donagi-P]
Choose $0 < \beta_\ell < \cdots < \beta_1 < 1$ and set $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant \ell$.
Then there is an isomorphism of moduli stacks

$$\mathfrak{M}^{ss}_\beta(S_{\underline{\xi}}; \underline{m}, c) \simeq \mathfrak{H}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, c + r(g-1)),$$

where $g$ is the genus of $C$ and $r = \sum_{i=1}^{\ell} m_i$.

**Note:** Holds for any $g \geqslant 0$ and any number of poles.

# CY3 and refined invariants (i)

- $Y_{\underline{\xi}} = S_{\underline{\xi}} \times \mathbb{A}^1$.

$$\mathfrak{M}^s_\beta(S_{\underline{\xi}}; \underline{m}, c) \simeq \mathfrak{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, c + r(g-1)) \times \mathbb{A}^1$$

- Stable pair theory [Pandharipande, Thomas]

$$\begin{aligned} PT_{Y_{\underline{\xi}}}(\underline{m}, c) = {} & \text{virtual number of pairs} \\ & \mathcal{O}_{Y_{\underline{\xi}}} \xrightarrow{s} F, \quad s \text{ generically surjective} \\ & \mathrm{ch}_1(F) = \Sigma_{\underline{m}}, \quad \chi(F) = c \\ & F \text{ compact support.} \end{aligned}$$

- Refined stable pair theory [Kontsevich and Soibleman]

$$\begin{aligned} PT_{Y_{\underline{\xi}}}(\underline{m}, c; y) = {} & \text{virtual Poincare polynomial of} \\ & \text{moduli space of such pairs.} \end{aligned}$$

# CY3 and refined invariants (ii)

## Generating function:

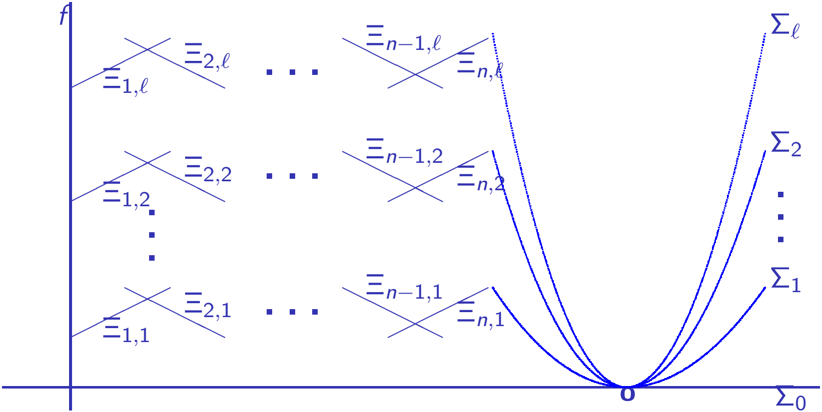
$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) =$$
$$1 + \sum_{\substack{c,\underline{m} \\ \underline{m}\neq(0,\ldots,0)}} q^c \prod_{i=1}^{\ell} Q_i^{m_i} \, PT_{Y_{\underline{\xi}}}(m_1, \ldots, m_\ell, c; y).$$

**Goal:** compute the generating function.

# Refined invariants via refined Chern-Simons theory (i)

- Specialize to $C = \mathbb{P}^1$, one marked point $p \in C$.
- Torus action $\mathbb{C}^\times \times S_{\underline{\xi}} \to S_{\underline{\xi}}$.
- Refined virtual localization [Nekrasov, Okounkov], [Maulik].
- Stable pair theory localizes on a finite collection of rational curves $\Sigma_1, \ldots, \Sigma_\ell$ in $S_{\underline{\xi}}$.
- Direct localization computations are possible in principle but are **hard**.

# Torus invariant rational curves on $S_{\underline{\xi}}$



# Refined invariants via refined Chern-Simons theory (ii)

**Note:** Local equation of the torus invariant curve $\Sigma_1 + \cdots + \Sigma_\ell$ at $\mathbf{o}$:

$$\prod_{i=1}^{\ell} (y - \lambda_i x^{n-2}) = 0.$$

**Oblomkov-Shende framework:**

$$\begin{pmatrix} \text{Refined invariants of} \\ \text{the singular plane curves} \\ \Sigma_1 + \cdots + \Sigma_\ell \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{Refined invariants of} \\ (\ell, (n-2)\ell)\text{-torus links} \end{pmatrix}$$

# Refined invariants via refined Chern-Simons theory (iii)

**Note:**

- Need a colored refined variant of this framework [Diaconescu,Hua,Soibelman].
- Colored refined invariants of torus links can be obtained from refined Chern-Simons theory [Aganagic, Shakirov], [Shakirov] together with some large $N$ duality considerations.

