THE GEOMETRY OF K3 SURFACES

LECTURES DELIVERED AT THE SCUOLA MATEMATICA INTERUNIVERSITARIA CORTONA, ITALY JULY 31—AUGUST 27, 1988

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1. Introduction

This is a course about K3 surfaces and several related topics. I want to begin by working through an example which will illustrate some of the techniques and results we will encounter during the course. So consider the following problem.

Problem. Find an example of $C \subset X \subset \mathbb{P}^3$, where C is a smooth curve of genus 3 and degree 8 and X is a smooth surface of degree 4.

Of course, smooth surfaces of degree 4 are one type of K3 surface. (For those who don't know, a *K3 surface* is a (smooth) surface X which is simply connected and has trivial canonical bundle. Such surfaces satisfy $\chi(\mathcal{O}_X) = 1$, and for every divisor D on X, $D \cdot D$ is an even integer.)

We first try a very straightforward approach to this problem. Let C be any smooth curve of genus 3, and let Z be any divisor on C of degree 8. (For example, we may take Z to be the sum of any 8 points on C.) I claim that the linear system $|Z|$ defines an embedding of C. This follows from a more general fact, which I hope you have seen before.

Theorem. *Let C be a smooth curve of genus g, and let Z be a divisor on C of degree $d > 2g$. Then the linear system $|Z|$ is base-point-free, and defines an embedding of C.*

Proof. First suppose that P is a base point of Z. We have an exact sequence

$$0 \to \mathcal{O}_C(Z - P) \to \mathcal{O}_C(Z) \to \mathcal{O}_P(Z) \to 0$$

and the assumption that P is a base point implies $H^0(\mathcal{O}_C(Z - P)) \cong H^0(\mathcal{O}_C(Z))$. But $\deg(Z-P) = d-1 > 2g-2$ and $\deg(Z) = d > 2g-2$ so that both of these divisors Z and $Z - P$ are *non-special*. By Riemann-Roch, $h^0(\mathcal{O}_C(Z - P)) = d - 1 - g + 1 \neq d - g + 1 = h^0(\mathcal{O}_C(Z))$, a contradiction.

Similarly, suppose that P and Q are not separated by the linear system $|Z|$. (We include the case $P = Q$, where we suppose that the maximal ideal of C at P is not embedded.) Then in the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_C(Z - P - Q) \to \mathcal{O}_C(Z) \to \mathcal{O}_{P+Q}(Z) \to 0$$

we must have that the map $H^0(\mathcal{O}_C(Z)) \to H^0(\mathcal{O}_{P+Q}(Z))$ is not surjective. This implies that $H^1(\mathcal{O}_C(Z - P - Q)) \neq (0)$. But

$$H^1(\mathcal{O}_C(Z - P - Q)) \cong H^0(\mathcal{O}_C(K_C - Z + P + Q))^*$$

and $\deg(K_C - Z + P + Q) = 2g - 2 - d + 2 < 0$ so this divisor cannot be effective (again a contradiction). Q.E.D.

I have included this proof because later we will study the question: on a *surface*, when does a linear system have base points, and when does it give an embedding?

To return to our problem, we have a curve C of genus 3 and a divisor Z of degree 8. This divisor is non-special and gives an embedding (since $8 > 2 \cdot 3$). By Riemann-Roch,

$$h^0(\mathcal{O}_C(Z)) = 8 - 3 + 1 = 6$$

so $|Z|$ maps C into $\mathbb{P}^5$.

The theory of generic projections guarantees that we can project C into $\mathbb{P}^3$ from $\mathbb{P}^5$ in such a way as to still embed C. So we assume from now on: $C \subset \mathbb{P}^3$ is a smooth curve of degree 8 and genus 3. (The linear system $|\mathcal{O}_C(1)|$ is not complete.)

We now consider the map $H^0(\mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(\mathcal{O}_C(k))$ for various degrees k. We have

$$h^0(\mathcal{O}_{\mathbb{P}^3}(k)) = \frac{(k+1)(k+2)(k+3)}{6}$$
$$h^0(\mathcal{O}_C(k)) = 8k - 3 + 1 = 8k - 2$$

(by Riemann-Roch).

It is easy to see that for $k \leq 3$ we have $h^0(\mathcal{O}_{\mathbb{P}^3}(k)) < h^0(\mathcal{O}_C(k))$ while for $k \geq 4$ we have $h^0(\mathcal{O}_{\mathbb{P}^3}(k)) > h^0(\mathcal{O}_C(k))$. This means that for $k \leq 3$, the linear system cut out on C by hypersurfaces of degree k is *incomplete*, while for $k \geq 4$ there must be a hypersurface of degree k *containing* C. In fact, for $k = 4$ we have

$$h^0(\mathcal{O}_{\mathbb{P}^3}(4)) = 35$$

$$h^0(\mathcal{O}_C(4)) = 30$$

so that there is at least a $\mathbb{P}^4$ of quartic hypersurfaces X containing C. By Bertini's theorem, the generic X is smooth *away from* C (the base locus of this $\mathbb{P}^4$). So we have almost solved our problem: we have constructed $C \subset X \subset \mathbb{P}^3$ with C smooth, but X is only known to be smooth away from C.

Where do we go from here? We could continue to pursue these methods of projective geometry in $\mathbb{P}^3$. For example, we might consider a generic pencil inside our $\mathbb{P}^4$

of quartics: the base locus of this pencil is $C \cup C'$ with C' another curve of degree 8. ($C' \neq C$ since the arithmetic genus would be wrong). We could try varying the pencil and showing that the induced family of divisors $C' \cap C$ on C has no base point. But the arguments are very intricate, and I'm not sure if they work! So we will try a different approach.

In this second approach, we construct X first instead of C. What we want is a K3 surface X together with two curves H and C on X. (H is the hyperplane section from the embedding $X \subset \mathbb{P}^3$). We want C to have genus 3, $|H|$ to define an embedding $X \subset \mathbb{P}^3$, and C should have degree 8 under this embedding. To translate these properties into *numerical* properties of H and C on X, notice that for any curve $D \subset X$ we have

$$\deg(K_D) = \deg((K_X + D)|_D) = \deg(D|_D) = D \cdot D$$

(using the adjunction formula, and the fact that K_X is trivial). Thus, $g(D) = \frac{1}{2}(D \cdot D) + 1$.

The numerical versions of our properties are: $C \cdot C = 4$ (so that $g(C) = 3$), $H \cdot H = 4$ (so that we get a *quartic* $X \subset \mathbb{P}^3$) and $H \cdot C = 8$ (so that C will have degree 8 in $\mathbb{P}^3$). In addition, we want C to be smooth and $|H|$ to define an embedding in $\mathbb{P}^3$.

The first step is to check that the topology of a K3 surface permits curves with these numerical properties to exist. The topological properties I have in mind concern the intersection pairing on $H^2(X, \mathbb{Z})$. Each curve D on X has a cohomology class $[D] \in H^2(X, \mathbb{Z})$, and the intersection number for curves coincides with the cup product pairing

$$H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Poincaré duality guarantees that this is a *unimodular* pairing (i. e. in a basis, the matrix for the pairing has determinant ± 1). In addition, the *signature* of the pairing (the number of $+1$ and -1 eigenvalues) can be computed as (3,19) for a K3 surface. The pairing is also *even*: this means $x \cdot x \in 2\mathbb{Z}$ for all $x \in H^2(X, \mathbb{Z})$. These three properties together imply that the isomorphism type of this bilinear form over $\mathbb{Z}$ is $(-E_8)^{\oplus 2} \oplus U^{\oplus 3}$ where E_8 is the unimodular even positive definite form of rank 8, and U is the hyperbolic plane. Let Λ denote this bilinear form; Λ is called the *K3 lattice*. We will return to study these topological properties in more detail (and define the terms!) in section 11.

In our example, we need a submodule of $H^2(X, \mathbb{Z}) \cong \Lambda$ generated by 2 elements h and c such that the matrix of the pairing on these 2 elements is

$$\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}.$$

The fact that such a submodule exists is a consequence of

Theorem (James [12]). *Given an even symmetric bilinear form L over the integers such that L has signature $(1, r-1)$ and the rank r of L is ≤ 10, there exists a submodule of Λ isomorphic to L.*

(Later[1] we will study refinements of this theorem in which the rank is allowed to be larger.)

So there is no topological obstruction in our case. Moreover, later[2] in the course we will partially prove the following:

Fact. Given a submodule L of Λ on which the form has signature $(1, r-1)$ with $r \leq 20$, there exists a $(20-r)$-dimensional family of K3 surfaces $\{X_t\}$, each equipped with an isomorphism $H^2(X_t, \mathbb{Z}) \cong \Lambda$ in such a way that elements of L correspond to cohomology classes of line bundles on X_t. Moreover, for t generic, these are the *only* line bundles on X_t, that is, the Néron-Severi group $\mathrm{NS}(X_t)$ [together with its intersection form] is isomorphic to L.

I said we would *partially* prove this: what we will not prove (for lack of time) is the global Torelli theorem and the surjectivity of the period map for K3 surfaces. This fact depends on those theorems.[3]

To return to our problem: we now have a K3 surface X and two line bundles $\mathcal{O}_X(H)$, $\mathcal{O}_X(C)$ with the correct numerical properties, which generate $\mathrm{NS}(X)$. For any line bundle $\mathcal{L}$ on X, we have $H^2(\mathcal{L}) \cong H^0(\mathcal{L}^*)^*$ (since K_X is trivial), from which follows the Riemann-Roch inequality:

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^*) \geq \frac{\mathcal{L} \cdot \mathcal{L}}{2} + 2$$

(since $\chi(\mathcal{O}_X) = 2$). Thus, if $\mathcal{L} \cdot \mathcal{L} \geq -2$ either $\mathcal{L}$ or $\mathcal{L}^*$ is effective.

In our situation we conclude: $\pm H$ is effective and $\pm C$ is effective. Replacing H by $-H$ if necessary we may assume H is effective. (The choice of C is then determined by $H \cdot C = 8$.)

To finish our construction, we need another fact which will be proved later.[4]

Fact. Let H be an effective divisor on a K3 surface X with $H^2 \geq 4$. If $|H|$ has base points or does not define an embedding, then there is a curve E on X with $E^2 = -2$ or $E^2 = 0$. Moreover, when it does define an embedding $H^1(\mathcal{O}_X(H)) = 0$ and $H^2(\mathcal{O}_X(H)) = 0$.

In our situation, we wish to rule out the existence of such an E. We have[5] $E \sim mH + nC$ since H and C generate $\mathrm{NS}(X)$. Thus, $E^2 = 4m^2 + 16mn + 4n^2$ is always

[1]Need cross-reference.
[2]Need cross-reference.
[3]For statements of the theorems and further discussion, see section 12.
[4]Need cross-reference: section 5?
[5]The symbol $\sim$ denotes linear equivalence.

divisible by 4, so $E^2 = -2$ is impossible. Moreover, if a rank 2 quadratic form $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ represents 0 then $-\det\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) = b^2 - ac$ is a *square*. In our case, $-\det\left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix}\right) = 48$ is *not* a square, so $E^2 = 0$ is impossible.

We conclude that $|H|$ is very ample. Then $H \cdot C = 8$ implies $-C$ is *not* effective, so that C *must* be effective. Furthermore, $|C|$ is then very ample, so this linear system contains a smooth curve (which we denote again by C). By Riemann-Roch, $h^0(\mathcal{O}_X(H)) = \frac{H \cdot H}{2} + 2 = 4$, so $|H|$ maps X into $\mathbb{P}^3$ as a smooth quartic surface, and we have $C \subset X \subset \mathbb{P}^3$ as desired.

2. K3 surfaces and Fano threefolds

We will use in this course a definition of K3 surfaces which is slightly different from the standard one. Namely, for various technical reasons which will appear later, it is convenient to allow K3 surfaces to have some singular points called *rational double points*. These will be the subject of a seminar later on;[6] if you are not familiar with them, I suggest that you ignore the singularities for the moment and concentrate on smooth K3 surfaces.

(We do not use the term "singular K3 surface" to refer to these surfaces, because that term has a different meaning in the literature: it refers to a smooth K3 surface with Picard number 20. Cf. [27].)

Here is a convenient definition of rational double points: a complex surface X has *rational double points* if the dualizing sheaf ω_X is locally free, and if there is a resolution of singularities $\pi : \widetilde{X} \to X$ such that $\pi^*\omega_X = \omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$. For those unfamiliar with the dualizing sheaf, what this means is: for every $P \in X$ there is a neighborhood U of P and a holomorphic 2-form $\alpha = \alpha(z_1, z_2) dz_1 \wedge dz_2$ defined on $U - \{P\}$ such that $\pi^*(\alpha)$ extends to a nowhere-vanishing holomorphic form on $\pi^{-1}(U)$.

The structure of rational double points (sometimes called *simple singularities*) is well-known: each such point must be analytically isomorphic to one of the following:

$$\begin{array}{ll} A_n\ (n \geq 1): & x^2 + y^2 + z^{n+1} = 0 \\ D_n\ (n \geq 4): & x^2 + yz^2 + z^{n-1} = 0 \\ E_6: & x^2 + y^3 + z^4 = 0 \\ E_7: & x^2 + y^3 + yz^3 = 0 \\ E_8: & x^2 + y^3 + z^5 = 0 \end{array}$$

and the resolution $\widetilde{X} \to X$ replaces such a point with a collection of rational curves of self-intersection -2 in the following configuration:

A_n (n curves)
D_n (n curves)
E_6
E_7

[6] See Appendix ??.

E_8

To return to the definition of K3 surfaces: a *K3 surface* is a compact complex analytic surface X with only rational double points such that $h^1(\mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$. (If X is smooth, the dualizing sheaf ω_X is the line bundle associated to the canonical divisor K_X, so this last condition says that the canonical divisor is trivial.)

If X is a K3 surface and $\pi : \widetilde{X} \to X$ is the minimal resolution of singularities (i. e. the one which appeared in the definition of rational double point) then it turns out that π^* establishes an isomorphism $H^1(\mathcal{O}_X) \cong H^1(\mathcal{O}_{\widetilde{X}})$, and also we have $\omega_{\widetilde{X}} = \pi^*\omega_X = \pi^*\mathcal{O}_X = \mathcal{O}_{\widetilde{X}}$. Thus, the smooth surface $\widetilde{X}$ is also a K3 surface.

We will concentrate on smooth K3 surfaces for quite a while, and only return to singular ones in several weeks.[7] I have included the singular case today so that we don't have to change the definitions later.

Here are some of the basic facts about smooth K3 surfaces. Let X be a smooth K3 surface.

(1) $\chi(\mathcal{O}_X) = 2$, because $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X))^* = h^0(\mathcal{O}_X) = 1$.

(2) $c_1^2(X) = 0$. [Remember that $c_1^2(X) = K_X \cdot K_X$.]

(3) Therefore, using Noether's formula

$$c_1^2(X) + c_2(X) = 12\chi(\mathcal{O}_X)$$

we find that $c_2(X) = 24$. Since $b_1(X) = 2h^1(\mathcal{O}_X)$ or $2h^1(\mathcal{O}_X) - 1$, we see that $b_1(X) = 0$. So the Betti numbers must be: $b_0 = 1$, $b_1 = 0$, $b_2 = 22$, $b_3 = 0$, $b_4 = 1$ giving 24 as the topological Euler characteristic.

[For those who haven't studied compact complex surfaces, I remind you that in the case of *algebraic* surfaces we always have $b_1(X) = 2h^1(\mathcal{O}_X)$. Kodaira proved that for non-algebraic complex surfaces this equality can only fail by 1, i. e., $b_1 = 2h^1(\mathcal{O}_X)$ or $2h^1(\mathcal{O}_X) - 1$.] (Cf. [BPV, Theorem II.6].[8])

(4) For any line bundle $\mathcal{L}$ on X, the Riemann-Roch theorem

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{\mathcal{L}\cdot\mathcal{L} - K_X\cdot\mathcal{L}}{2}$$

becomes:

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^0(\mathcal{L}^*) = 2 + \frac{\mathcal{L}\cdot\mathcal{L}}{2}$$

since $h^2(\mathcal{L}) = h^0(\mathcal{L}^*(K_X)) = h^0(\mathcal{L}^*)$.

In particular, if $\mathcal{L}\cdot\mathcal{L} \geq -2$ then $h^0(\mathcal{L}) + h^0(\mathcal{L}^*) > 0$, i. e., either $\mathcal{L}$ or $\mathcal{L}^*$ is effective.

(5) The intersection form on $H^2(X, \mathbb{Z})$ has a very explicit structure mentioned in the introduction. We postpone discussion of this structure until section 11.

[7] Need cross-reference: section 8?

[8] Per A. G.

(6) Let D be an irreducible reduced effective divisor on X. Then the adjunction formula $K_D = (K_X + D)|_D$ yields

$$\deg(K_D) = \deg(D|_D) = D \cdot D.$$

In particular,

$$g(D) = \frac{1}{2}(D \cdot D) + 1.$$

(7) $h^0(D) = g(D) + 1$.

But even more is true: if we consider the exact sequence (when D is smooth)

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$$

since $h^1(\mathcal{O}_X) = 0$ we have that $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_D(D))$ is surjective. Thus since $\mathcal{O}_D(D) \cong \mathcal{O}_D(K_D)$, the global sections of the line bundle $\mathcal{O}_X(D)$ induce the *canonical map* on D. So when D is not hyperelliptic, it must be embedded by $\mathcal{O}_X(D)$ and this embedding coincides with the canonical embedding of D.

To say this another way, if X is embedded in $\mathbb{P}^g$ by the complete linear system $|\mathcal{O}_X(D)|$, then a hyperplane section $D = X \cap \mathbb{P}^{g-1}$ is canonically embedded in $\mathbb{P}^{g-1} \subset \mathbb{P}^g$. In brief: "a hyperplane section of a K3 surface is a canonical curve".

This property can be considered as motivating the definition of a K3 surface. That is, we require $K_X \sim 0$ so that the normal bundle $\mathcal{O}_D(D)$ of a hyperplane section agrees with the canonical bundle $\mathcal{O}_D(K_D)$, and we require $h^1(\mathcal{O}_X) = 0$ so that the rational map $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_D(D))$ is surjective.

We now ask: What kind of threefold has a K3 surface as *its* hyperplane section? If Y is such a threefold, we must have $h^1(\mathcal{O}_Y) = 0$ to guarantee that $H^0(\mathcal{O}_Y(X)) \to H^0(\mathcal{O}_X(X))$ is surjective, and by adjunction

$$K_X = (K_Y + X)|_X$$

so that we need $K_Y = -X$. (Note that by the Lefschetz hyperplane theorem (cf. [9]), we have $h^1(\mathcal{O}_X) = 0$ so X is a K3 surface.) That is, Y is embedded by its anti-canonical linear system $|\mathcal{O}_Y(-K_Y)|$. This is called a *Fano threefold.*

We can ask the same question for higher dimension, of course.

Definition. A *Fano variety* is a complex projective variety Y with $\mathcal{O}_Y(-K_Y)$ ample. A Fano variety has *index* r if r is the maximum integer such that $-K_Y \sim rH$ for some ample H on Y. The *coindex* of Y is defined to be $c = \dim(Y) - r + 1$.

The linear system $|H|$ is called the *fundamental system* of the Fano variety.

Lemma. *If $|H|$ is very ample and we choose r general hyperplanes $H_1, \ldots, H_r$, then $Y \cap H_1 \cap \cdots \cap H_r$ is a variety of dimension $c - 1$ with trivial canonical bundle.*

(The proof is easy: use the adjunction formula.)

Thus, the Fano varieties related to elliptic curves are the ones of coindex 2; the ones related to K3 surfaces (and canonical curves) are the ones of coindex 3. To see that Fano varieties of coindex 3 are in fact related to K3 surfaces, rather than to some other surface with trivial canonical bundle, we need to recall the Kodaira vanishing theorem.