Combining all this we obtain the following conjectures.

# Stable pairs conjecture (i)

**Conjecture 1:** [Diaconescu-Donagi-P] The refined stable pair theory of $Y_{\underline{\xi}}$ is given by

$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) = \sum_{\mu_1,\ldots,\mu_\ell} \Bigg( \widetilde{W}^{(n-2)}_{\mu_1,\ldots,\mu_\ell}(s,t) \prod_{i=1}^{\ell} \left[ \left(ts^{-1}Q_i\right)^{|\mu_i|/2} f_{\mu_i}(s,t)^{n-1} P_{\mu_i^t}(t,s;\underline{s}) \right] \Bigg)\Bigg|_{s=qy,\ t=qy^{-1}}$$

where $\widetilde{W}^{(n-2)}_{\mu_1,\ldots,\mu_\ell}(s,t) =$

$$\sum_{\lambda_1,\ldots,\lambda_{\ell-1}} N^{\lambda_{\ell-1}}_{\mu_\ell,\lambda_{\ell-2}} N^{\lambda_{\ell-2}}_{\mu_{\ell-1},\lambda_{\ell-3}} \cdots N^{\lambda_2}_{\mu_3,\lambda_1} N^{\lambda_1}_{\mu_2,\mu_1} \cdot f_{\lambda_{\ell-1}}(s,t)^{2-n} P_{\lambda_{\ell-1}}(s,t;\underline{t}).$$

# Stable pairs conjecture (ii)

Here:

- $P_\lambda(t, s; \mathrm{x})$, $\mathrm{x} = (x_1, x_2, \ldots)$, are the $(t, s)$-Macdonald polynomials.
- $N^\sigma_{\nu,\lambda}$ are the $(s, t)$-Littlewood-Richardson coefficients

$$P_\nu(t, s; \mathrm{x}) P_\lambda(t, s; \mathrm{x}) = \sum_\sigma N^\sigma_{\nu,\lambda} P_\sigma(t, s; \mathrm{x}).$$

- $f_\lambda(s, t)$ are refined framing factors,

$$f_\lambda(s, t) = \prod_{\square \in \lambda} s^{a(\square)} t^{-l(\square)},$$

- $\underline{t} = (t^{1/2}, t^{3/2}, \ldots)$, $\underline{s} = (s^{1/2}, s^{3/2}, \ldots)$.

# Refined Gopakumar-Vafa expansion

**Conjecture 2:** [Diaconescu-Donagi-P]

$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) = \exp\left( -\sum_{k \geqslant 1} \sum_{\mu} \frac{m_\mu(Q_1^k, \ldots, Q_\ell^k, 0, \ldots)}{k} \frac{y^{-kr}(qy^{-1})^{kd_{\mu,n}/2} P_{\mu,n}((qy)^{-k}, -y^k))}{(1-(qy)^{-k})(1-(qy^{-1})^k)} \right)$$

- $m_\mu(x_1, \ldots)$ monomial symmetric functions.
- $P_{\mu,n}(u, v)$ perverse Poincaré polynomial of $\mathfrak{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$

for generic $\underline{\alpha}$, $\mu$ partition $r = m_1 + \cdots + m_\ell$.

# Mixed Poincaré polynomial

**Conjecture 3:** [Diaconescu-Donagi-P] For $C = \mathbb{P}^1$ with one marked point,

$$WP(M_G(Q, \mathbf{C}); u, v) = P_{\mu,n}(u, v)$$

where $\mu$ is the partition of multiplicities of eigenvalues of M.

**Remark:** • agrees with the HMW conjecture in many rank 2 and 3 examples with M regular

• For $\mu = (2, 1)$, $n = \{5, 6\}$, $P_{\mu,n}(1, v)$ agrees with Poincaré polynomial of Higgs bundle moduli space computed by localization.

**Question:** Can we prove the $v = 1$ specialization of Conjecture 3 by counting rational points on wild character varieties?

# Odds and ends

# Relative symplectic forms

## Setup:

- $X$ - stratified space.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks.
- $K_X \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ - Verdier's dualizing complex.

**Definition:** A closed relative $K_X[n]$-valued 2-form $\omega \in \mathbf{A}_X^{2,cl}(\mathscr{F})_{K_X[n]}$ is **symplectic** if for every $U \subset X$ and any point $z \in \Gamma(U, \mathscr{F})$, the map

$$\omega_z^\flat : \mathbb{T}_{\mathscr{F}_{|U},z} \longrightarrow \mathbb{L}_{\mathscr{F}_{|U},z} \otimes K_U[n]$$

is a quasi-isomorphism.

# Push-forwards (i)

Pushing forward also makes sense in this setting. Suppose

- $f : X \to Y$ is a stratified map of nicely stratified spaces;
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
- $E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$;

We have push-forwards

$$f_* \mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{dSt}_{\mathbb{C}}) \quad \text{and} \quad f_* E \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{Vect}_{\mathbb{C}}).$$

The pushforward of $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F})_E$ then is a closed relative form:

$$\boxed{f_* \omega \in \mathbf{A}_Y^{p,cl}(f_* \mathscr{F})_{f_* E}.}$$

# Push-forwards (ii)

**Theorem: [Arinkin-P-Toën]** Suppose that $f : X \to Y$ is proper and that $\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n]}$ is symplectic. Then the pushforward

$$\operatorname{tr} f_*\omega \in \mathbf{A}^{2,cl}_Y(f_*\mathscr{F})_{K_Y[n]}$$

is symplectic as well.

**Remark:**

- Here $\operatorname{tr} : f_*K_X \to K_Y$ denotes the canonical trace map.
- Most of the standard constructions of shifted symplectic structures arise as special cases of the above theorem.

# Deligne-Malgrange-Stokes sheaves (ii)

**Theorem: [Arinkin-P-Toën]** Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X^+, \mathrm{dSt}_{\mathbb{C}})$ be a DMS sheaf. Then

**(1)** The restriction map $\mathbf{A}^{2,cl}(\mathscr{F})_{K_X} \to \mathbf{A}^{2,cl}(\mathscr{F}_{\mathrm{in}})_{K_{X_{\mathrm{in}}}}$, is a homotopy equivalence.

**(2)** The extension $\omega$ of a form $\omega_{\mathrm{in}}$ is non-degenerate if and only if $\omega_{\mathrm{in}}$ is non-degenerate.

**Note:** Since $\mathscr{F}_{\mathrm{in}} = BG$ is **constant**, the form $\omega_{\mathrm{in}} = \omega_\kappa$ exists automatically since $G$ is assumed to be reductive.

# Deligne-Malgrange-Stokes sheaves (iii)

Suppose $\mathbf{Q}$ is an irregular type on $(\mathfrak{X}, D)$. Then:

- The data $\{L_i, \mathrm{Sto}_d\}$ determined by $\mathbf{Q}$ can be recorded equivalently in a Deligne-Malgrange-Stokes sheaf

$$\mathrm{DMS}_{G,\mathbf{Q}} \in \mathrm{Sh}^{\mathrm{str}}(X^+, \mathrm{dSt}_{\mathbb{C}});$$

- $\mathrm{DMS}_{G,\mathbf{Q}}$ classifies Stokes $G$-local systems $(\mathfrak{X}, D)$ of irregular type $\mathbf{Q}$, in the sense that

$$Loc_G(\mathbf{Q}) = \Gamma\left(X^+, \mathrm{DMS}_{G,\mathbf{Q}}\right).$$

# Deligne-Malgrange-Stokes sheaves (iv)

Suppose

$\mathscr{F}$ is a DMS sheaf of stacks euipped with a $K_{X^+}$-valued relative symplectic form $\omega$.

$f : X^+ \to (0, 1]$ is the stratified map which collapses $X^+ - X_{\text{out}}$ to 1 and projects each cylinder component of $X_{\text{out}}$ onto its ruling $(0, 1)$.

Then $f$ is a proper stratified map and

**Pushforward theorem** $\Longrightarrow$ $\operatorname{tr} f_* \omega$ is a relative $K_{(0,1]}$-valued symplectic structure on $f_* \mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}((0, 1], \mathrm{dSt}_{\mathbb{C}})$.