Theorem (Kodaira). *Let L be an ample divisor on Y. Then $H^i(\mathcal{O}_Y(K_Y+L))=0$ for $i>0$.*

Corollary. *Let Y be a Fano variety. Then $H^i(\mathcal{O}_Y)=0$ for $i>0$.*

Proof. Take $L=-K_Y$, which is ample, so that $\mathcal{O}_Y(K_Y+L)=\mathcal{O}_Y$. then $H^i(\mathcal{O}_Y)=(H^i(\mathcal{O}_Y(K_Y+L))=0$ for $i>0$. Q.E.D.

Corollary. *Let Y be a Fano variety of coindex c, and let $X=Y\cap H_1\cap\cdots\cap H_r$ be a linear section of dimension $c-1$ (which has trivial canonical bundle). Then $H^i(\mathcal{O}_X)=0$ for $0<i<\dim X=c-1$.*

[This justifies the statement made earlier that Fano varieties of coindex 3 have linear surface sections which are K3 surfaces.]

Proof. Let $Z=Y\cap H_1\cap\cdots\cap H_{r-1}$ be a linear section of dimension c so that $X=Z\cap H$; then $\mathcal{O}_Z(K_Z)=\mathcal{O}_Z(-H)$ and Z is again a Fano variety. We apply Kodaira vanishing (in its dual form) to conclude that $H^i(\mathcal{O}_Z(-H))=0$ for $0\le i<c=\dim Z$. Thus, in the long exact cohomology sequence associated to

$$0\to\mathcal{O}_Z(-H)\to\mathcal{O}_Z\to\mathcal{O}_X\to 0$$

we find that $H^i(\mathcal{O}_Z)\cong H^i(\mathcal{O}_X)$ for $i+1<c$. The statement now follows from the previous corollary. Q.E.D.

3. Examples of canonical curves, K3 surfaces, and Fano varieties of coindex 3

Before beginning the examples, let us recall the theory of the Hirzebruch surfaces $\mathbb{F}_n$ (or Σ_n). These are defined as $\mathbb{F}_n=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n))$, and are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. $\mathbb{F}_n$ has a distinguished section of its $\mathbb{P}^1$-bundle structure (when $n>0$), given by $\sigma_\infty=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n))\subset\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n))$. [When $n=0$, the section exists but is no longer unique.] σ_∞ satisfies: $\sigma_\infty^2=-n$, $\sigma_\infty\cdot f=1$ (where f is a general fiber) and also $f^2=0$; moreover, σ_∞ and f generate $\operatorname{Pic}(\mathbb{F}_n)$. There are other sections $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1})\subset\mathbb{F}_n$ which are linearly equivalent to $\sigma_\infty+nf$.

We need to know the canonical bundle formula for $\mathbb{F}_n$, and it is derived as follows. Write

$$K_{\mathbb{F}_n}=a\sigma_\infty+bf$$

and use the fact that σ_∞ and f are smooth rational curves:

$$\begin{array}{rclclcl} -2 &=& \deg(K_{\sigma_\infty}) &=& \deg((a+1)\sigma_\infty + bf)|_{\sigma_\infty} &=& -n(a+1)+b \\ -2 &=& \deg(K_f) &=& \deg(a\sigma_\infty + (b+1)f)|_f &=& a. \end{array}$$

This implies $a=-2$, $b=-n-2$ so $K_{\mathbb{F}_n} = -2\sigma_\infty - (n+2)f$.

Example 1. Write $-2K_{\mathbb{F}_4} = \sigma_\infty + (3\sigma_\infty + 12f)$ and choose a smooth divisor $D \in |3\sigma_\infty + 12f|$. (We will see in a moment[9] that this can be done.) Let X be the double cover branched on $D + \sigma_\infty \in |-2K_{\mathbb{F}_4}|$. Then $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{\mathbb{F}_4}) = 0$ and

$$K_X = \pi^*(K_{\mathbb{F}_4} + \frac{1}{2}(D+\sigma_\infty)) = \pi^*(K_{\mathbb{F}_4} - K_{\mathbb{F}_4}) \sim 0$$

so that X is a K3 surface.[10]

We have the following picture of the branch locus:

In fact, the double cover of f is branched in 4 points (because $-2K_{\mathcal{F}_4}\cdot f = 4$), and so is a smooth elliptic curve for general f by the Hurwitz formula. Thus, X is a K3 surface with a pencil of curves of genus 1.

To see that D exists and to describe this all more explicitly, consider the complement of σ_∞ in $\mathbb{F}_4$. This is a $\mathbb{C}$-bundle over $\mathbb{P}^1$, isomorphic to the total space of the (bundle associated to the) sheaf $\mathcal{O}_{\mathbb{P}^1}(4)$. If we restrict further to $\mathbb{C} \subset \mathbb{P}^1$, we can choose coordinates: x in the fiber direction and t in the base.

There will be another set of coordinates: $\boxed{s = \frac{1}{t}}$ and y in the fiber direction. To see how these are related, let e be a nonvanishing section of $\mathcal{O}_{\mathbb{P}^1}(4)|_{t\text{-chart}}$ and let f be one of $\mathcal{O}_{\mathbb{P}^1}(4)|_{s\text{-chart}}$. There is a global section of $\mathcal{O}_{\mathbb{P}^1}(4)$ with a zero of order 4 at $t=0$ and no other zero: this must be given by t^4e and $1\cdot f$ when restricted to the two charts. So we have $t^4 e = 1\cdot f$ which determines the transition $e = t^{-4}f$. Since arbitrary sections are to be represented by xe or yf we have $yf = xe = xt^{-4}f$ so that $\boxed{xt^{-4} = y}$. The 2 boxed equations are the transition functions.

Now on this space we have the line bundle $\mathcal{L} = \mathcal{O}(\sigma_\infty + 4f)$ which has a section vanishing at $(x=0)\cup(y=0)$, and D is the zero-locus of a section of $\mathcal{L}^{\otimes 3}$. If we let ε and φ be trivializing sections for $\mathcal{L}$ in the (x,t) and (y,s) charts respectively, then $x\varepsilon = y\varphi$ so that the transition is $\varepsilon = \frac{y}{x}\varphi = t^{-4}\varphi$.

[9] I. e., below.

[10] See also the appendix on double covers.

Let us write the section of $\mathcal{L}^{\otimes 3}$ whose zero-locus is D in the form $f(x,t)\varepsilon^{\otimes 3} = g(y,s)\varphi^{\otimes 3}$. Then $f(x,t)t^{-12} = g(y,s)$. A monomial $x^i t^j$ can appear in f only if

$$x^i t^{j-12} = (ys^{-4})^i (s^{-1})^{j-12} = y^i s^{12-4i-j}$$

is holomorphic, i. e., $12 - ri - j \geq 0$. This implies that f has the form

$$f(x,t) = kx^3 + a_4(t)x^2 + b_8(t)x + c_{12}(t).$$

We take $k \neq 0$ so that there are truly three points of intersection with the fiber; then by a coordinate change we may assume $k = 1$.

Finally we may describe the double cover: it has the form

$$z^2 = x^3 + a_4(t)x^2 + b_8(t)x + c_{12}(t)$$

in one chart, and

$$w^2 = y^3 + (s^4 a_4(\frac{1}{s}))y^2 + (s^8 b_8(\frac{1}{s}))y + (s^{12} c_{12}(\frac{1}{s}))$$

in the other chart. As is well-known, this compactifies nicely when $x \to \infty$ or $y \to \infty$, and branches at ∞ in the fiber direction, giving a family of elliptic curves with a section.

What remains to be checked is that D is smooth when a, b, c are chosen generically. We leave this as an exercise.

Example 2. Let $X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched in a smooth curve C of degree 6. Then[11]

$$h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{\mathbb{P}^2}) = 0$$
$$K_X = \pi^*(K_{\mathbb{P}^2} + \frac{1}{2}C) = \pi^*(-3H + \frac{1}{2}(6H)) \sim 0$$

so that X is a K3 surface. π expresses the inverse image of a general line in $\mathbb{P}^2$ as a double cover of the line branched in 6 points, so the inverse image is a curve of genus 2. Thus, X is a K3 surface with a curve of genus 2.

Examples 3, 4, 5. Let us find all complete intersections in projective space $\mathbb{P}^{n+k}$ which are either a canonical curve, a K3 surface, or a Fano variety of coindex 3. Let X be the variety in question, and notice that in all three cases we want

$$K_X = (2-n)H|_X$$

where $n = \dim(X)$. [In the Fano case, $r = n-2$ so $c = n - (n-2) + 1 = 3$.]

Now if X is the intersection of hypersurfaces $V_1, \dots, V_k$ in $\mathbb{P}^{n+k}$ of degrees $d_1, \dots, d_k$, we may assume $d_i \geq 2$. An easy induction with the adjunction formula gives

$$\begin{aligned} K_X &= (K_{\mathbb{P}^{n+k}} + V_1 + \cdots + V_k)|_X \\ &= ((-n-k-1) + d_1 + \cdots + d_k)H|_X. \end{aligned}$$

[11] See the appendix on double covers.

So we need

$$-n-k-1+d_1+\cdots+d_k=2-n$$

or

$$\sum_{i=1}^{k}(d_i-1)=n+1+2-n=3.$$

The solutions are $3=3$, $3=2+1$ and $3=1+1+1$ corresponding to

Example 3. A quartic hypersurface in $\mathbb{P}^{n+1}$ (this is a K3 with curve section of genus 3 when $n=2$);

Example 4. The intersection of a quadric and a cubic in $\mathbb{P}^{n+2}$ (this is a K3 with curve section of genus 4 when $n=2$); and

Example 5. The intersection of three quadrics in $\mathbb{P}^{n+3}$ (this is a K3 with curve section of genus 5 when $n=2$).

Example 6. The canonical bundle of the Grassmannian $\mathrm{Gr}(2,5)$ satisfies $K_{\mathrm{Gr}(2,5)}=-5\Sigma$, where Σ is the Schubert cycle of codimension 1. The linear system $|\Sigma|$ induces the Plucker embedding of the Grassmannian $\mathrm{Gr}(2,5)\to\mathbb{P}(\Lambda^2\mathbb{C}^5)=\mathbb{P}^9$.

Let Y be the double cover of $\mathrm{Gr}(2,5)$ branched along a divisor $D\in|2\Sigma|$. Then $K_Y=\pi^*(K_{\mathrm{Gr}(2,5)}+\frac{1}{2}D)=\pi^*(-5\Sigma+\Sigma)=-4\pi^*(\Sigma)$ so that Y has index 4. Since $\dim(Y)=6$, the coindex is $6-4+1=3$.

The fundamental system $|\pi^*(\Sigma)|$ satisfies:

$$\pi^*(\Sigma)^6=2\cdot\Sigma^6=2\cdot 5=10$$

and so has degree 10. Moreover, since $H^0(\pi^*\Sigma)\cong H^0(\Sigma)\oplus H^0(\Sigma-\frac{1}{2}D)$ and $H^0(\Sigma-\frac{1}{2}D)=H^0(\mathcal{O}_{\mathrm{Gr}(2,5)})\cong\mathbb{C}$, the fundamental system maps Y to $\mathbb{P}H^0(\pi^*\Sigma)\cong\mathbb{P}^{10}$. The genus of a linear curve section is 6, since $2g-2=10$.

Examples 7, 8, 9, 10. Mukai has investigated the question: which compact complex homogeneous spaces are Fano varieties of coindex 3? (Note that $\mathbb{P}^n$ and the quadric $Q^n\subset\mathbb{P}^{n+1}$, which are the Fano varieties of coindex 0 and 1 respectively, *are* complex homogeneous spaces). The answers Mukai found (cf. [21]) are the following (we give no details about his methods):

	Variety	Dimension	Degree of $\lvert H\rvert$	Ambient space	Genus of C
7)	$S0(10)/U(5)$	10	12	$\mathbb{P}^{15}$	7
8)	$U(6)/U(2)\times U(4)=\mathrm{Gr}(2,6)$	8	14	$\mathbb{P}^{14}$	8
9)	$Sp(3)/U(3)$	6	16	$\mathbb{P}^{13}$	9
10)	G_2/P	5	18	$\mathbb{P}^{13}$	10

where G_2 is the exceptional Lie group of that name, and P is the maximal parabolic associated to the long root in the Dynkin diagram $\bullet \Rrightarrow \bullet$.

[In the table, H represents the ample generator of the Picard group (which is isomorphic to $\mathbb{Z}$), the ambient space refers to the embedding $\varphi_{|H|}$ (it turns out that $|H|$ is in fact very ample), and C is a linear curve section of the embedded variety.]

In the examples we have given with linear curve section of genus g, $2 \le g \le 10$, it will turn out later[12] that the general K3 surface with a (primitive) curve of that genus belongs to the family we have described. It is also possible to give such a description for $g = 12$ (we give the example below), but *not* for $g = 11$ or $g \ge 13$. We will give several examples of Fano varieties of genus 11 and coindex 3, but all have Picard number ≥ 2 and so only give proper subsets of the set of all K3 surfaces of genus 11.

Example 11a. $\mathbb{P}^3 \times \mathbb{P}^3$.

$\mathrm{Pic}(\mathbb{P}^3 \times \mathbb{P}^3)$ is generated by $H_1 = \mathbb{P}^2 \times \mathbb{P}^3$ and $H_2 = \mathbb{P}^3 \times \mathbb{P}^2$, with $K_{\mathbb{P}^3 \times \mathbb{P}^3} = -4H_1 - 4H_2$. Thus, $r = 4$, $c = 6 - 4 + 1 = 3$ and of course the dimension is 6. The fundamental system is $H = H_1 + H_2$, and $H^6 = \binom{6}{3} H_1^3 H_2^3 = \binom{6}{3} = 20$. The mapping $\varphi_{|H|}$ is the Segre embedding $\mathbb{P}^3 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{15}$. Since $2g - 2 = 20$ we have $g = 11$.

Example 11b. $\mathbb{P}^2 \times Q^3$, where $Q^3 \subset \mathbb{P}^4$ is a quadric.

$\mathrm{Pic}(\mathbb{P}^2 \times Q^3)$ is generated by $H_1 = \mathbb{P}^1 \times Q^3$ and $H_2 = \mathbb{P}^2 \times (\mathbb{P}^3 \cap Q^3)$, with $K_{\mathbb{P}^2 \times Q^3} = -3H_1 - 3H_2$. Thus, $r = 3$, $c = 5 - 3 + 1 = 3$ and the dimension is 5. The fundamental system is $H = H_1 + H_2$ of degree $H^5 = \binom{5}{2} H_1^2 H_2^3 = \binom{5}{2} \cdot 2 = 20$. The mapping $\varphi_{|H|}$ is induced by the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^4$:

$$\mathbb{P}^2 \times Q^3 \subset \mathbb{P}^2 \times \mathbb{P}^4 \hookrightarrow \mathbb{P}^{14}.$$

Since $2g - 2 = 20$ we have $g = 11$.

Example 11c. $\mathbb{P}^1 \times V_5^3$, where V_5^3 is a 3-dimensional linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ (and thus has degree 5 in $\mathbb{P}^6$).

$\mathrm{Pic}(\mathbb{P}^1 \times V_5^3)$ is generated by $H_1 = \mathbb{P}^0 \times V_5^3$ and $H_2 = \mathbb{P}^1 \times (V_5^3 \cap \mathbb{P}^5)$ with $K_{\mathbb{P}^1 \times V_5^3} = -2H_1 - 2H_2$. ($V_5^3$ is a Fano 3-fold of index 2.) Thus, $r = 2$, $c = 4 - 2 + 1 = 3$ and the dimension is 4. The fundamental system is $H = H_1 + H_2$ of degree $H^4 = \binom{4}{1} H_1 \cdot H_2^3 = \binom{4}{1} \cdot 1 \cdot 5 = 20$. The mapping $\varphi_{|H|}$ is induced by the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^6$:

$$\mathbb{P}^1 \times V_5^3 \subset \mathbb{P}^1 \times \mathbb{P}^6 \hookrightarrow \mathbb{P}^{13}.$$

Since $2g - 2 = 20$, we once again have $g = 11$.

Example 11d. We give no details on this, but Mori and Mukai have found that if we take a smooth conic $C \subset \mathbb{P}^2$ and choose a degree 5 map $C \to \mathbb{P}^1$ so that the induced curve $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ has bidegree (5,2), then the blowup of $\mathbb{P}^1 \times \mathbb{P}^2$ with center

[12] Need cross-reference.

C is a Fano 3-fold of index 1 with degree of the fundamental system $= 20$ and so $g = 11$. This is an example with Picard number 3.

Example 12. Let F_1, F_2, F_3 be general skew-symmetric bilinear forms on $\mathbb{C}^7$. Let

$$Y = \{w \in \mathrm{Gr}(3,7) \mid F_1(w,w) = F_2(w,w) = F_3(w,w) = 0\}.$$

Mukai has shown that this Y is a smooth Fano 3-fold of degree 22 (and $g = 12$). In fact, the map $\varphi_{|H|}$ associated to the fundamental system factors through the inclusion $Y \subset \mathrm{Gr}(3,7)$ and we have

$$\begin{array}{ccl} Y & \xrightarrow{\varphi_{|H|}} & \mathbb{P}^{13} = \text{span of } Y \\ \bigcap & & \bigcap \text{[linear subspace]} \\ \mathrm{Gr}(3,7) & \xrightarrow{\text{Plucker}} & \mathbb{P}^{34} \end{array}$$

Example Km. Our final example of K3 surfaces will include some non-algebraic ones. Let $T = \mathbb{C}^2/\Gamma$ be a complex torus of complex dimension 2. (Thus, $\Gamma \subset \mathbb{C}^2$ is an additive subgroup such that there is an isomorphism of $\mathbb{R}$-vector spaces $\Gamma \otimes \mathbb{R} \cong_{\mathbb{R}} \mathbb{C}^2$. In particular, Γ is a free $\mathbb{Z}$-module of rank 4.) Let (z,w) be coordinates on $\mathbb{C}^2$, and define $i(z,w) = (-z,-w)$. Since Γ is a subgroup under addition, $i(\Gamma) = \Gamma$. Thus, i descends to an automorphism $\tilde{i} : T \to T$.

What are the fixed points of $\tilde{i}$? To find them, we need to know the solutions to $i(z,w) \equiv (z,w) \mod \Gamma$. These solutions are $\{(z,w) \mid (2z,2w) \in \Gamma\}$ and so $\tilde{i}$ has as fixed points $\frac{1}{2}\Gamma/\Gamma$. (There are 16 of these.)

Let $X = T/\tilde{i}$; X is called a *Kummer surface.* This surface has 16 singular points at the images of the fixed points of $\tilde{i}$. To see the structure of these singular points, consider the action of i on a small neighborhood U of (0,0) in $\mathbb{C}^2$. Then U/i is isomorphic to a neighborhood of a singular point of X.

To describe U/i, we note that the invariant functions on U are generated by z^2, zw, and w^2. Thus, if we let $r = z^2$, $s = zw$ and $t = w^2$ we can write

$$U/i \cong \{(r,s,t) \text{ near } (0,0,0) \mid rt = s^2\}.$$

This is a rational double point of type A_1.

$dz \wedge dw$ is a global holomorphic 2-form on $\mathbb{C}^2$, invariant under the action of Γ, and so descends to a form on T. It is also invariant under the action of i (since $d(-z) \wedge d(-w) = dz \wedge dw$), so we get a form $dz \wedge dw$ on $X - \{\text{singular points}\}$. In local coordinates, $dr = 2zdz$, $dt = 2wdw$ so that

$$dz \wedge dw = \frac{dr \wedge dt}{4zw} = \frac{dr \wedge dt}{4s}.$$

It is easy to check that this form induces a global nowhere vanishing holomorphic 2-form on the minimal resolution $\widetilde{X}$ of X.

To finish checking that X is a K3 surface, we use the fact that $H^1(\mathcal{O}_{\widetilde{X}}) \cong H^1(\mathcal{O}_X) \cong \{\text{elements of } H^1(\mathcal{O}_T) \text{ invariant under } \tilde{i}\}$. Now $H^1(\mathcal{O}_T) \cong H^{0,1}(T)$, the space of global differential forms of type (0,1). This space is generated by $d\bar{z}$ and $d\bar{w}$; since $\tilde{i}^*(d\bar{z}) = -d\bar{z}$ and $\tilde{i}^*(d\bar{w}) = -d\bar{w}$ there are no invariants. It follows that $H^1(\mathcal{O}_X) \cong H^1(\mathcal{O}_{\widetilde{X}}) = (0)$, and that X and $\widetilde{X}$ are both K3 surfaces. Notice that when T (or equivalently Γ) is chosen generally, then T is not algebraic, nor are X or $\widetilde{X}$.

3.1. Addendum to section 3. There is a beautiful 3-fold, which I believe was first constructed by Segre, which shows that projective K3 surfaces (specifically quartics in $\mathbb{P}^3$) can have 15 or 16 singularities of type A_1.

The threefold is defined in $\mathbb{P}^5$ by two equations

$$x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2x_1y_1x_2y_2 - 2x_1y_1x_3y_3 - 2x_2y_2x_3y_3 = 0$$
$$x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 0.$$

(Of course this is really in $\mathbb{P}^4$, but the equations are more symmetric this way.) This is a quartic 3-fold which has 15 singular lines:

(a) 8 lines of the form $\ell_1 = \ell_2 = \ell_3 = \ell'_1 + \ell'_2 + \ell'_3 = 0$ where $\ell_\alpha \in \{x_\alpha, y_\alpha\}$ and ℓ'_α is different from ℓ_α for each α (e.g. $x_1 = x_2 = x_3 = y_1 + y_2 + y_3 = 0$)
(b) 6 lines of the form $x_i = y_i = \ell_j + \ell_k = \ell'_j + \ell'_k = 0$ where $\ell_\alpha \in \{x_\alpha, y_\alpha\}$, ℓ'_α is different from ℓ_α and (i, j, k) is a permutation of (1,2,3).
(c) the line $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_1 + x_2 + x_3 = 0$.

A general hyperplane section of this 3-fold is a quartic surface with 15 A_1 singularities. On the other hand, for the general point P of this variety Y, if $T_P(Y)$ is the (projective) tangent plane to Y at P then $T_P(Y) \cap Y$ is a quartic surface with 16 A_1 singularities: one at P and 15 at the intersection with the lines.

Note.

(i) For the general point P, the rank of $\mathrm{Pic}(\widetilde{T_P(Y) \cap Y}$ is 17, where $\widetilde{T_P(Y) \cap Y}$ is the minimal resolution of $T_P(Y) \cap Y$.
(ii) $T_P(Y) \cap Y$ is a (quartic) Kummer surface (see Nikulin's theorem in section 8).
(iii) Every Kummer surface coming from a principally polarized abelian surface can be represented as such a $T_P(Y) \cap Y$. This has been "sort of" proved by Van der Geer.[13]

3.2. Appendix: Double covers. A double cover is constructed from a variety X, a line bundle $\mathcal{L}$, and a section $s \in H^0(\mathcal{L}^{\otimes 2})$ whose zero-locus is D. (This is called the double cover of X branched along D.) For simplicity we assume X and D smooth.

To describe the construction, we need an open cover $\{U_i\}$ of X such that $\mathcal{L}|_{U_i}$ is trivial. Let $\vec{t}_i$ denote coordinates on U_i, and choose a nowhere zero section $e_i \in H^0(U_i, \mathcal{L}|_{U_i})$ to trivialize $\mathcal{L}|_{U_i}$. Then every section of $\mathcal{L}|_{U_i}$ may be written in the form $s_i(\vec{t}_i)e_i$ for some functions s_i on U_i.

[13]Need a literature reference, and an explanation of "sort of."

The sections e_i are related by the transition functions: $e_i = \lambda_{ij} e_j$. Thus, if $s = \{s_i(\vec{t}_i)e_i\}$ is a *global* section, we must have

$$s_i(\vec{t}_i)e_i = s_i(\vec{t}_i)\lambda_{ij}e_j = s_j(\vec{t}_i)e_j$$

so that: $s_i(\vec{t}_i)\lambda_{ij} = s_j(\vec{t}_j)$.

In the case of the double cover construction, we have $s = \{s_i(\vec{t}_i)e_i^{\otimes 2}\}$, a section of $\mathcal{L}^{\otimes 2}$ and so

$$s_i(\vec{t}_i)\lambda_{ij}^2 = s_j(\vec{t}_j)$$

where $\{\lambda_{ij}\}$ are transition functions for L. Define

$$V_i = \{(\vec{t}_i, x_i) \in U_i \times \mathbb{C} \,|\, x_i^2 = s_i(\vec{t}_i)\}.$$

The *double cover* is $Y = \bigcup V_i$, with projection map $\pi : Y \to X$ given by $\pi(\vec{t}_i, x_i) = \vec{t}_i$. The coordinate charts V_i are to be patched by:

$$(\vec{t}_j, x_j) = (\vec{t}_j(t_i), x_i\lambda_{ij})$$

[so that $x_i^2\lambda_{ij}^2 = s_i\lambda_{ij}^2 = s_j = x_j^2$.]

Lemma.

(1) *$K_Y = \pi^*(K_X + L)$, i. e., $\mathcal{O}_Y(K_Y) = \pi^*(\mathcal{O}_X(K_X) \otimes \mathcal{L})$.*

(2) *If $\mathcal{M}$ is another line bundle on X, then*

$$H^0(\pi^*\mathcal{M}) \cong H^0(\mathcal{M}) \oplus H^0(\mathcal{M} \otimes \mathcal{L}^{-1}).$$

[Property #1 is often written: $K_Y = \pi^*(K_X + \frac{1}{2}D)$, which is a bit sloppy.]

Proof of (1). Let $d\vec{t}_i$ denote the differential form $dt_i^1 \wedge \cdots \wedge dt_i^n$ on U_i. $d\vec{t}_i$ is a section of $\mathcal{O}_X(K_X)|_{U_i}$ which trivializes that bundle; since

$$d\vec{t}_i = \left|\frac{\partial(\vec{t}_i)}{\partial(\vec{t}_j)}\right| d\vec{t}_j$$

[the notation means the "Jacobian determinant"], we see that $w_{ij} = \left|\frac{\partial(\vec{t}_i)}{\partial(\vec{t}_j)}\right|$ give transition functions for $\mathcal{O}_X(K_X)$.

Since X and D are smooth, we may assume (after shrinking the U_i's) that $s_i(\vec{t}_i) = t_i^1$ (the first coordinate). Then $x_i^2 = t_i^1$ so that $(x_i, t_i^2, \ldots, t_i^n)$ form coordinates on V_i. Moreover,

$$\begin{aligned} d\vec{t}_i &= dt_i^1 \wedge dt_i^2 \wedge \cdots \wedge dt_i^n \\ &= 2x_i dx_i \wedge dt_i^2 \wedge \cdots \wedge dt_i^n. \end{aligned}$$

Thus, if e_i is a trivializing section of $\mathcal{O}_X(K_X)|_{U_i}$ and $\tilde{e}_i$ is a trivializing section of $\mathcal{O}_Y(K_Y)|_{V_i}$ we have $\pi^*(e_i) = 2x_i\tilde{e}_i$. It follows that

$$\begin{aligned} \tilde{e}_i &= \frac{1}{2x_i}\pi^*(e_i) = \frac{1}{2}\frac{\lambda_{ij}}{x_j}\pi^*(w_{ij})\pi^*(e_j) \\ &= \pi^*(\lambda_{ij}w_{ij})\tilde{e}_j \end{aligned}$$

(because $\pi^*(\lambda_{ij}) = \lambda_{ij}$ due to this function being independent of x_i). So the transition functions for $\mathcal{O}_Y(K_Y)$ and $\pi^*\mathcal{O}_X(K_X) \otimes \mathcal{L}$ are the same. Q.E.D.

Proof of (2). Let μ_{ij} be transition functions for $\mathcal{M}$, and let ε_i be a trivializing section of $\mathcal{M}|_{U_i}$. A global section of $\pi^*(\mathcal{M})$ is given by $\{f_i(\vec{t}_i, x_i)\varepsilon_i\}$ with

$$f_i(\vec{t}_i, x_i)\mu_{ij} = f_j(\vec{t}_j, x_j).$$

Now the maps $(\vec{t}_i, x_i) \mapsto (\vec{t}_i, -x_i)$ are compatible and give an automorphism of Y whose quotient is X. We let this automorphism act on $H^0(\pi^*\mathcal{M})$ and write

$$f_i(\vec{t}_i, x_i) = f_i^+(\vec{t}_i, x_i) + f_i^-(\vec{t}_i, x_i)$$

where we have decomposed according to $+1$ and -1 eigenspaces. $\{f_i^+\}$ and $\{f_i^-\}$ give global sections [since the automorphism was global].

Now $f_i^+(\vec{t}_i, x_i)$ involves only *even* powers of x_i: we may write

$$f_i^+(\vec{t}_i, x_i) = g_i(\vec{t}_i, x_i^2) = g_i(\vec{t}_i, s_i(\vec{t}_i))$$

and so we get a section of $\mathcal{M}$.

Similarly, $f_i^-(\vec{t}_i, x_i)$ involves only *odd* powers of x_i: if we write

$$f_i^-(\vec{t}_i, x_i) = x_i h_i(\vec{t}_i, x_i^2) = x_i h_i(\vec{t}_i, s_i(\vec{t}_i))$$

then $\{h_i\}$ give a section of $\mathcal{M} \otimes \mathcal{L}^{-1}$ since

$$h_j = \frac{f_j^-}{x_j} = \frac{f_i^-\mu_{ij}}{x_i\lambda_{ij}} = h_i(\frac{\mu_{ij}}{\lambda_{ij}}).$$

Q.E.D.

An explicit example of all of this is given in Example C1 in the next section.

4. Elliptic K3 surfaces

I will give a very brief[14] sketch of the following fact: if X is a K3 surface with a nonsingular connected elliptic curve E and a smooth rational curve C such that $C \cdot E = 1$, then X is constructed from a Weierstrass equation as in example 1. Note that $h^0(E) = 2$, by basic fact (7) from section 2.

[14]For more details, see Hartshorne II.3.2, Deligne's "Formulaire", or Tate.

Consider X as an elliptic curve $\mathcal{E}$ over the function field $k(\Gamma)$, where Γ is the base of the elliptic fibration $X \xrightarrow{f} \Gamma$, one of whose fibers is E. The curve C (which is a section of f) can be considered as a point $P \in \mathcal{E}$.

Now $\mathcal{O}_{\mathcal{E}}(P)$ is a line bundle of degree 1 on $\mathcal{E}$. By Riemann-Roch, we have $h^0(nP) = n$. Thus, we have sections

$$\begin{array}{rlll} & 1 & \text{generating} & H^0(\mathcal{O}_{\mathcal{E}}(P)) \\ & 1^2, x & \text{generating} & H^0(\mathcal{O}_{\mathcal{E}}(2P)) \\ & 1^3, 1\cdot x, y & \text{generating} & H^0(\mathcal{O}_{\mathcal{E}}(3P)) \\ & 1^4, 1^2x, 1\cdot y, x^2 & \text{generating} & H^0(\mathcal{O}_{\mathcal{E}}(4P)) \\ & 1^5, 1^3x, 1^2y, 1x^2, xy & \text{generating} & H^0(\mathcal{O}_{\mathcal{E}}(5P)) \\ \text{and} & 1^6, 1^4x, 1^3y, 1^2x^2, 1xy, x^3y^2 & \text{contained in} & H^0(\mathcal{O}_{\mathcal{E}}(6P)). \end{array}$$

It follows that there is an equation relating all of these sections of $\mathcal{O}_{\mathcal{E}}(6P)$, of the form

$$c_1y^2 + a_1xy + a_3y = c_2x^3 + a_2x^2 + a_4x + a_6.$$

Some standard linear algebra reduces this to an equation of the form

$$y^2 = x^3 + b_4x + b_6$$

in Weierstrass form.

To complete the analysis, we must see how to do all of this in affine charts, and how the different equations relate in the overlap; one eventually gets that X is the double cover of $\mathbb{P}(\mathcal{O}_\Gamma \oplus \mathcal{L})$ for a line bundle $\mathcal{L}$ which is divisible by 2, and that b_4, b_6 are sections of $\mathcal{L}^{\otimes 2}$ and $\mathcal{L}^{\otimes 3}$ respectively. Computing the canonical bundle forces the construction to be that of example 1.

If we want X to have only rational double points, we must take the branch locus to have only simple singularities.[15]

Every elliptic surface which is a K3 surface must be of the type described above. In particular, the parameter curve for the elliptic fibration must be isomorphic to $\mathbb{P}^1$. In fact, the canonical bundle formula for an elliptic fibration $\pi : X \to C$ implies that $\kappa(X) = 1$ either if $g(C) \geq 2$ or if $g(C) = 1$ and π is not a trivial fibration. (Here, κ denotes the Kodaira dimension.[16]) Moreover, if $g(C) = 1$ and π is trivial, then $h^1(\mathcal{O}_X) = 2$, which prevents X from being a K3 surface. Thus, when X is a K3 surface C must have genus 0.

4.1. Elliptic K3 surfaces, continued. We now use elliptic K3 surfaces to produce some examples of badly-behaved linear systems on K3 surfaces. First, we recall the Kodaira-Ramanujan vanishing theorem (to be proved in section 5).

[15]Need references to literature.

[16]This is occurring for the first time.

Theorem (Kodaira-Ramanujan). *Let X be a smooth projective surface, let L be a divisor on X which is nef and big. (That is, $L \cdot C \geq 0$ for all curves C on X, and $L^2 > 0$.) Then $H^i(\mathcal{O}_X(K_X + L)) = 0$ for $i > 0$.*

In section 1, we saw three important properties which many linear systems $|K_X + L|$ have: (a) the higher cohomology may vanish, i. e., $H^i(\mathcal{O}_X(K_X + L)) = 0$ for $i > 0$, (b) $|K_X + L|$ may have no base points, and (c) $\varphi_{|K_X+L|}$ may give an embedding. We now give some examples of linear systems on K3 surfaces for which these properties *fail*. The first example shows that the hypothesis "$L^2 > 0$" in the Kodaira-Ramanujan theorem cannot be relaxed.

Example A. Let X be an elliptic K3 surface, and let E be a fiber of the elliptic pencil. Consider $L = kE$. The map $\varphi_{|L|}$ factors through the elliptic pencil $f : E \to \Gamma \cong \mathbb{P}^1$ and in fact $\mathcal{O}_X(L) = f^*\mathcal{O}_{\mathbb{P}^1}(k)$. Thus, $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_{\mathbb{P}^1}(k)) = k + 1$. Moreover, since $-kE$ is not effective, $h^2(\mathcal{O}_X(L)) = 0$. But then

$$h^0(\mathcal{O}_X(L)) - h^1(\mathcal{O}_X(L)) = \frac{L \cdot L}{2} + 2 = 2$$

which implies that $h^1(\mathcal{O}_X(L)) = k - 1$, which is nonzero for $k \geq 2$. Note that $L^2 = 0$ in this example, and L is nef.

Example B. Let $f : X \to \Gamma$ be an elliptic K3 surface with fiber E, and let C be a section of f. (In particular, X has a Weierstrass form, but we shall not need it for this example.) Consider $L = C + kE$; note that $L^2 = 2k - 2$.

Let us check that L is nef for $k \geq 2$. If D is any irreducible curve on X, then $D \cdot (C + kE) < 0$ implies that D is a component of $C + kE$, that is, $D = C$ or $D = E$. But

$$C \cdot (C + kE) = k - 2 \geq 0$$

$$E \cdot (C + kE) = 1$$

so that $D \cdot (C + kE) \geq 0$. Hence $L = C + kE$ is nef.

We may then use the Kodaira-Ramanujan vanishing theorem and Riemann-Roch to compute:

$$h^0(\mathcal{O}_X(L)) = \frac{L \cdot L}{2} + 2 = k + 1.$$

On the other hand, $h^0(\mathcal{O}_X(kE)) = k + 1$ as well by Example A. Since the exact sequence

$$0 \to \mathcal{O}_X(kE) \to \mathcal{O}_X(C + kE) \to \mathcal{O}_C(C + kE) \to 0$$

induces an injection

$$0 \to H^0(\mathcal{O}_X(kE)) \to H^0(\mathcal{O}_X(C + kE)),$$

we see that $H^0(\mathcal{O}_X(kE)) \cong H^0(\mathcal{O}_X(L))$ and that C is a fixed component of $|L|$. In particular, every point of C is a base point of $|L|$. Note that in this example L is nef, $L^2 > 0$ and there exists an E with $E^2 = 0$ and $L \cdot E = 1$.

Example C1. Here are two examples of linear systems without base points which do not give embeddings. Let X be the double cover of $\mathbb{P}^2$ branched along a curve of degree 6. In the notation of the double cover appendix, we have $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$ and $K_{\mathbb{P}^2} = -3H$ so the canonical bundle of the double cover is trivial. If we take $\mathcal{M} = \mathcal{O}_{\mathbb{P}^2}(n)$, then $H^0(\pi^*\mathcal{M})$ contains 2 pieces: $H^0(\mathcal{M})$, coming from the pullback from $\mathbb{P}^2$, and $H^0(\mathcal{M} \otimes \mathcal{L}^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^2}(n-3))$. We conclude: all sections of $\pi^*\mathcal{O}(1)$ and $\pi^*\mathcal{O}(2)$ come from $\mathbb{P}^2$, so the maps $\varphi_{|\pi^*\mathcal{O}(1)|}$ and $\varphi_{|\pi^*\mathcal{O}(2)|}$ factor through the projection to $\mathbb{P}^2$. That is, if X is the double cover of $\mathbb{P}^2$ branched along a smooth curve of degree 6, then $|\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$ and $|\pi^*\mathcal{O}_{\mathbb{P}^2}(2)|$ both factor through the map $\pi : X \to \mathbb{P}^2$. In particular, these linear systems do not give embeddings. It is only with $|\pi^*\mathcal{O}(3)|$ that we can embed the K3 surface.

Example C2. To see another example of this phenomenon, let D be a smooth divisor in $|-2K_{\mathbb{F}_0}|$. (Note that $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong$ a smooth quadric in $\mathbb{P}^3$, and that D is a curve of "type (4,4)": the complete intersection of $\mathbb{F}_0$ with a quartic surface.) Let $\pi : X \to \mathbb{F}_0$ be the double cover branched on D. Then X is a K3 surface. If we let C be the graph in $\mathbb{P}^1 \times \mathbb{P}^1$ of a degree k map $\mathbb{P}^1 \to \mathbb{P}^1$, then C is a smooth rational curve of "type $(1,k)$", and $C \cdot D = 4k+4$. Thus, $\pi^{-1}(C)$ is a hyperelliptic curve of genus $2k+1$ by the Hurwitz formula. In particular, if $L = \pi^{-1}(C)$ then the map $\varphi_{|L|}$ induces the canonical map $|K_{\pi^{-1}(C)}|$ on $\pi^{-1}(C)$. Since that map has degree 2, $\varphi_{|L|}$ cannot be an embedding.

Note that in this case L is nef, $L^2 > 0$, and there is a curve $E = \pi^{-1}(F)$ (where F is a "type (0,1) fibre" of $\mathbb{F}_0$) which is elliptic with $E^2 = 0$ and $L \cdot E = 2$.