# Deligne-Malgrange-Stokes sheaves (v)

Hence

- tr $f_*\omega$ defines a 1-shifted symplectic structure on $\Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}})$.
- tr $f_*\omega$ defines a 0-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X^+ - X_{\text{out}}, \mathscr{F}) \to \Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}}).$$

# Deligne-Malgrange-Stokes sheaves (vi)

For $\mathscr{F} = \mathrm{DMS}_{G,\mathbf{Q}}$ we get

$$\begin{aligned} Loc_G(\mathbf{Q}) &= \Gamma(X^+ - X_{\mathrm{out}}, \mathscr{F}) = \Gamma(X^+, \mathscr{F}) \\ Loc_{\mathcal{L}}(\partial\mathfrak{X}) &= \Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}}) \end{aligned}$$

where $\mathcal{L}$ is the local system of Levi subgroups on $X_{\mathrm{out}}$ for which $\mathscr{F}_{\mathrm{out}} = B\mathcal{L}$.

Moreover in this setting the map

$$r_{\mathbf{Q}} : Loc_G(X, \mathbf{Q}) \to Loc_{\mathcal{L}}(\partial\mathfrak{X})$$

assigns to each Stokes filtered local system its formal monodromy at $\infty$.

# Deligne-Malgrange-Stokes sheaves (vii)

Since $\mathcal{L}$ is a locally constant sheaf we again have that fixing a flat section $\lambda \in \Gamma(\partial\mathfrak{X}, \mathcal{L})$ gives a Lagrangian map

$$\prod_{i=1}^{k} BG_{\lambda_i} \to \prod_{i=1}^{k} [\mathcal{L}_i/\mathcal{L}_i] = Loc_{\mathcal{L}}(\partial\mathfrak{X}).$$

The push forward theorem again implies that the intersection

$$Loc_G(\mathbf{Q}, \mathbf{C})$$

of this Lagrangian with the Lagrangian map $r_{\mathbf{Q}}$ is the $\mathbf{C}$-restricted moduli of Stokes $G$-local systems of type $\mathbf{Q}$ and is therefore symplectic.

**Note:** Here $\mathbf{C}_i$ is the conjugacy class of $\lambda_i$.

Back

# Macdonald polynomials (i)

**Recall:** **[Garsia-Haiman;Haiman]** The modified Macdonald polynomial $\widetilde{H}_\mu\left(z^2, w^2; \boldsymbol{x}^i\right)$ has a geometric interpretation in terms of $\mathbf{Hilb}^r(\mathbb{C}^2)$, $r = |\mu|$.

# Macdonald polynomials (i)

**Recall:** Haiman showed that the reduction $^{\mathbf{iso}}\mathbf{Hilb}^{r}(\mathbb{C}^2)_{\mathrm{red}}$ of the **isospectral Hilbert scheme** $^{\mathbf{iso}}\mathbf{Hilb}^{r}(\mathbb{C}^2)$ is normal, irreducible, and Cohen-Macaulay.

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Defined as the fiber product

$$\begin{array}{ccc} {}^{\text{iso}}\,\mathbf{Hilb}^r(\mathbb{C}^2) & \longrightarrow & (\mathbb{C}^2)^{\times r} \\ \downarrow & & \downarrow \\ \mathbf{Hilb}^r(\mathbb{C}^2) & \longrightarrow & \mathrm{Sym}^r\,\mathbb{C}^2 \end{array}$$

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induced from the standard action of $\mathbb{T}$ on $\mathbb{C}^2$

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Consider $\mathbf{Hilb}^r(\mathbb{C}^2) \circlearrowleft \mathbb{T} = \mathbb{C}^\times \times \mathbb{C}^\times$

**Note:** The $\mathbb{T}$-fixed points on $\mathbf{Hilb}^r(\mathbb{C}^2)$ are the monomial ideals in $\mathbb{C}[x, y]$: in one-to-one correspondence with partitions $\mu$ of $r$.

# Macdonald polynomials (ii)

Given $[I_\mu] \in \mathbf{Hilb}^r(\mathbb{C}^2)$ we can decompose the fiber of $\mathscr{R}$ at $[I_\mu]$ under the action of $\mathbb{T} \times \mathbf{S}_r$:

Back

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**Theorem: [Haiman]**

$$\widetilde{H}_\mu(q_1, q_2; \boldsymbol{x}) = \sum_{\lambda \in \mathscr{P}_r} \mathrm{ch}_{\mathbb{T}}(V_{\lambda,\mu}) s_\lambda(\boldsymbol{x}).$$

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**Note:** For us Haiman's formula will be the definition of $\widetilde{H}_\mu(q_1, q_2; \boldsymbol{x})$.

Back