Example C3. The genus of the curves in the previous example was always odd; to get an even genus case, start with $\mathbb{F}_1 \cong$ blowup of $\mathbb{P}^2$ at a point P. Let $\bar{D}$ be a curve in $\mathbb{P}^2$ of degree 6 with a node at P so that the proper transform D of $\bar{D}$ is a smooth curve in $|-2K_{\mathbb{F}_1}|$. Let $\bar{C}$ be an irreducible curve in $\mathbb{P}^2$ of degree k with a point of multiplicity $(k-1)$ at P. [For example, if $P = [1,0,0]$, take an equation for C which is the general linear combination of monomials $x_0^a x_1^b x_2^c$ with $a+b+c = k$, $a \leq 1$.] The proper transform C of $\bar{C}$ is a smooth rational curve. Since the local intersection multiplicity of $\bar{C}$ and $\bar{D}$ at P is $2(k-1)$, we get $C \cdot D = 4k - 2(k-1) = 4k+2$.

Now we repeat the construction of example C2: $\pi : X \to \mathbb{F}_1$ branched on D is a K3 surface with a hyperelliptic curve $\pi^{-1}(C)$ of genus $2k$. The linear system $|L| = |\pi^{-1}(C)|$ cannot embed X, and we have: L is nef, $L^2 > 0$, and there is an elliptic $E = \pi^{-1}(\text{fiber on } \mathbb{F}_1)$ with $E^2 = 0$ and $L \cdot E = 2$.

We will see in section 6 that all examples with $\varphi_{|L|}$ not birational are of these types.

5. Reider's method

Reider's method, only a few years old, is now one of the most important tools for studying linear systems on algebraic surfaces. It has almost completely supplanted the older method of studying "d-connectedness" of divisors, although it is closely related to that method. We present Reider's method here for arbitrary surfaces, and then give a refinement for K3 surfaces in the next section.

We begin with a version of the Hodge index theorem for surfaces.

Theorem. *Let L, D be divisors on a smooth projective surface X with $L^2 > 0$. Then either*

(a) $L^2D^2 < (L \cdot D)^2$, *or*
(b) $L^2D^2 = (L \cdot D)^2$ *and* $D \approx sL$ *for some* $s \in \mathbb{Q}$.

(Here, $\approx$ denotes numerical equivalence.)

"Proof": (Based on the version of the Hodge index theorem given in Hartshorne). Consider the intersection form on $(\mathrm{Pic}(X)/\approx) \otimes \mathbb{Q}$. Hartshorne's version of the index theorem says: if H is ample and $H \cdot \Delta = 0$ then either $\Delta^2 < 0$ or $\Delta \approx 0$. If we choose a basis $H, \Delta_1, \dots, \Delta_{r-1}$ of $(\mathrm{Pic}(X)/\approx) \otimes \mathbb{Q}$ with $H \cdot \Delta_i = 0$ for all i, the index theorem says that the intersection form on $(\mathrm{Pic}(X)/\approx) \otimes \mathbb{Q}$ has signature $(1, r-1)$. [It must be negative definite on the span of $\Delta_1, \dots, \Delta_{r-1}$.]

Now consider the span of D and L inside $(\mathrm{Pic}(X)/\approx) \otimes \mathbb{Q}$. If this span has dimension 2, then since $L^2 > 0$ it must have signature (1,1). [There can be at most one positive eigenvalue.] This is true if and only if

$$\det \begin{pmatrix} L^2 & L \cdot D \\ L \cdot D & D^2 \end{pmatrix} < 0,$$

giving case (a).

If the span has dimension 1, then $D \approx sL$ for some $s \in \mathbb{Q}$ and an easy computation shows $L^2D^2 = (L \cdot D)^2$. Q.E.D.

We will apply this to prove a very technical-looking lemma, which contains the key computations for the Reider method.

Let us say that 2 divisors L and D on X *satisfy condition* $(*)_d$ if:

$$(*)_d \qquad \begin{cases} L \cdot D \geq 0 \\ L \cdot (L - 2D) \geq 0 \\ D \cdot (L - D) \leq d. \end{cases}$$

Technical Lemma. *Let X be a smooth projective surface and let L and D be divisors such that $L^2 > 0$, $D \not\approx 0$, and L and D satisfy condition $(*)_d$. Then either*

(i) $0 < L \cdot D \leq \min\{2d, \frac{1}{2}L^2\}$, $\max\{0, -d + L \cdot D\} \leq D^2 \leq \frac{(L \cdot D)^2}{L^2}$, *or*
(ii) $0 \leq L \cdot D \leq \min\{d - 1, \frac{1}{2}L^2\}$, $-d + L \cdot D \leq D^2 < 0$.

Moreover, if $D^2 = \frac{(L \cdot D)^2}{L^2}$ *in case (i), then* $L \approx sD$ *where* $s = \frac{L^2}{L \cdot D}$*.*

Proof. We have

$$L \cdot D \leq \frac{1}{2} L^2 \tag{1}$$

$$-d + L \cdot D \leq D^2 \tag{2}$$

which account for 2 of the inequalities in either case.

Case 1. $L \cdot D = 0$.

Here, we must be in case (ii) and what must be shown is $L \cdot D < d$ and $D^2 < 0$. The first is a consequence of the second, in light of (2).

If $D^2 = 0$, then $L^2 D^2 = (L \cdot D)^2 = 0$ so by Hodge index, $D \approx sL$. This means $D \approx 0$, contrary to hypothesis.

In any case, $D^2 \leq (L \cdot D)^2 / L^2 = 0$, proving this case.

Case 2. $L \cdot D > 0$, $D^2 \geq 0$.

Multiply eq. (2) by $(L \cdot D)$: $(-d + L \cdot D)(L \cdot D) \leq D^2 (L \cdot D)$

Multiply eq. (1) by D^2: $D^2 (L \cdot D) \leq D^2 (\frac{1}{2} L^2)$

Use Hodge index: $\frac{1}{2} L^2 D^2 \leq \frac{1}{2} (L \cdot D)^2$.

Thus, since $L \cdot D > 0$ we get

$$-d + L \cdot D \leq \frac{1}{2} (L \cdot D)$$

or $L \cdot D \leq 2d$.

The remaining inequalities are clear.

Case 3. $L \cdot D > 0$, $D^2 < 0$.

All inequalities are clear in this case.

Q.E.D.

Corollary. *Under the hypotheses* $L^2 > 0$, $D \not\approx 0$ *there are the following possible solutions for* $(*)_0$, $(*)_1$, $(*)_2$*:*

Solution for $(*)_0$*: None.*

Solution for $(*)_1$*:*

$L \cdot D = 2,\ D^2 = 1,\ L^2 = 4,\ L \approx 2D$

$L \cdot D = 1,\ D^2 = 0$

$L \cdot D = 0,\ D^2 = -1$

Solution for $(*)_2$*: all solutions for* $(*)_1$*, and:*

$L \cdot D = 4,\ D^2 = 2,\ L^2 = 8,\ L \approx 2D$

$L \cdot D = 3,\ D^2 = 1,\ 6 \leq L^2 \leq 9$, *(if* $L^2 = 9$ *then* $L \approx 3D$*)*

$L \cdot D = 2,\ D^2 = 0$

$L \cdot D = 1,\ D^2 = -1$

$$L \cdot D = 0,\ D^2 = -2.$$

(The proof consists of enumerating cases in the conclusion of the "technical lemma".)

For the next step in the Reider method, we need to construct some vector bundles on the surface X. The construction proceeds by means of extensions of sheaves, which are now review.

Suppose that $0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$ is a short exact sequence of sheaves of $\mathcal{O}_X$-modules, and consider the functor $\mathrm{Hom}(-,\mathcal{B})$. This is a half-exact contravariant functor, and leads to a long exact sequence which begins

$$0 \to \mathrm{Hom}(\mathcal{A},\mathcal{B}) \to \mathrm{Hom}(\mathcal{E},\mathcal{B}) \to \mathrm{Hom}(\mathcal{B},\mathcal{B}) \to \mathrm{Ext}^1(\mathcal{A},\mathcal{B}) \to$$

where $\mathrm{Ext}^1(-,\mathcal{B})$ is the first derived functor. The *extension class* of the sequence $0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$ is the image of the identity map $1_{\mathcal{B}} \in \mathrm{Hom}(\mathcal{B},\mathcal{B})$ in $\mathrm{Ext}^1(\mathcal{A},\mathcal{B})$. [Conversely, any element of $\mathrm{Ext}^1(\mathcal{A},\mathcal{B})$ is the extension class of *some* extension.] Notice that $1_{\mathcal{B}}$ maps to zero in $\mathrm{Ext}^1(\mathcal{A},\mathcal{B})$ if and only if there is some map $\varphi : \mathcal{E} \to \mathcal{B}$ such that the composite $\mathcal{B} \subset \mathcal{E} \xrightarrow{\varphi} \mathcal{B}$ is the identity on $\mathcal{B}$. That is, the extension class is 0 if and only if the sequence is split, so that $\mathcal{E} = \mathcal{A} \oplus \mathcal{B}$.

The key to the bundle constructions we need is to use Serre duality (for sheaves which may not be locally free) to interpret an H^1 cohomology group as the dual of an Ext^1 group, and then build an extension. So recall this form of Serre duality (proved in Hartshorne's book), which we state only for surfaces.

Theorem (Serre Duality). *Let X be a smooth projective surface and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then*

$$H^1(\mathcal{F} \otimes \omega_X) \cong \mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X)^*.$$

We use these techniques to build two kinds of vector bundles. First, if L is a line bundle on X with $H^1(\mathcal{O}_X(K_X+L)) \neq 0$, and $e \in H^1(\mathcal{O}_X(K_X+L))^*$ is a nonzero element, define $\mathcal{E}_{e,L}$ to be the extension

$$0 \to \mathcal{O}_X \to \mathcal{E}_{e,L} \to \mathcal{O}_X(L) \to 0$$

with extension class $e \in H^1(\mathcal{O}_X(K_X+L))^* = \mathrm{Ext}^1(\mathcal{O}_X(L),\mathcal{O}_X)$.

Second, if L is a line bundle on X and Z is a zero-cycle with $H^1(\mathcal{I}_Z(K_X+L)) \neq 0$, we define a "universal" extension as follows. For *any* complex vector space V,

$$\mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X \otimes V) \cong \mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X) \otimes V.$$

In particular, this holds for $V = H^1(\mathcal{I}_Z(K_X+L)) \cong \mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X)^*$. We may regard the identity mapping on $\mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X)$ as an element

$$\mathrm{id} \in \mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X) \otimes \mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X)^* \cong \mathrm{Ext}^1(\mathcal{I}_Z(L),\mathcal{O}_X \otimes H^1(\mathcal{I}_Z(K_X+L))$$

and thus get an extension

$$0 \to \mathcal{O}_X \otimes H^1(\mathcal{I}_Z(K_X+L)) \to \mathcal{E}(Z,L) \to \mathcal{I}_Z(L) \to 0.$$

Before proceeding, let's pause and show why this particular cohomology group $H^1(\mathcal{I}_Z(K_X+L))$ is so interesting. Suppose we have a line bundle L with $H^1(\mathcal{O}_X(K_X+L)) = 0$, and we consider the exact sequence

$$0 \to \mathcal{I}_Z(K_X + L) \to \mathcal{O}_X(K_X + L) \to \mathcal{O}_Z(K_X + L) \to 0.$$

The long exact cohomology sequence has 4 interesting terms:

$$\begin{array}{ccccccc} 0 & \to & H^0(\mathcal{I}_Z(K_X + L)) & \to & H^0(\mathcal{O}_X(K_X + L)) & \to & H^0(\mathcal{O}_Z(K_X + L)) \\ & \to & H^1(\mathcal{I}_Z(K_X + L)) & \to & 0. & & \end{array}$$

We say that Z *fails to impose independent conditions on* $|K_X+L|$ if $H^1(\mathcal{I}_Z(K_X+L)) \neq 0$. The key cases for the Reider method will be (1) $Z = P$ is a point (in which case this condition means that P is a base point) or (2) $Z = P + Q$ is a pair of points, possibly infinitely near (in which case this condition means that either P or Q is a base point, or the map $\varphi_{|K_X+L|}$ fails to separate P and Q [in the infinitely near case: fails to have injective differential at P]). We will use the sheaf $\mathcal{E}(Z, L)$ to extract information about these situations.

We need the following lemma, whose proof we do not give here.

Lemma. *Let X be a smooth projective surface, L be a nef and big divisor*

(i) *If e is a general element of $H^1(\mathcal{O}_X(K_X + L))^*$ then $\mathcal{E}_{e,L}$ is locally free.*
(ii) *If for every $Z' \subsetneqq Z$ we have $h^1(\mathcal{I}_{Z'}(K_X + L)) < h^1(\mathcal{I}_{Z'}(K_X + L))$ then $\mathcal{E}(Z, L)$ is locally free.*

In these cases, we let $E_{e,L}$ and $E(Z, L)$ denote the corresponding vector bundles, and write

$$\begin{aligned} \mathcal{E}_{e,L} &= \mathcal{O}_X(E_{e,L}); \\ \mathcal{E}(Z, L) &= \mathcal{O}_X(E(Z, L)). \end{aligned}$$

We will be primarily interested in the case of rank 2 bundles; notice that in this case we have a sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{I}_Z(L) \to 0$$

where $E = E_{e,L}$ or $E(Z, L)$, and $Z = \emptyset$ in the first case (assuming rank $E = 2$).

Definition. Let E be a rank 2 vector bundle on a smooth projective surface X.

(i) We say that E has the *strong Bogomolov property* if there are a zero-cycle A, line bundles $\mathcal{M}, \mathcal{N} \in \mathrm{Pic}(X)$ and an exact sequence

$$0 \to \mathcal{M} \to \mathcal{O}_X(E) \to \mathcal{I}_A \otimes \mathcal{N} \to 0$$

such that $h^0((\mathcal{M} \otimes \mathcal{N}^{-1})^{\otimes k}) > 0$ for some $k > 0$.

(ii) If $L = c_1(E)$ is nef, we say that E has the *weak Bogomolov property* if there are a zero-cycle A, line bundles $\mathcal{M}, \mathcal{N} \in \mathrm{Pic}(X)$ and an exact sequence

$$0 \to \mathcal{M} \to \mathcal{O}_X(E) \to \mathcal{I}_A \otimes \mathcal{N} \to 0$$

such that $L \cdot (\mathcal{M} \otimes \mathcal{N}^{-1}) \geq 0$.

[Clearly, when $c_1(E)$ is nef the strong Bogomolov property implies the weak one. Notice also: $\mathcal{O}_X(L) = \mathcal{M} \otimes \mathcal{N}$.]

The reason for making this somewhat strange looking definition is

Bogomolov's Theorem. *If E is a rank 2 vector bundle on a smooth projective surface X with $c_1^2(X) > 4c_2(X)$ then E has the strong Bogomolov property.*

The proof of this theorem is far beyond the scope of this course; it is essential for doing Reider's method on arbitrary surfaces, but as we will see in the next section, for K3 surfaces we get better results by using a slightly different method (which avoids Bogomolov's theorem).

The final ingredient in Reider's method is the following proposition:

Proposition. *Let E be a rank 2 vector bundle on a smooth projective surface X such that $L = c_1(E)$ is nef and big. Suppose that there is a section $s : \mathcal{O}_X \to \mathcal{O}_X(E)$ whose zero-locus Z has dimension 0. [This is the case if and only if we have an exact sequence of sheaves*

$$(*) \qquad 0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{O}_X(E) \to \mathcal{I}_Z(L) \to 0.]$$

If E has the weak Bogomolov property with associated sequence

$$(**) \qquad 0 \to \mathcal{M} \to O_X(E) \to \mathcal{I}_A \otimes \mathcal{N} \to 0,$$

then there is an effective divisor D containing Z such that $\mathcal{N} = \mathcal{O}_X(D)$. Moreover, if $D = 0$ then $Z = \emptyset$ and the sequence (*) splits.

Proof. We assemble (*) and (**) into a diagram and consider the induced map α :

$\mathcal{O}_X \to \mathcal{I}_A \otimes \mathcal{N}$ shown below.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_X \arrow[d, "s"] \arrow[dr, "\alpha"] & & \\
0 \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{O}_X(E) \arrow[r] \arrow[d] & \mathcal{I}_A \otimes \mathcal{N} \arrow[r] & 0 \\
 & & \mathcal{I}_Z(L) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Suppose first that α is identically 0. Then the image of s lies in $\mathcal{M}$, and there is an induced map $\mathcal{O}_X \to \mathcal{M}$. For $x \notin Z \cup A$, the maps on fibers $\mathcal{O}_{X,x} \to E_x$ and $\mathcal{M}_x \to E_x$ are injective; thus, $\mathcal{O}_{X,x} \to \mathcal{M}_x$ is an isomorphism for such values of x. But a map between line bundles which is an isomorphism away from a codimension 2 set like $Z \cup A$ must be an isomorphism everywhere; it follows that $\mathcal{M} \cong \mathcal{O}_X$.

But now $\mathcal{O}_X(L) = \mathcal{M} \otimes \mathcal{N} = \mathcal{N}$ so that $\mathcal{M} \otimes \mathcal{N}^{-1} = \mathcal{O}_X(-L)$. The weak Bogomolov property then implies that $L \cdot (-L) \geq 0$, contradicting the assumption "L is big".

Thus, α is not identically zero, so it defines a non-trivial section of $\mathcal{I}_A \otimes \mathcal{N}$; composing with the inclusion $\mathcal{I}_A \otimes \mathcal{N} \subset \mathcal{N}$ we get a section of $\mathcal{N}$. The zero-locus D of this section contains Z (since the map s, through which our section $\mathcal{O}_X \to \mathcal{N}$ factors, vanishes on Z), and satisfies $\mathcal{N} = \mathcal{O}_X(D)$.

It remains to prove the last statement. Suppose that $D = 0$. Since D is by definition the subset of X where the composite map $\mathcal{O}_X \xrightarrow{\alpha} \mathcal{I}_A \otimes \mathcal{N} \subset \mathcal{N}$ fails to be surjective on fibers, we see that this map is an isomorphism. (In particular, $\mathcal{I}_A = \mathcal{O}_X$

and $A = \emptyset$.) Now in the diagram

$$\begin{tikzcd}
0 \arrow[d] & & \\
\mathcal{O}_X \arrow[d] \arrow[dr, "\alpha"] & & \\
\mathcal{O}_X(E) \arrow[r, "\beta"] \arrow[d] & \mathcal{N} \arrow[r] & 0 \\
\mathcal{I}_Z(L) \arrow[d] & & \\
0 & &
\end{tikzcd}$$

the map $\alpha^{-1} \circ \beta : \mathcal{O}_X(E) \to \mathcal{O}_X$ gives a splitting of the sequence. This implies that $\mathcal{O}_X(E) = \mathcal{O}_X \oplus \mathcal{I}_Z(L)$; since E is locally free, Z must be empty as well. Q.E.D.

We can now give Mumford's proof of the Kodaira-Ramanujan vanishing theorem, which is the "0^{th} case" of Reider's method. (In retrospect, this proof is a special case of Reider's method, but in fact it preceded Reider's work by about 10 years.)

Mumford's proof of Kodaira-Ramanujan vanishing. Let L be nef and big, and suppose $H^1(\mathcal{O}_X(K_X + L)) \neq 0$. Choose $e \neq 0$ to be a general element of $H^1(\mathcal{O}_X(K_X + L))^*$ and consider the vector bundle $E_{e,L}$, which has a defining sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E_{e,L}) \to \mathcal{O}_X(L) \to 0.$$

Since $e \neq 0$, this sequence is not split.

Now $c_1(E_{e,L}) = L$ and $c_2(E_{e,L}) = 0$. Thus, $c_1^2(E_{e,L}) = L^2 > 0 = 4c_2(E_{e,L})$ so that Bogomolov's theorem applies, and we have the strong (and the weak) Bogomolov property:

$$0 \to \mathcal{M} \to \mathcal{O}_X(E_{e,L}) \to \mathcal{I}_A \otimes \mathcal{N} \to 0.$$

By the proposition, $\mathcal{N} = \mathcal{O}(D)$ for some effective divisor D; moreover, since the defining sequence is not split, $D \neq 0$. This implies that $D \not\approx 0$, since D is effective.

Now $L \cdot D \geq 0$ since L is nef and D is effective; $L \cdot ((L - D) - D) \geq 0$ since $\mathcal{M} = \mathcal{O}_X(L - D)$, $\mathcal{N} = \mathcal{O}_X(D)$; and

$$\begin{aligned}
0 &= c_2(E_{e,L}) = c_2(\mathcal{M}) + c_2(\mathcal{I}_A \otimes \mathcal{N}) + c_1(\mathcal{M}) \cdot c_1(\mathcal{I}_A \otimes \mathcal{N}) \\
&= \deg A + (L - D) \cdot D \geq (L \cdot D) \cdot D
\end{aligned}$$

so that L and D satisfy $(*)_0$. But by the technical lemma, there are *no* solutions to $(*)_0$ with $D \not\approx 0$ and $L^2 > 0$, a contradiction.

Hence, $H^1(\mathcal{O}_X(K_X + L)) = 0$. Q.E.D.

We assemble all of our pieces of the Reider method for $E(Z,L)$ in the rank 2 case into the following theorem.

Theorem (Reider's method). *Let X be a smooth projective surface, let L be a nef and big line bundle on X, and let Z be a zero-cycle of degree $d>0$ such that $h^1(\mathcal{I}_Z(K_X+L))=1$ but for every $Z' \subsetneqq Z$, $h^1(\mathcal{I}_{Z'}(K_X+L))=0$. Suppose that the vector bundle $E(Z,L)$ satisfies the weak Bogomolov property. Then there is an effective divisor D containing Z such that L and D satisfy $(*)_d$ [with $L^2>0$ and $D \not\approx 0$]. In particular, L and D satisfy the conclusion of the "technical lemma": either*

(i) $0 < L\cdot D \le \min\{2d, \frac{1}{2}L^2\}$, $\max\{0, -d+L\cdot D\} \le D^2 \le \frac{(L\cdot D)^2}{L^2}$, *or*
(ii) $0 \le L\cdot D \le \min\{d-1, \frac{1}{2}L^2\}$, $-d+L\cdot D \le D^2 < 0$.

Proof. Since $h^1(\mathcal{I}_Z(K_X+L))=1$ and $h^1(\mathcal{I}_{Z'}(K_X+L))=0$ for all $Z' \subsetneqq Z$, the vector bundle $E(Z,L)$ exists and has rank 2. Its defining sequence has the form

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E(Z,L)) \to \mathcal{I}_Z(L) \to 0.$$

Since $E(Z,L)$ satisfies the weak Bogomolov property, there is another sequence

$$0 \to \mathcal{M} \to \mathcal{O}_X(E(Z,L)) \to \mathcal{I}_A \otimes \mathcal{N} \to 0$$

with $L\cdot(\mathcal{M}\otimes\mathcal{N}^{-1}) \ge 0$. By the proposition, $\mathcal{N} = \mathcal{O}_X(D)$ for an effective divisor D containing Z. Since $Z \ne \emptyset$, $D \ne 0$ so $D \not\approx 0$.

We compute Chern classes of $E = E(Z,L)$:

$$L = c_1(E) = c_1(\mathcal{M}) + c_1(\mathcal{I}_A\otimes\mathcal{N}) = c_1(\mathcal{M}) + D$$

which implies $c_1(\mathcal{M}) = L - D$, and

$$\begin{aligned} d &= \deg Z = c_2(E) = c_2(\mathcal{M}) + c_2(\mathcal{I}_A\otimes\mathcal{N}) + c_1(\mathcal{M})\cdot c_1(\mathcal{I}_A\otimes\mathcal{N}) \\ &= \deg(A) + c_1(\mathcal{M})\cdot c_1(\mathcal{N}). \end{aligned}$$

which implies $d = \deg(A) + (L-D)\cdot D$.

It remains to verify $(*)_d$. Since L is nef and D is effective, $L\cdot D \ge 0$. By the weak Bogomolov property, since $\mathcal{M}\otimes\mathcal{N}^{-1} = \mathcal{O}_X((L-D)-D)$ we have $L\cdot(L-2D)\ge 0$. And finally, by the computation of c_2,

$$d = \deg A + (L-D)\cdot D \ge (L-D)\cdot D.$$

Q.E.D.

As an application, we prove Reider's original theorem.

Reider's Theorem. *Let X be a smooth projective surface, and let L be a nef line bundle on X.*

(I) *If P is a base point of $|K_X+L|$ and $L^2 \ge 5$ then there is an effective divisor D containing P such that either*

(a) $L \cdot D = 0$, $D^2 = -1$
or
(b) $L \cdot D = 1$, $D^2 = 0$.

(II) *If P and Q are not base points of $|K_X + L|$, and P and Q are not separated by the map $\varphi_{|K_X+L|}$ (including the infinitely mear case in which the differential of $\varphi_{|K_X+L|}$ at P has a kernel in the direction corresponding to the infinitely near point Q), and if $L^2 \geq 9$, then there is an effective divisor D containing $P + Q$ such that either*
(a) $L \cdot D = 0$, $D^2 = -1$ *or* -2
(b) $L \cdot D = 1$, $D^2 = 0$ *or* -1
(c) $L \cdot D = 2$, $D^2 = 0$
or
(d) $L \cdot D = 3$, $D^2 = 1$, $L^2 = 9$, $L \approx 3D$.

Proof. (I) Let $Z = P$ and note that $H^0(\mathcal{O}_Z(K_X + L)) \cong \mathbb{C}$. In view of the exact sequence

$$0 \to H^0(\mathcal{I}_Z(K_X + L)) \to H^0(\mathcal{O}_X(K_X + L)) \to \mathbb{C} \to H^1(\mathcal{I}_Z(K_X + L)) \to 0$$

we have $h^1(\mathcal{I}_Z(K_X + L)) = 1$ if and only if P is a base point of $|K_X + L|$. Moreover,

$$c_1^2(E(Z, L)) = L^2 > 4 \deg(Z) = 4c_2(E(Z, L)),$$

so by Bogomolov's theorem, $E(Z, L)$ satisfies the Bogomolov property. The statement now follows from the previous theorem together with the list of solutions to $(*)_1$. (We omitted all cases with $L^2 \leq 4$.)

(II) This time, if $Z = P + Q$ we have $H^0(\mathcal{O}_Z(K_X + L)) \cong \mathbb{C}^2$. If neither P nor Q is a base point, the image of the map $H^0(\mathcal{O}_X(K_X + L)) \to H^0(\mathcal{O}_X(K_X + L))$ has dimension at least 1, and it has dimension exactly 1 if and only if P and Q are not separated by $\varphi_{|K_X+L|}$. In this case, $h^1(\mathcal{I}_Z(K_X + L)) = 1$ while for $Z' \subsetneqq Z$ (i. e. $Z' = P$ or $Z' = Q$), $h^1(\mathcal{I}_{Z'}(K_X + L)) = 0$ since neither is a base point.

In this case, $c_1^2(E(Z, L)) = L^2 > 4 \deg(Z) = 4c_2(E(Z, L))$ so the statement again follows from Bogomolov's theorem, the previous theorem, and the list of solutions to $(*)_2$ (with $L^2 \geq 9$). Q.E.D.

Corollary 1. *Let L be a nef line bundle on a smooth projective surface X such that $L^2 \geq 10$. If $\varphi_{|K_X+L|}$ is not birational, then there is a (possibly irrational) pencil $\{D_t\}$ such that $D_t^2 = 0$ and $L \cdot D_t = 1$ or 2.*

Proof. Assume first that the base locus of $|K_X + L|$ is a proper subvariety of X. If $\varphi_{|K_X+L|}$ is not birational, then there is some Zariski-open subset $U \subset X$ such that no point of U is a base point of $|K_X + L|$, but for every $P \in U$ there is some $Q \neq P$, $Q \in U$ which is not separated from P by the map $\varphi_{|K_X+L|}$. Thus, every point $P \in U$ must be contained in some curve D from cases (a), (b), or (c) of part (II) of Reider's theorem. Since each such D contains a 1-parameter family of points but U has dimension 2, there must be at least a 1-parameter family of such curves D. If we taken an irreducible component of this family of dimension ≥ 1, it cannot consist of curves D with $D^2 < 0$, since such curves do not move in algebraic families. Thus, there is a family of curves $\{D_t\}$ with parameter space of dimension ≥ 1 such that $D_t^2 = 0$ and $L \cdot D_t = 1$ or 2.

A similar argument in the case that the base locus of $|K_X + L|$ is all of X shows the existence of a family $\{D_t\}$ with $D_t^2 = 0$ and $L \cdot D_t = 1$ (from part (I) of Reider's theorem). Q.E.D.

Corollary 2.

(i) *If $-K_X$ is nef and big then $|-3K_X|$ is birational. (Del Pezzo surfaces)*

(ii) *if K_X is nef and big then $|5K_X|$ is birational. (Surfaces of general type)*

Proof. Take $L = \mp 4K_X$. Then $L^2 = 16K_X^2 \geq 16$. Furthermore, $L \cdot D$ is always divisible by 4, so the cases $D^2 = 0$, $L \cdot D = 1$ or 2 cannot occur. By Corollary 1, $|K_X + L|$ is birational. Q.E.D.

Remark. If we take $L = \mp 3K_X$ then $L^2 = 9K_X^2 \geq 9$. A similar argument shows: if in addition $K_X^2 > 1$, then $|-2K_X|$ resp. $|4K_X|$ is birational.

5.1. Addendum to section 5. There is some further information on generalized Del Pezzo surfaces which can easily be obtained from Reider's theorem.

Proposition. *Let X be a generalized Del Pezzo surface, that is, a surface for which $-K_X$ is nef and big, and let $m \geq 1$.*

(1) *If $|-mK_X|$ has a base point, then $m = 1$ and $K_X^2 = 1$.*

(2) *If there are two points P and Q which are not separated by $\varphi_{|-mK_X|}$ and which do not lie on smooth rational curves with self-intersection -2, then either $m = 1$, $K_X^2 \leq 2$ or $m = 2$, $K_X^2 = 1$.*

Proof. Let $L = -(m+1)K_X$, so that $|K_X + L| = |-mK_X|$.

(1) $L^2 \geq 5$ if and only if $m = 1$, $K_X^2 \geq 2$ or $m \geq 2$. In this case, if there is a base point then there is an effective divisor D with either $L \cdot D = 0$, $D^2 = -1$ or $L \cdot D = 1$, $D^2 = 0$. In the first case, $K_X \cdot D + D^2 = -1$ while in the second case, $K_X \cdot D + D^2 = -\frac{1}{m+1}$. In neither case can $K_X \cdot D + D^2$ be an even integer, so such a D cannot exist.

2) $L^2 \geq 10$ if and only if $m = 1$, $K_X^2 \geq 3$ or $m = 2$, $K_X^2 \geq 2$ or $m \geq 3$. In this case, if there exist such points P and Q which are not base points, then there is an effective divisor D with either one of the properties in (1) [which is impossible] or $L \cdot D = 1$, $D^2 = -1$ or $L \cdot D = 2$, $D^2 = 0$. We compute again: $K_X \cdot D + D^2 = \frac{-1}{m+1} - 1$ in the first case, and $= \frac{-2}{m+1}$ in the second case; again, neither can be an even integer. Q.E.D.

Before leaving the topic of Reider's method in general, we give a bit more information about the condition $h^1(\mathcal{I}_{Z'}(K_X+L)) < h^1(\mathcal{I}_Z(K_X+L))$ for $Z' \subsetneqq Z$ in a slightly special case.

Suppose that X is regular (so that $H^1(\mathcal{O}_X(K_X)) \cong H^1(\mathcal{O}_X)^* = 0$) and that there is a smooth curve $C \in |L|$ containing Z. Then we can make a big diagram of exact sheaf sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{I}_{C/X}(K_X+L) \arrow[r, equal] \arrow[d] & \mathcal{I}_{C/X}(K_X+L) \arrow[d] & & \\
0 \arrow[r] & \mathcal{I}_{Z/X}(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_X(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_Z(K_X+L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{I}_{Z/C}(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_C(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_Z(K_X+L) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Using the standard isomorphism $\mathcal{I}_{C/X}(K_X+L) \cong \mathcal{O}_X(K_X)$, $\mathcal{I}_{Z/C}(K_X+L) \cong \mathcal{O}_C(K_C-Z)$, and $\mathcal{O}_C(K_X+L) \cong \mathcal{O}_C(K_C)$, this becomes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_X(K_X) \arrow[r, equal] \arrow[d] & \mathcal{O}_X(K_X) \arrow[d] & & \\
0 \arrow[r] & \mathcal{I}_{Z/X}(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_X(K_X+L) \arrow[r] \arrow[d] & \mathcal{O}_Z(K_X+L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_C(K_C-Z) \arrow[r] \arrow[d] & \mathcal{O}_C(K_C) \arrow[r] \arrow[d] & \mathcal{O}_Z(K_C) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

This has the following interpretation: $H^1(\mathcal{I}_{Z/X}(K_X+L))$ measures the failure of the points Z to impose independent conditions on the linear system $|K_X+L|$. Now $|K_X+L|$ induces the canonical linear system $|K_C|$ on C, so we are measuring the linear dependence relations among the points Z in the canonical space $\mathbb{P}H^0(\mathcal{O}_C(K_C))$. The "geometric version of Riemann-Roch" (see Griffiths-Harris), relates this to $H^0(\mathcal{O}_C(Z))$. In our case, looking at the long exact sequence associated to the left vertical sequence we find (since $H^1(\mathcal{O}_X(K_X)) = 0$, $H^2(\mathcal{O}_X(K_X)) \cong$

$H^0(\mathcal{O}_X)^* \cong \mathbb{C}$, and $H^2(\mathcal{I}_{Z/X}(K_X + L)) \cong H^2(\mathcal{O}_X(K_X + L)) \cong H^0(\mathcal{O}_X(-L))^* = (0))$:

$$\begin{array}{ccccccccc} 0 & \to & H^1(\mathcal{I}_{Z/X}(K_X + L)) & \to & H^1(\mathcal{O}_C(K_C - Z)) & \to & \mathbb{C} & \to & 0 \\ & & & & \| \wr & & & & \\ & & & & H^0(\mathcal{O}_C(Z))^* & & & & \end{array}$$

In particular, if $r = h^1(\mathcal{I}_{Z/X}(K_X + L))$ then $h^0(\mathcal{O}_C(Z) = r + 1$ and $E(Z, L)$ has rank $r + 1$.

The condition $h^1(\mathcal{I}_{Z'/X}(K_X + L)) < h^1(\mathcal{I}_{Z/X}(K_X + L))$ for all $Z' \subsetneqq Z$ can now be interpreted in the following way: $h^0(\mathcal{O}_{Z'}(C)) < h^0(\mathcal{O}_Z(C))$. In other words, this condition means that the linear system $|Z|$ on X has no base points.

5.2. Second addendum to section 5. When the bundle $E(Z, L)$ comes from a base-point-free linear system $|Z|$ on a smooth curve $C \in |L|$, there is an easy way to guarantee that $E(Z, L)$ is generated by its global sections (when X is regular). Namely, $Z \subset C \subset X$ gives rise to the exact sequence

$$0 \to \mathcal{I}_{C/X}(L) \to \mathcal{I}_{Z/X}(L) \to \mathcal{I}_{Z/C}(L) \to 0$$

which can also be written as

$$(*) \qquad 0 \to \mathcal{O}_X \to \mathcal{I}_{Z/X}(L) \to \mathcal{O}_C(K_C - Z) \to 0.$$

Now the map $H^0(\mathcal{O}_X(E(Z, L)) \to H^0(\mathcal{I}_{Z/X}(L))$ is surjective since X is regular; thus, $(*)$ induces an extra section of $E(Z, L)$ and there is an exact sequence

$$0 \to (\mathcal{O}_X)^{\oplus r} \oplus \mathcal{O}_X \to \mathcal{O}_X(E(Z, L)) \to \mathcal{O}_C(K_C - Z) \to 0.$$

This sequence is exact on global sections; thus, $E(Z, L)$ is generated by global sections if $\mathcal{O}_C(K_C - Z)$ is. The latter happens whenever $|K_C - Z|$ has no base points. Thus, if both $|Z|$ and $|K_C - Z|$ have no base points, $\mathcal{O}_C(E(Z, L))$ is locally free and generated by global sections.

6. Linear systems on K3 surfaces

We will use Reider's method to investigate linear systems on K3 surfaces, but with one difference: instead of Bogomolov's theorem, we will find another technique for ensuring that the vector bundles $E(Z, L)$ have the weak Bogomolov property. To begin, we need to give the computation of $\chi(E \otimes E^*)$ due to Mukai and Lazarsfeld.

Computing Euler characteristics for vector bundles requires the Hirzebruch-Riemann-Roch theorem, which we now review. Let E be a vector bundle of rank m on a smooth projective variety X of dimension n, and let

$$c_t(E) = 1 + c_1(E)t + \cdots + c_n(E)t^n$$

be the Chern polynomial. There is a "splitting principle" for calculating Chern classes which states that any formula which can be proved under the assumption that E is a direct sum of line bundles in fact holds in general. If we pretend that

$E = L_1 \oplus \cdots \oplus L_m$ and write $c_t(L_j) = 1 + \lambda_j t$, then by the multiplicativity of $c_t(E)$ this splitting principle corresponds to a formal factorization

$$c_t(E) = \prod_{j=1}^{m} (1 + \lambda_j t) \mod t^{n+1}$$

and so $c_j(E)$ is the j^{th} elementary symmetric function in $\{\lambda_1, \ldots, \lambda_m\}$.

Let us work now in the graded ring $\bigoplus H^{2i}(X, \mathbb{Z})$, in which we denote an element by $[a_0, a_1, \ldots]$ and the i^{th} component by $[a_0, a_1, \ldots]_i = a_i$. (The grading is given by $\deg H^{2i}(X, \mathbb{Z}) = i$.) We may regard λ_j as an element of $H^2(X, \mathbb{Z})$, and e^{λ_j} as an element of our ring (via truncated power series): $[1, \lambda_j, \lambda_j^2/2!, \ldots]$. With these conventions, define the *Chern character of E* to be

$$\operatorname{ch}(E) = \sum_{j=1}^{m} e^{\lambda_j} \in \bigoplus H^{2i}(X, \mathbb{Z}).$$

This definition is to be interpreted in the non-split case as follows: it is symmetric in $\{\lambda_1, \ldots, \lambda_m\}$, and so can be written in terms of the elementary symmetric functions, and hence in terms of the Chern classes $c_i(E)$.

The first few terms in the Chern character are

$$\operatorname{ch}(E) = [\operatorname{rank}(E), c_1(E), \frac{c_1^2(E) - 2c_2(E)}{2}, \frac{c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)}{6}, \ldots].$$

Here are some properties of $\operatorname{ch}(E)$ which can easily be verified from the definition (and the splitting principle):

(1) $\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F)$ (More generally, ch is additive in exact sequences)
(2) $\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \cdot \operatorname{ch}(F)$
(3) $\operatorname{ch}(E^*)_i = (-1)^i \operatorname{ch}(E)_i$.

The Hirzebruch-Riemann-Roch theorem also involves the Todd class of X, another element of $\bigoplus H^{2i}(X, \mathbb{Z})$. This is a bit complicated to define, but it begins as

$$\operatorname{td}(X) = [1, \frac{c_1(X)}{2}, \frac{c_1^2(X) + c_2(X)}{12}, \frac{c_1(X)c_2(X)}{24}, \ldots]$$

where $c_i(X) = c_i(T_X)$ are the Chern classes of the tangent bundle. The Hirzebruch-Riemann-Roch theorem says:

$$\chi(E) = (\operatorname{ch}(E) \cdot \operatorname{td}(X))_n.$$

As an example, consider a line bundle L on a surface X. We have $c_t(L) = 1 + c_1(L)t$ and so $\operatorname{ch}(L) = [1, L, \frac{1}{2}L^2]$. In addition, for a surface we have $\operatorname{td}(X) = [1, -\frac{1}{2}K_X, X(\mathcal{O}_X)]$. Thus,

$$\chi(L) = \chi(\mathcal{O}_X) + -\frac{1}{2}K_X \cdot L + \frac{1}{2}L^2$$

giving the familiar formula.

We return to the case of a K3 surface X, and consider the vector bundle $E = E(Z,L)$ on X. We define $r+1 = \operatorname{rank} E$, $d = \deg Z$, and $2g-2 = L^2 = c_1^2(E)$. In particular, in the special case in which there is a smooth curve $C \in |L|$ containing Z, the linear system $|Z|$ on C satisfies: genus$(C) = g$, $\deg(\mathcal{O}_C(Z)) = d$, $h^0(\mathcal{O}_C(Z)) - 1 = r$. (It is also base-point-free, to get the vector bundle $E(Z,L)$.)

Now comes the miraculous computation of Mukai and Lazarsfeld:

$$\begin{aligned}
\operatorname{ch}(E) &= [r+1, c_1(E), \frac{c_1^2(E) - 2c_2(E)}{2}] \\
\operatorname{ch}(E^*) &= [r+1, -c_1(E), \frac{c_1^2(E) - 2c_2(E)}{2}] \\
\operatorname{ch}(E \otimes C^*) &= [(r+1)^2, 0, rc_1^2(E) - (2r+2)c_2(E)] \\
\operatorname{td}(X) &= [1, 0, 2]
\end{aligned}$$

since X is a K3 surface. Thus

$$\begin{aligned}
\chi(E \otimes E^*) &= 2(r+1)^2 + rc_1^2(E) - (2r+2)c_2(E) \\
&= 2(r+1)^2 + r(2g-2) - (2r+2)d \\
&= 2 - 2[-(r+1)^2 - (r+1)(g-1) + g + (r+1)d] \\
&= 2 - 2[g - (r+1)((r+1) + (g-1) + d)] \\
&= 2 - 2(g - (r+1)(r-d+g)).
\end{aligned}$$

The miracle is this: the number

$$\rho(g,r,d) = g - (r+1)(r-d+g)$$

is called the *Brill-Noether number*, and is very important in the theory of special linear systems on curves. As an example, we have:

Part of the Brill-Noether Theorem. *If C is a general curve of genus g, then every g_d^r in C (that is, a linear system with $h^0 = r+1$ and degree d) satisfies $\rho(g,r,d) \geq 0$.*

So the Mukai-Lazarsfeld computation says: $\chi(E \otimes E^*) = 2 - 2\rho(g,r,d)$, where $g = \frac{1}{2}c_1^2(E) + 1$, $r = \operatorname{rank}(E) - 1$ and $d = c_2(E)$. But even more is true: by Serre duality, since $K_X = 0$ we have

$$\begin{aligned}
H^2(E \otimes E^*) &\cong H^0((E \otimes E^*)^* \otimes K_X)^* \\
&\cong H^0(E^* \otimes E)^*
\end{aligned}$$

so that $h^2(E \otimes E^*) = h^0(E \otimes E^*)$. In particular

$$2h^0(E \otimes E^*) - h^1(E \otimes E^*) = 2 - 2\rho.$$

The conclusion is: if $\rho < 0$ then $h^0(E \otimes E^*) > 1$. (To restate in the case $Z \subset C$: if the linear system $\mathcal{O}_C(Z)$ has negative Brill-Noether number, then the vector bundle $E(Z, C)$ has an extra endomorphism.) It is this extra endomorphism which gives us the weak Bogomolov property.

Proposition. *Let E be a vector bundle of rank 2 on a smooth projective surface X. Suppose that $L = c_1(E)$ is nef, and that $h^0(E \otimes E^*) > 1$. Then E has the weak Bogomolov property.*

Proof. $H^0(E \otimes E^*) \cong \mathrm{Hom}(E, E)$; let $\varphi : E \to E$ be a (sheaf) homomorphism which is not a scalar multiple of the identity 1_E. (This exists since $\dim \mathrm{Hom}(E, E) > 1$.)

If we choose a point $x \in X$, and choose an eigenvalue λ of the fiber map $\varphi_x : E_x \to E_x$, then $\tilde{\varphi} = \varphi - \lambda \cdot 1_E$ has the property that $\mathrm{rank}(\tilde{\varphi})$ is less than 2 somewhere, while $\tilde{\varphi} \not\equiv 0$. We replace φ by $\tilde{\varphi}$, and assume this about φ.

Now $\mathrm{Ker}(\varphi)$ is a subsheaf of the locally free sheaf $\mathcal{O}_X(E)$, and so is torsion-free. Thus, it has a *rank*, and away from a set of codimension 2 on X it is locally free of that rank. [In fact, since $E/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi) \subset E$ is also torsion-free, $\mathrm{Ker}(\varphi)$ is reflexive and hence locally free.] Since

$$0 \subsetneqq \mathrm{Ker}(\varphi) \subsetneqq E$$

we see that $\mathrm{rank}(\mathrm{Ker}(\varphi)) = 1$; this implies that $\mathrm{Ker}(\varphi)$ is a line bundle.

We have naturally $\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\varphi^2) \subset E$, and $\mathrm{Ker}(\varphi^2)$ is locally free. Thus, either it has rank 1 and $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi^2)$, or it has rank 2 and $\mathrm{Ker}(\varphi^2) = E$.

If $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi^2)$, then I claim that $\varphi : \mathrm{Im}(\varphi) \to E$ is injective. For if $x \in \mathrm{Im}(\varphi)$ with $\varphi(x) = 0$ then $x = \varphi(y)$ for some y and thus $\varphi^2(y) = 0$. But then $\varphi(y) = 0$ so that $x = 0$, proving the claim. Thus, φ gives a splitting of the sequence

$$0 \to \mathrm{Ker}(\varphi) \to E \overset{\varphi}{\leftrightarrows} \mathrm{Im}(\varphi) \to 0$$

and we have $E = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$. If $L \cdot c_1(\mathrm{Ker}\,\varphi) \geq L \cdot c_1(\mathrm{Im}(\varphi))$ then

$$0 \to \mathrm{Ker}(\varphi) \to E \to \mathrm{Im}(\varphi) \to 0$$

provides a Bogomolov sequence; if $L \cdot c_1(\mathrm{Ker}\,\varphi) \leq L \cdot c_1(\mathrm{Im}\,\varphi)$ then

$$0 \to \mathrm{Im}(\varphi) \to E \to \mathrm{Ker}(\varphi) \to 0$$

is the desired sequence.

On the other hand, if $\mathrm{Ker}(\varphi^2) = E$ then $\mathrm{Im}\,\varphi \subset \mathrm{Ker}\,\varphi$ so there is a non-trivial section $\mathcal{O} \to \mathcal{O}(\mathrm{Ker}\,\varphi) \otimes \mathcal{O}(\mathrm{Im}\,\varphi)^*$. Since L is nef, this implies $L \cdot (\mathcal{O}(\mathrm{Ker}\,\varphi) \otimes \mathcal{O}(\mathrm{Im}\,\varphi)^*) \geq 0$. But then

$$0 \to \mathrm{Ker}\,\varphi \to E \to \mathrm{Im}\,\varphi \to 0$$

has the weak Bogomolov property. Q.E.D.

We now come to the first main theorem about linear systems on K3 surfaces.

Theorem. *Let X be a smooth projective K3 surface, and let L be a nef line bundle on X.*

(I) *If P is a base point of $|L|$ and $L^2 \geq 2$ then there is an effective divisor D containing P such that $L \cdot D = 1$, $D^2 = 0$.*

(II) *If P and Q are* not *base points of $|L|$, and P and Q are not separated by the map $\varphi_{|L|}$ (including the infinitely near case in which the differential of $\varphi_{|L|}$ at P has a kernel in the direction corresponding to the infinitely near point Q), and if $L^2 \geq 4$ then there is an effective divisor D containing $P + Q$ such that either*

 (a) $L \cdot D = 0$, $D^2 = -2$,

 (b) $L \cdot D = 1$ *or* 2, $D^2 = 0$, *or*

 (c) $L \cdot D = 4$, $D^2 = 2$, $L^2 = 8$, $L \approxeq 2D$.

Proof. As in the proof of Reider's theorem, the key step is to show that bundles $E(Z, L)$ satisfy the weak Bogomolov property when $\deg Z = 1$ (for part (I)), and $\deg Z = 2$ (for part (II)). (We have $Z = P$ and $Z = P + Q$, respectively.) We have $r + 1 = 2$ and for $\deg Z = d$:

$$\rho = g - 2(1 - d + g) = 2d - 2 - g.$$

Thus, when $g > 0$ in the case $d = 1$ (i. e. $L^2 > 0$ to have nef and big) or when $g > 2$ in case $d = 2$ (i. e. $L^2 \geq 4$) we have $\rho < 0$; by the "miraculous computation" of $\chi(E \otimes E^*)$ and the previous proposition, E satisfies the weak Bogomolov property. Thus, by the "Reider's method theorem", there is an effective divisor D containing Z satisfying $(*)_1$ resp. $(*)_2$. Noting that $D^2 \in 2\mathbb{Z}$ for a K3 surface, we get the solutions listed in the theorem. Q.E.D.

To complete the story of linear systems on K3 surfaces we need some converse statements.

Proposition 1. *Let X be a smooth projective K3 surface, let L be a nef and big line bundle on X, and suppose there is an effective divisor D such that $L \cdot D = 1$, $D^2 = 0$. Then $|L|$ has a fixed component.*

Proof. Consider the divisor $L - gD$, where $L^2 = 2g - 2$. We use the following:

Standard Trick. On a K3 surface, if Γ is a divisor with $\Gamma^2 \geq -2$ then Γ or $-\Gamma$ is effective. If $\Gamma^2 \geq -2$ and $L \cdot \Gamma > 0$ for some nef divisor L, then it is Γ which is effective.

Proof of the Trick.

$$\begin{aligned} H^2(\mathcal{O}_X(\Gamma)) &\cong H^0(\mathcal{O}_X(K_X - \Gamma))^* \\ &\cong H^0(\mathcal{O}_X(-\Gamma))^*. \end{aligned}$$

Thus, $h^0(\Gamma) + h^0(-\Gamma) \geq \chi(\Gamma) = \frac{1}{2}(\Gamma^2) + 2 \geq 1$ so either Γ or $-\Gamma$ is effective. If $L \cdot \Gamma > 0$ and L is nef then $-\Gamma$ cannot be effective. Q.E.D.

Resuming the proof of the proposition, we have

$$(L - gD)^2 = 2g - 2 - 2g = -2$$

$$L \cdot (L - gD) = 2g - 2 - g = g - 2.$$

If $g \geq 3$ then $L - gD$ is effective; if $g = 2$ then either $L - 2D$ or $2D - L$ is effective. In the latter case let $\widetilde{D} = L - D$. Then $L \cdot \widetilde{D} = 1$, $(\widetilde{D})^2 = 0$ and $L - 2\widetilde{D} = 2D - L$ is effective. Thus, replacing D by $\widetilde{D}$ if necessary, we may assume that $L - gD$ is effective.

But now $h^0(D) \geq 2$ implies $h^0(gD) \geq g + 1$ while $h^0(L) = g + 1$. Thus, $L - gD$ is a fixed component of $|L|$. Q.E.D.

Proposition 2. *Let $|L|$ be a nef and big linear system on a smooth K3 surface without base points. Suppose that there is an effective divisor D on X such that $L \cdot D = 2$ and $D^2 = 0$. Then $\varphi_{|L|}$ has degree 2; moreover, any smooth $C \in |L|$ is hyperelliptic.*

Proof. By Bertini's theorem, the general $C \in |L|$ is smooth. Now $\varphi_{|L|}|_C$ induces the canonical map $\varphi_{|K_C|}$ on C. On the other hand, the linear system $\mathcal{O}_C(D)$ has degree 2 and dimension 1, so that C is hyperelliptic. Thus, $\varphi_{|K_C|} = \varphi_{|L|}|_C$ has degree 2; since this is true for the general hyperplane section $\varphi_{|L|}$ itself has degree 2. Q.E.D.

In the course of proving the next proposition, we will need some facts about rational double points, whose proofs we omit.

Facts about rational double points. Let $C_1, \dots, C_n$ be a collection of smooth rational curves on a smooth surface X such that $C_i^2 = -2$ and $\cup C_i$ is connected, and suppose that the intersection matrix $(C_i \cdot C_j)$ is negative definite. Then there is a linear combination $C = \sum n_i C_i$ with $n_i \in \mathbb{Z}$, $n_i > 0$ such that $-C \cdot C_i \geq 0$ for all i, and $C^2 = -2$. Moreover, there is a contraction map $\pi : X \to \bar{X}$ such that $\pi(\cup C_i) = P$ is a point, and $\pi|_{X - \cup C_i} : X - \cup C_i \to \bar{X} - P$ is an isomorphism. $P \in \bar{X}$ is a rational double point.

Proposition 3. *Let X be a smooth K3 surface, let $|L|$ be a nef and big base-point-free linear system on X, and suppose there is an effective curve D such that $L \cdot D = 0$, $D^2 = -2$. Then every irreducible component D_i of D satisfies $L \cdot D_i = 0$, $D_i^2 = -2$.*

Moreover, if $C_1, \dots, C_n$ is a maximal connected set of irreducible curves such that $L \cdot D_i = 0$, $C_i^2 = -2$, then there is a contraction $\pi : X \to \bar{X} of \cup C_i$ to a rational double point, and the map $\varphi_{|L|}$ factors through π.

Proof. If we write $D = \sum n_i D_i$ with $n_i > 0$, then $0 = L \cdot D = \sum n_i L \cdot D_i$ and each $L \cdot D_i \geq 0$ implies $L \cdot D_i = 0$ for all i (since L is nef). By Hodge index, since $L \cdot D_i = 0$ we have $D_i^2 < 0$. But then $D_i^2 = -2$ and D_i is a smooth rational curve. Since

Hodge index implies that these curves have a negative-definite intersection matrix, the maximal connected components can be contracted to rational double points.

Now suppose that $C_1, \ldots, C_n$ is a maximal connected set of such curves. Also suppose $L^2 \geq 4$. Suppose that $C = \sum n_i C_i$ satisfies $C \cdot C_i \leq 0$ for all i, and $C^2 = -2$. Consider the linear system $|L - C|$. Note that $(L-C)^2 = L^2 - 2 \geq 0$ and $L \cdot (L - C) = 2g - 2$, so $L - C$ is effective.

Suppose that $L - C$ is not nef, and let Γ be an irreducible curve such that $(L - C) \cdot \Gamma < 0$. If $\Gamma^2 \geq 0$ then $|\Gamma|$ moves, so that $(L-C)\cdot\Gamma$ cannot be negative since $L - C$ is effective. Thus, $\Gamma^2 = -2$. We have $C \cdot \Gamma > L \cdot \Gamma \geq 0$, so that Γ cannot be a component of C, and must be connected to $\mathrm{Supp}(C)$. By our assumption about the maximality of C, it follows that $L \cdot \Gamma > 0$. If we let $x = L \cdot \Gamma$, $y = C \cdot \Gamma$ and $2g - 2 = L^2$ (so that $g \geq 3$) then $0 < x < y$ and the intersection matrix for L, C, Γ is:

$$\begin{bmatrix} 2g-2 & 0 & x \\ 0 & -2 & y \\ x & y & -2 \end{bmatrix}$$

By Hodge index, this must have determinant ≥ 0. Thus, (using also the relation $0 < x < y$):

$$0 \leq 2x^2 + (4 - y^2)(2g-2) < 2y^2 + (4 - g^2)(2g - 2)$$

which implies (since $g > 2$):

$$y^2 < 4\left(\frac{g-1}{g-2}\right) = 4\left(\frac{1}{g-2}\right) \leq 8.$$

It follows that $y = 2$ and $x = 1$. But then $(C + \Gamma)^2 = 0$ and $L \cdot (C + \Gamma) = 1$ so that $|L|$ has a fixed component, contrary to hypothesis.

Thus, $|L - C|$ is nef. But now by Riemann-Roch,

$$h^0(L - C) = \frac{1}{2}(L - C)^2 + 2 = \frac{1}{2}L^2 + 2 - 1 = h^0(L) - 1.$$

Since C imposes only 1 condition on $H^0(L)$, $\varphi_{|L|}(C)$ must be a point, so $\varphi_{|L|}$ factors through the contraction $\pi : X \to \bar{X}$ of C.

In the case $L^2 = 2$, $h^0(L - C) \geq \frac{1}{2}(L - C)^2 + 2 = 2$. Since C is not a fixed component of L, we have

$$2 \leq h^0(L - C) < h^0(L) = 3$$

so that $h^0(L - C) = h^0(L) - 1$ and the result follows as before. Q.E.D.

In sum, we have almost finished the proof of:

Theorem. [17] *Let L be a nef and big linear system on a K3 surface.*

[17] Reference for this theorem?

(1) $|L|$ *has base points if and only if there is a divisor* D *such that* $L \cdot D = 1$, $D^2 = 0$.
(2) *In the case of no base points:*
   (a) *If* $\pi : X \to \bar{X}$ *denotes the contraction of all effective curves* C *with* $L \cdot C = 0$, $C^2 = -2$ *to rational double points, then* $\varphi_{|L|}$ *factors through* π.
   (b) *The induced map* $\bar{\varphi} : \bar{X} \to \mathbb{P}^g$ *has degree 2 if and only if either* $L^2 = 2$, *or* $L \approx 2D$ *for a divisor* D *with* $D^2 = 2$, *or there is a divisor* D *with* $L \cdot D = 2$, $D^2 = 0$. *Otherwise,* $\bar{\varphi} : \bar{X} \to \mathbb{P}^g$ *is an embedding.*

Notice that in the degree 2 case, the image surface $\bar{\varphi}(\bar{X})_{\mathrm{red}}$ has degree $\frac{1}{2}(2g-2) = g-1$ in $\mathbb{P}^g$. It is thus a surface of minimal degree, and so is a scroll or the Veronese. (The Veronese case corresponds to $L \approx 2D$.)

Proof: The statement does not say "D is effective", but that follows in each case since $L \cdot D > 0$, $D^2 \geq -2$.

The only remaining thing to prove is that $\bar{\varphi}$ is an embedding when the degree is not 2. We omit the proof for infinitely near points (which is complicated for the rational double points), and simply show that $\bar{\varphi}$ separates distinct points.

Let $\bar{P}, \bar{Q} \in \bar{X}$ with $\bar{P} \neq \bar{Q}$ and suppose they are not separated by $\bar{\varphi}$. Choose $P \in \pi^{-1}(\bar{P})$, $Q \in \pi^{-1}(\bar{Q})$; then P and Q are not separated by $\varphi_{|L|}$. If $\bar{\varphi}$ (and φ) do not have degree 2, there is an effective divisor D containing P and Q with $L \cdot D = 0$, $D^2 = -2$. But since P and Q do not belong to the same maximal connected set of irreducible curves C_i with $L \cdot C_i = 0$, $C_i^2 = -2$, this is impossible: any such D would have $D^2 \leq -4$ (being supported on 2 different connected components). (This is another fact about rational double points.) "Q.E.D."

6.1. Addendum to section 6.

Corollary. *If* $|L|$ *is a nef and big linear system on a K3 surface, then* $|3L|$ *induces an embedding of* X *into projective space.*

Proof. If not, either $2 = (3L)^2 = 9L^2$ (impossible), or $3L \approx 2D$ with $D^2 = 2$ so that $L^2 = \frac{4}{9}D^2$ (impossible), *or* there is some D with $D^2 = 0$ and $3L \cdot D = 1$ or 2 (also impossible). Q.E.D.

7. The geometry of canonical curves and K3 surfaces

We now wish to discuss some of the connections between the geometry of K3 surfaces in projective space $\mathbb{P}^g$, and the geometry of their hyperplane sections, which are canonical curves. We begin by mentioning a theorem of Lazarsfeld, which can be partially proved with the tools we have developed.

Theorem (Lazarsfeld [13]**).** *Let X be a smooth K3 surface, and let L be a nef and big line bundle which generates the Picard group* $\mathrm{Pic}(X)$*. (It is easy to see that this implies that $|L|$ is base-point-free.) Then every smooth $C \in |L|$ satisfies the Brill-Noether property in the following form: for each line bundle $\mathcal{L}$ on C, $\rho(\mathcal{L}) \geq 0$.*

Partial Proof. We treat the case $h^0(\mathcal{L}) = 2$: suppose that $\mathcal{L}$ satisfies $h^0(\mathcal{L}) = 2$ and $\rho(\mathcal{L}) < 0$ and write $\mathcal{L} = \mathcal{O}_C(Z)$ for some effective divisor Z on C. The bundle $E(Z, L)$ has rank 2, and since $\rho(E(Z, L)) = \rho(\mathcal{L}) < 0$ it satisfies the weak Bogomolov property. Thus, there is an effective divisor D containing Z such that L and D satisfy $(*)_d$, where $d = \deg \mathcal{L}$. (Note that $Z \neq \emptyset$, so that $D \neq 0$.) By the assumption on $\mathrm{Pic}(X)$, $D \sim kL$ for some $k \in \mathbb{Z}$. But now $L \cdot D \geq 0$ implies $k \geq 0$ while $L \cdot (L - 2D) \geq 0$ implies $k \leq \frac{1}{2}$. We find that $k = 0$ and so $D = 0$, a contradiction. Q.E.D.

Thus, the general principle is that the existence of a g^1_d on C with $\rho < 0$ forces the existence of some divisor D on X in addition to C. We make this more explicit in the low degree cases.

Proposition. *Let L be a nef and big linear system without base points on a K3 surface.*

(1) *If there is a smooth $C \in |L|$ which is hyperelliptic, then $\varphi_{|L|}$ has degree 2. Conversely, if $\varphi_{|L|}$ has degree 2 then every smooth $C \in |L|$ is hyperelliptic.*
(2) *Suppose $L^2 = 8$ or $L^2 \geq 12$. If there is a smooth $C \in |L|$ which is trigonal (that is, which has a g^1_3) but not hyperelliptic, then there is an effective divisor D with $L \cdot D = 3$, $D^2 = 0$. Conversely, when such a divisor exists, every smooth $C \in |L|$ is trigonal.*
(3) *Suppose $L^2 = 12$, 14 or 18. If there is a smooth $C \in |L|$ which is tetragonal (that is, which has a g^1_4) but not hyperelliptic or trigonal, then there is an effective divisor D such that either $L \cdot D = 4$, $D^2 = 0$ or $L \cdot D = 6$, $D^2 = 2$. If $L^2 = 12$ or 14, the converse also holds: if there is a divisor of either kind, then every smooth $C \in |L|$ is tetragonal.*

(We will return to the case of $L^2 = 18$ and a g^1_4 on C a bit later.)

Proof. In each case, we are given a g^1_d on C, call it $|Z|$, with d the minimum possible value for such systems on C. Because it is the minimum, $|Z|$ has no base points. We may thus choose $Z \in |Z|$ consisting of distinct points.

Consider the bundle $E(Z, L)$: this has rank 2, and $\rho = g - 2(1 - d + g) = 2d - 2 - g$. Thus, for $g > 2d - 2$ (which is equivalent to $L^2 = 2g - 2 > 4d - 6$) we have $\rho < 0$. For $d = 2, 3, 4$ this is implied by $L^2 > 0$, $L^2 \geq 8$, $L^2 \geq 12$ respectively.

By the Reider method, there is an effective divisor D containing Z such that L and D satisfy $(*)_d$. Notice that $C \in |L|$ is irreducible and is not a component of D (for $D - C$ effective would imply

$$0 \leq C \cdot (D - C) = C \cdot D - C^2 \leq -C \cdot D \leq 0$$

(using $C^2 \geq 2C \cdot D$) and hence $C \cdot D = C^2 = 0$, a contradiction). Thus, $C \cdot D \geq \# \operatorname{Supp}(C \cap D) \geq \# \operatorname{Supp}(Z) = d$.

By induction on d, we can see that L and D do not satisfy $(*)_{d-1}$: this follows from the "converse statement" for $d-1$ in each case. (The "converse statement" for $d-1 = 1$ is the statement that $|L|$ has a fixed component when there is a D satisfying $(*)_1$.) Thus, we need solutions to $(*)_d$ under the additional conditions: $D \cdot (L-D) = d$, $L \cdot D \geq d$, $D^2 \in 2\mathbb{Z}$. We have done the case $d = 2$ before: the $d = 3$ and $d = 4$ cases are:

	Solution	Restrictions
$d=3$	$L \cdot D = 5,\ D^2 = 2$	$10 \leq L^2 \leq \frac{25}{2}$
	$L \cdot D = 3,\ D^2 = 0$	$6 \leq L^2$
$d=4$	$L \cdot D = 8,\ D^2 = 4$	$16 \leq L^2 \leq 16$
	$L \cdot D = 6,\ D^2 = 2$	$12 \leq L^2 \leq \frac{36}{2}$
	$L \cdot D = 4,\ D^2 = 0$	$8 \leq L^2$

To match the numerical statements given in to proposition, we need a lemma:

Lemma. *If $|L|$ is a base-point-free linear system on a K3 surface with $L^2 = 12$ and if there is a divisor D such that $L \cdot D = 5$, $D^2 = 2$ then smooth curves $C \in |L|$ are hyperelliptic.*

The proof of this is easy: $(L-2D)^2 = 0$ and $L \cdot (L-2D) = 2$ so $\varphi_{|L|}$ must have degree 2.

To finish the proof of the proposition, notice that the existence of a curve D with $D^2 = 0$, $L \cdot D = d$ implies that each smooth $C \in |L|$ has a g^1_d, namely $\mathcal{O}_C(D)$. The only remaining thing to check is the converse statement when $d = 4$, $D^2 = 2$, $L \cdot D = 6$. In that case, each smooth C has a g^2_6. But since $L^2 \neq 18$ (i. e. $g \neq 10$), the image of C under this g^2_6 cannot be a smooth plane sextic. The pencil residual to a singular point of the image is a g^1_4 (or smaller—but if there were something smaller, every $C \in |L|$ would be hyperelliptic or trigonal). Q.E.D.

In order to further study the trigonal case, we recall the Enriques-Babbage-Petri theorem.

Theorem (Enriques-Babbage-Petri; cf. ACGH). *A (non-hyperelliptic) canonical curve $C \subset \mathbb{P}^{g-1}$ is cut out by the quadrics containing it if and only if C is not trigonal and not a smooth plane quintic. In the case in which C is not cut out* *by quadrics, the quadrics containing C cut out a surface of minimal degree; in the trigonal case, this is a scroll ruled by the trisecant lines of C spanned by the g^1_3.*

The following theorem was originally proved by Saint-Donat using different techniques, and including the case $g = 6$ (in which the statement must be modified to include the smooth plane quintic hyperplane section case).

Theorem (Saint-Donat). *If a K3 surface $X \subset \mathbb{P}^g$, $g \geq 5$, is not cut out by quadrics and $g \neq 6$, then every smooth hyperplane section of X is trigonal, and there is a family of curves D of degree 3 with $D^2 = 0$ which cut out the trigonal series on the hyperplane sections. Moreover, the $\mathbb{P}^2$'s spanned by the plane cubics D sweep out a threefold scroll (of minimal degree), the base locus of the quadrics through X.*

Proof. Let $P \notin X$ be a point contained in all quadrics through X. A general hyperplane H through P meets X in a smooth curve C (by Bertini's theorem).

Since $H^1(\mathcal{O}_{\mathbb{P}^g}(1)) = 0$ and the natural map $H^0(\mathcal{O}_{\mathbb{P}^g}(1)) \to H^0(\mathcal{O}_X(1))$ is an isomorphism, it follows from the exact sequence

$$0 \to \mathcal{I}_X(1) \to \mathcal{O}_{\mathbb{P}^g}(1) \to \mathcal{O}_X(1) \to 0$$

that $H^0(\mathcal{I}_X(1)) = H^1(\mathcal{I}_X(1)) = 0$. Now from the exact sequence

$$0 \to \mathcal{I}_X(1) \to \mathcal{I}_X(2) \to \mathcal{I}_C(2) \to 0$$

it follows that $H^0(\mathcal{I}_X(2)) \cong H^0(\mathcal{I}_C(2))$. In particular, P is contained in all quadrics through C.

Thus, by the Enriques-Babbage-Petri theorem C is trigonal. By the previous proposition, there is a D with $L \cdot D = 3$, $D^2 = 0$ where $\mathcal{O}_X(L) = \mathcal{O}_X(1)$; every smooth $C \in |L|$ is then trigonal.

We sketch a proof of the "moreover" statement. The base locus of the quadrics through C is the intersection of H with the base locus of the quadrics through X. Since the former is a scroll swept by the trisecant lines, the latter is a scroll swept by the linear span of the D's (which cut C in the trisecant lines)—i. e., by the $\mathbb{P}^2$'s containing the plane cubics D. Q.E.D.

To ensure that you are not left with the wrong impression about special linear systems on hyperplane sections of K3 surfaces, I return to the example of $L^2 = 18$ $(g = 10)$ and g^1_4's. I need a lemma.

Lemma. *A smooth plane sextic curve has no g^1_4.*

Proof. We first show: any set of $k \leq 4$ distinct points $P_1, \ldots, P_k$ in $\mathbb{P}^2$ impose independent conditions on cubics. This is a fairly straightforward fact, but it is amusing to prove it as an application of Reider's method. If it were false, after re-ordering the points if necessary, upon blowing up $P_1, \ldots, P_{k-1}$ with a map $\pi : X \to \mathbb{P}^2$ the anti-canonical series $|-K_X|$ would have a base point at P_k. It is easy to check that $-K_X$ is nef (the worst case is three collinear points which prevents $-K_X$ from being ample). But since $K_X^2 \geq 6$, the application of Reider's theorem to generalized Del Pezzo surfaces given in section 5 implies that $|-K_X|$ can have no base point.

Now given a smooth plane sextic with a g_k^1, $k \leq 4$, base-point-free, we would have k distinct points *not* imposing independent conditions on $|K_C|$. (Choosing an element in the g_k^1 with distinct points.) Since $|K_C|$ coincides with $\mathcal{O}_{\mathbb{P}^2}(3)$ restricted to C, any such set of points must impose independent conditions, a contradiction. Q.E.D.

Example (due to Donagi and the author). Consider again the example we gave in the appendix to section 3 (on double covers): the double cover of $\mathbb{P}^2$ branched along a curve of degree 6 with map $\pi : X \to \mathbb{P}^2$. Define L by $\mathcal{O}_X(L) = \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$. We have $L \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1) = 6$ so that for a smooth $C \in |L|$, either $\pi(C)$ is a curve of degree 6 or π maps C onto a cubic curve and $\deg(\pi|_C) = 2$.

The computation we made earlier shows that both cases occur: on a linear space of codimension 1 in $|L|$, $\deg(\pi|C) = 2$ but the generic curve $C \in |L|$ is not the pullback of a curve in $\mathbb{P}^2$. Now the curves which are double covers of elliptic curves all carry a g_4^1, which they inherit from a g_2^1 on the elliptic curve. (In fact, there is a 1-parameter family of g_4^1's.) On the other hand, for general $C \in |L|$, $\pi(C)$ is a smooth plane sextic and so carries *no* g_4^1. We conclude from our proposition that there is a curve D such that $L \cdot D = 6$, $D^2 = 2$, so that every curve carries a g_6^2; something which is obvious from the geometry. (In fact, Hodge index implies $L \approx 3D$ so that the g_6^2's produced by the Reider method are *exactly* those induced by the map $\pi : X \to \mathbb{P}^2$.)

Quite a bit more is known about the connections between special linear systems on hyperplane sections of a fixed K3 surface than we have covered here. To briefly indicate what else is known, consider our basic setup: a linear system $|Z|$ on C with $\rho < 0$, the bundle $E(Z, L)$ (where $C \in |L|$) and the corresponding divisor D on X. Suppose for simplicity that $H^1(\mathcal{O}_X(D - L)) = 0$ so that the natural map $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_C(D))$ is an isomorphism. (It is easy to see that $D - L$ cannot be effective.) In this case, the basic inequality $D \cdot (C - D) \leq d$ which forms a part of $(*)_d$ can be interpreted as follows: $h^0(\mathcal{O}_C(D)) = \frac{1}{2}D^2 + 2$ and $\deg(\mathcal{O}_C(D)) = D \cdot C$ so that

$$d \geq D \cdot (C - D) = \deg(\mathcal{O}_C(D)) - 2h^0(\mathcal{O}_C(D)) + 4.$$

If we define for any line bundle $\mathcal{L}$ on C the *Clifford index*

$$\nu(\mathcal{L}) = \deg(\mathcal{L}) - 2(h^0(\mathcal{L}) - 1)$$

(which is "$d - 2r$" for a g_d^r) then this says:

$$\nu(\mathcal{O}_C(Z)) \geq \nu(\mathcal{O}_C(D)).$$

The *Clifford index of* C is defined to be:

$$\nu(C) = \min\{\nu(\mathcal{L}) \,|\, \mathcal{L} \in \mathrm{Pic}(C), h^0(\mathcal{L}) \geq 2, h^1(\mathcal{L}) \geq 2\}.$$

The theorems are then:

Theorem (Donagi-Morrison). *If C is a smooth curve on smooth K3 surface X with $C^2 > 0$ and $|Z|$ is a base-point-free g^1_d on C with $\rho < 0$ then there is a divisor D on X containing Z such that $\nu(\mathcal{O}_C(D)) \le \nu(\mathcal{O}_C(Z))$ and the function $C' \mapsto \nu(\mathcal{O}_{C'}(D))$ is constant for smooth $C' \in |C|$.*

Theorem (Green-Lazarsfeld). *If C is a smooth curve on a smooth K3 surface X with $C^2 > 0$ then $\nu(C') = \nu(C)$ for every smooth $C' \in |C|$. Moreover if $\nu(C) < \left[\frac{g-1}{2}\right]$ (which implies that linear systems at the minimum have $\rho < 0$) then there is a divisor D on X such that $\nu(C) = \nu(\mathcal{O}_C(D))$.*

For the proofs (which use techniques related to the ones we have discussed here). I refer you to the original papers (J. Diff. Geo. 1988 and Inventiones Math. 1987, resp.).

We now apply our investigations of the geometry of canonical curves and K3 surfaces to give characterizations of K3 surfaces of low degree, showing that they almost always coincide with the examples we have constructed.

Theorem. *Let L be a nef line bundle on a smooth K3 surface X. Let $\pi : X \to \bar{X}$ be the contraction of all irreducible curves C_i on X with $C_i^2 = -2$, $C_i \cdot L = 0$ to rational double points.*

1. *If $L^2 = 2$ and there does not exist a divisor D with $D^2 = 0$, $L \cdot D = 1$ then $\varphi_{|L|}$ induces a map $\bar{X} \to \mathbb{P}^2$ of degree 2, expressing $\bar{X}$ as the double cover of $\mathbb{P}^2$ branched on a curve with only simple singularities.*
2. *If $L^2 = 4$ and there does not exist a divisor D with $D^2 = 0$, $L \cdot D = 1$ or 2 then $\varphi_{|L|}$ embeds $\bar{X}$ as a quartic surface in $\mathbb{P}^3$ with only rational double points.*
3. *If $L^2 = 6$ and there does not exist a divisor D with $D^2 = 0$, $L \cdot D = 1$ or 2 then $\varphi_{|L|}$ embeds $\bar{X}$ as a generically transverse intersection of a quadric and a cubic in $\mathbb{P}^4$ with only rational double points.*
4. *If $L^2 = 8$ and there does not exist a divisor D with $D^2 = 0$, $L \cdot D = 1$, 2, or 3 then either $\varphi_{|L|}$ embeds $\bar{X}$ as a generically transverse intersection of three quadrics in $\mathbb{P}^5$ with only rational double points, or $L \approx 2D$ for some divisor D and $\varphi_{|L|}$ induces a map $\bar{X} \to V \subset \mathbb{P}^5$ of degree 2 from $\bar{X}$ to the Veronese V.*

Proof. First notice that in all cases we have assumed there is no D with $L \cdot D = 1$, $D^2 = 0$ so that $|L|$ has no base points. In addition, our previous results about when $\varphi_{|L|}$ has degree 2 agree with the statements made here. In fact, the only thing left to prove for (1) is the statement that the branch curve has simple singularities. But these are exactly the singularities producing only rational double points on the double cover $\bar{X}$.

In the case $L^2 = 4$, since $\varphi_{|L|}$ is an embedding (X is not hyperelliptic), it embeds $\bar{X}$ as a hypersurface in $\mathbb{P}^3$, which has degree $L^2 = 4$.

In the case $L^2 = 6$, consider the sequence

$$0 \to H^0(\mathcal{I}_{\bar{X}}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_{\bar{X}}(2)).$$

Since $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$ and $h^0(\mathcal{O}_{\bar{X}}(2)) = \frac{1}{2}(2L)^2 + 2 = 14$, we have $h^0(\mathcal{I}_{\bar{X}}(2)) \geq 1$. Let $q \in H^0(\mathcal{I}_{\bar{X}}(2))$ and let Q be the corresponding quadric, so that $\varphi_{|L|}(\bar{X}) \subset Q$. Note that Q is irreducible since $\varphi_{|L|}(\bar{X})$ is not contained in any hyperplane of $\mathbb{P}^4$.

If $x_0, \ldots, x_4$ denote coordinates in $\mathbb{P}^4$, then $x_i q \in H^0(\mathcal{I}_{\bar{X}}(3))$ for $i = 0, \ldots, 4$. On the other hand, since $h^0(\mathcal{O}_{\mathbb{P}^4}(3)) = 35$ and $h^0(\mathcal{O}_{\bar{X}}(3)) = \frac{1}{2}(3L)^2 + L = 29$ we have $h^0(\mathcal{I}_{\bar{X}}(3)) \geq 6$. Thus, there is some section $r \in H^0(\mathcal{I}_{\bar{X}}(3))$ whose associated cubic R does not contain Q. Then $R \cap Q$ is a surface of degree 6 containing $\varphi_{|L|}(\bar{X})$; since $\varphi_{|L|}(\bar{X})$ also has degree $L^2 = 6$ it follows that $\varphi_{|L|}(\bar{X}) = R \cap Q$ is a generically transverse intersection.

Finally, if $L^2 = 8$ and $\varphi_{|L|}$ is an embedding we have $h^0(\mathcal{O}_{\mathbb{P}^5}(2)) = 21$ while $h^0(\mathcal{O}_{\bar{X}}(2)) = \frac{1}{2}(2L)^2 + L = 18$ so that $h^0(\mathcal{I}_{\bar{X}}(2)) \geq 3$. By our assumptions (that there is no D with $L \cdot D = 3$, $D^2 = 0$), $\varphi_{|L|}(\bar{X})$ is cut out by the quadrics containing it. Thus, there are 3 elements Q_1, Q_2, Q_3 in the linear system $|\mathcal{I}_{\bar{X}}(2)|$ whose intersection is generically transverse so that $\dim(Q_1 \cap Q_2 \cap Q_3) = 2$. Since $\varphi_{|L|}(\bar{X}) \subset Q_1 \cap Q_2 \cap Q_3$ and $\deg \varphi_{|L|}(\bar{X}) = 8 = \deg(Q_1 \cap Q_2 \cap Q_3)$ it follows that $\varphi_{|L|}(\bar{X}) = Q_1 \cap Q_2 \cap Q_3$. Q.E.D.

Mukai has given some further characterizations using vector bundle techniques; we will describe his method, and do the case of $g = 8$ in detail.

A vector bundle E is *simple* if it has no endomorphisms other than scalar multiplies of the identity, that is, if $h^0(E \otimes E^*) = 1$. Mukai originally made the computation of $X(E \otimes E^*)$ for bundles on a K3 surface X because of the fact that the tangent space to the moduli space of vector bundles on X at the point $[E]$ can be naturally identified with $H^1(E \otimes E^*)$. Thus, in the simple case

$$X(E \otimes E^*) = 2 - h^1(E \otimes E^*) = 2 - 2\rho(g, r, d)$$

when $E = E(Z, L)$ for a g^r_d (namely Z) on a smooth $C \in |L|$ of genus g. It follows that

$$h^1(E \otimes E^*) = 2\rho(g, r, d).$$

We have used this computation previously to see that $E(Z, L)$ cannot be simple when $\rho < 0$. Now, however we consider the case of $\rho = 0$. The bundles in this case will be *rigid* (that is, will have no local deformations), and so form some kind of discrete invariants of the K3 surface X. It is natural to expect that the embeddings into such bundles will yield information about the geometry of X.

To apply this idea, we need to recall:

Another Part of the Brill-Noether Theorem (cf. ACGH). *Suppose that*

$$\rho(g, r, d) = g - (r + 1)(r - d + g) \geq 0$$

and that $r - d + g \geq 0$, $r \geq 0$. *Then every smooth curve of genus* g *has a line bundle* $\mathcal{L}$ *with* $\deg \mathcal{L} = d$ *and* $h^0(\mathcal{L}) \geq r + 1$.

Corollary. *Every smooth curve of genus 8 has a complete base-point-free* g^1_d *for some* $d \leq 5$.

Proof. For $g = 8$, $r = 1$, $d = 5$ we have $\rho = 0$; thus, there is a line bundle $\mathcal{L} = \mathcal{O}_C(Z)$ with $\deg Z = 5$, $h^0(\mathcal{O}_C(Z)) \geq 2$.

Suppose that $h^0(\mathcal{O}_C(Z)) = k+2$ with $k \geq 0$. Pick points $P_1, \ldots, P_k$ on C such that P_i is not a base point of $|Z - P_1 - \cdots - P_{i-1}|$. Then $h^0(\mathcal{O}_C(Z - P_1 - \cdots - P_k)) = 2$ and degree $(z - P_1 - \cdots - P_k) = 5 - k$. Now if $|Z - P_1 - \cdots - P_k|$ has as its base points $Q_1 + \cdots + Q_\ell$, we have $h^0(\mathcal{O}_C(Z - \sum P_i - \sum Q_j)) = 2$ and $\deg(Z - \sum P_i - \sum Q_j) = 5 - k - \ell \leq 5$. Q.E.D.

Theorem (A more precise version of a theorem of Mukai).
Let L *be a nef line bundle on a smooth K3 surface* X, *and let* $\pi : X \to \bar{X}$ *be the contraction of all irreducible curves* C_i *on* X *with* $C_i^2 = -2$, $C_i \cdot L = 0$ *to rational double points. Suppose that* $L^2 = 14$ *and there does not exist a divisor* D *with* $D^2 = 0$, $L \cdot D = 1, 2, 3$ *or 4 or* $D^2 = 2$, $L \cdot D = 6$. *Then* $\varphi_{|L|}$ *embeds* $\bar{X}$ *in* $\mathbb{P}^8 \cap \mathrm{Gr}(2,6)$ *of the Grassmannian* $\mathrm{Gr}(2,6)$ *in its Plucker embedding* $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$.

Proof. [18] Since there is no D with $D^2 = 0$, $L \cdot D = 1$ or 2, the linear system $|L|$ has no base points, and $\varphi_{|L|}$ defines an embedding of $\bar{X}$ into $\mathbb{P}^8$.

Let $C \in |L|$ be a smooth curve. By the corollary above, C has a linear system $|Z|$ which is a complete base-point-free g^1_d for some $d \leq 5$. Our assumptions about non-existence of divisors D imply that d cannot be less than 5, so $d = 5$. Thus, we get a vector bundle $E(Z, L)$ for which $\rho = 0$.

I claim that the linear system $|K_C - Z|$ on C has no base points. For if P were a base point of this system, then $|Z + P|$ would be a g^2_6 on C. Since C does not have genus 10, $\varphi_{|Z+P|}$ cannot be an embedding of C. But this means for some $Q \in C$, $|Z + P - 2Q|$ is a g^1_4. We do not have g^1_4's on C by our assumptions on the non-existence of divisors D.

It follows that $E(Z, L)$ is generated by its global sections. To see how many sections, we prove a

Lemma. *If* $|Z|$ *is a* g^r_d *on* $C \in |L|$ *of genus* g, *then*

$$\begin{aligned} h^0(\mathcal{O}_X(E(Z,L))) &= g + 1 - \nu(\mathcal{O}_C(Z)) \\ &= g + 1 - d + 2r. \end{aligned}$$

[18] Mukai's theorem covers the case of a generic K3 surface, and his proof is quite different.

Proof. The sequence

$$0 \to H^0((\mathcal{O}_X)^{\oplus r}) \to H^0(\mathcal{O}_X(E(Z,L))) \to H^0(\mathcal{I}_Z(L)) \to 0$$

is exact since X is regular. Now

$$\begin{aligned} h^0(\mathcal{I}_Z(L)) &= h^0(\mathcal{O}_X(L)) - h^0(\mathcal{O}_Z(L)) + h^1(\mathcal{I}_Z(L)) \\ &= g+1-d+r. \end{aligned}$$

So $h^0(\mathcal{O}_X(E(Z,L))) = r + h^0(\mathcal{I}_Z(L)) = g+1-d+2r$. Q.E.D.

In the case of the theorem we are proving, $g = 8$, $r = 1$ and $d = 5$ so that $h^0(E) = 6$.

Since $E = E(Z,L)$ is generated by its global sections, there is a regular map $\pi : X \to \mathrm{Gr}(2, H^0(E)^*) = \mathrm{Gr}(2,6)$ defined by

$$x \mapsto \{\varphi \in H^0(E)^* \,|\, \text{if } v \in H^0(E) \text{ vanishes at } x \text{ then } \varphi(v) = 0\}$$

(cf. Griffiths and Harris, p. 207).

Moreover, since $\mathcal{O}_X((\Lambda^2 E)^{**}) = \mathcal{O}_X(L)$, we have

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{|L|}} & \mathbb{P}(H^0(\mathcal{O}_X(L))^*) = \mathbb{P}(H^0(\bigwedge^2 E)^*) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(2, H^0(E)^*) & \xrightarrow{\mathrm{P}\ell} & \mathbb{P}(\bigwedge^2 H^0(E)^*) \end{array}$$

where $\mathrm{P}\ell$ is the Plucker embedding.

(The map between projective spaces is given by the dual of the natural map

$$\textstyle\bigwedge^2 H^0(E) \to H^0(\bigwedge^2 E).$$

In non-intrinsic terms, the diagram becomes

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{|L|}} & \mathbb{P}^8 \\ {\scriptstyle\varphi_E}\downarrow & & \downarrow \\ \mathrm{Gr}(2,6) & \xrightarrow{\mathrm{P}\ell} & \mathbb{P}^{14} \end{array}$$

where the map $\mathbb{P}^8 \to \mathbb{P}^{14}$ is the one referred to above.)

Now $\dim \mathrm{Gr}(2,6) = 8$ and $\mathrm{P}\ell$ maps it to $\mathbb{P}^{14}$; intersecting with the linear $\mathbb{P}^8 = \mathbb{P}(H^0(\mathcal{O}_X(L))^*)$ produces $\mathbb{P}^8 \cap \mathrm{Gr}(2,6)$ of codimension 6 in $\mathbb{P}^8$. That is, $\mathbb{P}^8 \cap \mathrm{Gr}(2,6)$ is a surface containing $\varphi_{|L|}(\bar{X})$. Since $\deg(\mathbb{P}^8 \cap \mathrm{Gr}(2,6)) = \deg \mathrm{Gr}(2,6) = 14$ and $\deg(\varphi_{|L|}(\bar{X})) = L^2 = 14$ it follows that $\varphi_{|L|}(\bar{X}) = \mathbb{P}^8 \cap \mathrm{Gr}(2,6)$. Q.E.D.

8. Kummer surfaces

In this section we will prove a theorem of Nikulin which says that a K3 surface is a Kummer surface if and only if it has 16 singular points of type A_1. The proof is rather combinatorial in nature, but the combinatorial analysis has a nice byproduct: with it, we will be able to prove a Torelli-type theorem for Kummer surfaces, which is an important step in proving the Torelli theorem for all K3 surfaces.

We start with a K3 surface X which has A_1 singularities at points $P_1, \ldots, P_k$ and is smooth elsewhere. Let $\pi : \widetilde{X} \to X$ be the minimal desingularization, let $E_i = \pi^{-1}(P_i)$, and let e_i be the class of E_i in $H^2(\widetilde{X}, \mathbb{Z})$. The combinatorial analysis is devoted to the following problem: for which subsets $J \subset \{1, \ldots, k\}$ is $\sum_{i \in J} E_i$ divisible by 2 in $\mathrm{Pic}(\widetilde{X})$? (By the Lefschetz (1,1) theorem, this is equivalent to asking when $\frac{1}{2}\sum_{i\in J} e_i \in H^2(\widetilde{X}, \mathbb{Z})$.) The final step in showing that X is a Kummer surface when $k = 16$ will be to use the divisibility by 2 of $\sum_{i=1}^{16} E_i$ to construct a double cover of $\widetilde{X}$ and thus recover the complex torus out of which the Kummer surface is constructed; our goal must therefore be to show that $\sum_{i=1}^{16} e_i$ is in fact divisible by 2.

Definition. Let L be a free $\mathbb{Z}$-module of finite rank equipped with a symmetric bilinear form $L \times L \to \mathbb{Z}$ (which we denote by $(x, y) \mapsto x \cdot y$). We define

$$L^{\#} = \{x \in L \otimes \mathbb{Q} \,|\, x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$$

and note the natural map $L^{\#} \to \mathrm{Hom}(L, \mathbb{Z})$ which sends x to the function $y \mapsto x \cdot y$. The form on L is *nondegenerate* if this map is an isomorphism, and in that case the cokernel $L^{\#}/L$ is called the *discriminant-group* of the form. (This is necessarily a finite group, since $L \subset L^{\#}$ and they become isomorphic after tensoring with $\mathbb{Q}$.)

As an example, let L be the $\mathbb{Z}$-span of $e_1, \ldots, e_k$ in $H^2(\widetilde{X}, \mathbb{Z})$, for our K3 surface X as above. Then $e_i \cdot e_j = -2\delta_{ij}$ so that $L^{\#}$ is generated by $\frac{1}{2}e_1, \ldots, \frac{1}{2}e_k$. It follows that $L^{\#}/L \cong (\mathbb{Z}/2\mathbb{Z})^k$, generated by $\{\frac{1}{2}e_i\}$.

In our example, if we augment L by including the elements $\frac{1}{2}\sum_{i\in J} e_i$ which are contained in $H^2(\widetilde{X}, \mathbb{Z})$ we get an inclusion $L \subset M$ of free $\mathbb{Z}$-modules of the same rank. This leads to:

Property 1 of Discriminant-Groups. *Let $L \subset M$ be an inclusion of free $\mathbb{Z}$-modules of the same (finite) rank, let M be equipped with a nondegenerate symmetric bilinear form, and consider $L \subset M \subset M^{\#} \subset L^{\#}$. Then*

$$|L^{\#}/L| = |M^{\#}/M| \cdot [M{:}L]^2,$$

where $|G|$ denotes the order of a finite group and $[G{:}H]$ denotes the index of H in G.

Proof. We have

$$|L^{\#}/L| = [L^{\#}{:}M^{\#}][M^{\#}{:}M][M{:}L]$$

so it suffices to show that $[L^{\#}{:}M^{\#}] = [M{:}L]$. Now

$$L^{\#}/M^{\#} \cong \operatorname{Hom}(L,\mathbb{Z})/\operatorname{Hom}(M,\mathbb{Z}),$$

so we must compute the order of this latter group.

There is an exact sequence

$$0 \to \operatorname{Hom}(M,\mathbb{Z}) \to \operatorname{Hom}(L,\mathbb{Z}) \xrightarrow{\delta} \operatorname{Hom}(M/L,\mathbb{Q}/\mathbb{Z}) \to 0$$

constructed as follows: pick an integer n such that $nM \subset L$, and define

$$\delta(\varphi)(x) = \frac{1}{n}\varphi(nx) \mod \mathbb{Z}$$

for $\varphi \in \operatorname{Hom}(L,\mathbb{Z})$, $x \in M \bmod L$. If $x \in L$ then $\frac{1}{n}\delta(nx) = \delta(x) \in \mathbb{Z}$, so that this is well-defined. It is easy to see that $\operatorname{Ker}(\delta) = \operatorname{Hom}(M,\mathbb{Z})$: if φ is in the kernel, then $\frac{1}{n}\varphi(nx) = \varphi(x) \in \mathbb{Z}$ for all $x \in M$, and conversely. To see that δ is surjective, pick a basis $e_1,\dots,e_r$ of M and for $\psi \in \operatorname{Hom}(M/L,\mathbb{Q}/\mathbb{Z})$ pick $g_i \in \mathbb{Q}$ such that $g_i \equiv \psi(e_i) \bmod \mathbb{Z}$ for all i. Then defining $\varphi \in \operatorname{Hom}(M,\mathbb{Q})$ by $\varphi(e_i) = g_i$, we see that φ is integer-valued on L, so $\varphi|_L \in \operatorname{Hom}(L,\mathbb{Z})$, and that $\delta(\varphi|_L) = \psi$.

Finally, to compute the order of $\operatorname{Hom}(M/L,\mathbb{Q}/\mathbb{Z})$: it is well known that this has the same order as M/L itself. To see this, it suffices to check it for a cyclic group $\mathbb{Z}/d\mathbb{Z}$; the homomorphisms $\mathbb{Z}/d\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ are classified by the image of 1, which must go to some element of $\frac{1}{d}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Q.E.D.

Consider now an inclusion $L \subset \Lambda$ of free $\mathbb{Z}$-modules with symmetric bilinear forms which do not necessarily have the same rank. (We assume that the form on Λ restricts to the form on L.) *The saturation of L in Λ* is defined to be $M = (L \otimes \mathbb{Q}) \cap \Lambda$; this has the same rank as L. If M is saturated, and the form on M is nondegenerate, then $N := M^{\perp}$ is also saturated and $M \oplus N \subset \Lambda$ is an inclusion of modules of the same rank.

A form Λ is *unimodular* if $\Lambda = \Lambda^{\#}$, or equivalently, if the map $\Lambda \to \operatorname{Hom}(\Lambda,\mathbb{Z})$ is an isomorphism.

Property 2 of Discriminant-Groups. *Let Λ be a free $\mathbb{Z}$-module with a unimodular form, let M be a saturated submodule on which the form is non-degenerate, and let $N = M^{\perp}$. Then there is a natural isomorphism $M^{\#}/M \xrightarrow{\cong} N^{\#}/N$.*

Proof. For a given $x \in M^{\#}$, there is an associated $\varphi_x \in \operatorname{Hom}(M,\mathbb{Z})$ defined by $\varphi_x(y) = x \cdot y$. Now M is saturated means that Λ/M is torsion-free, so every homomorphism $M \to \mathbb{Z}$ can be extended to a homorphism $\Lambda \to \mathbb{Z}$. Pick such an extension $\varphi \in \operatorname{Hom}(\Lambda,\mathbb{Z})$; since Λ is unimodular there is some $\lambda \in \Lambda$ corresponding to φ: we have $x \cdot y = \lambda \cdot y$ for all $y \in M$.

Now $\lambda - x \in N \otimes \mathbb{Q}$, and for all $z \in N$ we have $(\lambda - x)\cdot z = \lambda \cdot z \in \mathbb{Z}$. Sending $x \mapsto \lambda - x$ defines the homomorphism $M^{\#}/M \to N^{\#}/N$. It is easy to check that it is well-defined (i. e. does not depend on the choice of extension φ). To see that it is injective, suppose that $\lambda - x \in N$. Then $x \in \Lambda \cap (M \otimes \mathbb{Q})$ and so (since M is saturated) $x \in M$.

Finally, since we have $M^\#/M \hookrightarrow N^\#/N$, if we reverse the roles of M and N we get $N^\#/N \hookrightarrow M^\#$. This implies that the groups have the same order, and that the inclusions are isomorphisms. Q.E.D.

To return to our example of a K3 surface with A_1 singularities, the minimal resolution $\widetilde{X}$ has a cohomology group $H^2(\widetilde{X}, \mathbb{Z})$ with a symmetric bilinear form, which is *unimodular* by Poincaré duality. We have the submodule L of $\Lambda = H^2(\widetilde{X}, \mathbb{Z})$ generated by $e_1, \ldots, e_k$; its saturation $M = (L \otimes \mathbb{Q}) \cap \Lambda$; and the orthogonal complement $N = M^\perp$. $L^\#/L \cong (\mathbb{Z}/2\mathbb{Z})^k$ is naturally a vector space over $\mathbb{F}_2$, as is any subgroup of it.

Lemma. *Let $\alpha = \dim_{\mathbb{F}_2}(M/L)$. Then*

$$k - 2\alpha \leq 22 - k.$$

Proof. Since $|M/L| = 2^\alpha$ and $|L^\#/L| = 2^k$, it follows from property 1 that $|M^\#/M| = 2^{k-2\alpha}$. Since $M^\#/M$ is a sub-quotient of $L^\#/L \cong (\mathbb{Z}/2\mathbb{Z})^k$, it follows that $M^\#/M \cong (\mathbb{Z}/2\mathbb{Z})^{k-2\alpha}$.

Now by property 2, $N^\#/N \cong (\mathbb{Z}/2\mathbb{Z})^{k-2\alpha}$. On the other hand, the second Betti number of a smooth K3 surface is 22, which implies that N has rank $22 - k$ (since L and M have rank k). Thus, $N^\#$ and $N^\#/N$ can both be generated by $22 - k$ (or fewer) elements; since $(\mathbb{Z}/2\mathbb{Z})^{k-2\alpha}$ requires at least $k - 2\alpha$ elements to generate it, $k - 2\alpha \leq 22 - k$. Q.E.D.

Corollary. *If the K3 surface X has 16 singular points of type A_1, there is at least a five-dimensional $\mathbb{F}_2$-vector space of linear combinations $\sum_{i \in J} E_i$ which are divisible by 2 in the Picard group.*

(A nice way to think about this result is: the topology of the K3 surface, as represented by $H^2(\widetilde{X}, \mathbb{Z})$, forces the existence of certain double covers of $\widetilde{X}$ [and X], corresponding to the divisors divisible by 2.)

To proceed further, we need to investigate the *kinds* of double covers which can occur. One kind we already know: the double cover of a Kummer surface by a complex torus. To investigate the double covers in general, consider the following set-up: given a subset $J \subset \{1, \ldots, k\}$ such that $\frac{1}{2}{}_i \sum_{i \in J} e_i \in M$, let $X_J \to X$ be the resolution of the singular points *not* in J. If we form the double cover $\widetilde{Y}$ of $\widetilde{X}$ branched on $\sum_{i \in J} E_i$, we get a diagram

$$\begin{array}{ccccc} \widetilde{Y} & & & & \\ \big\downarrow \eta & & & & \\ \widetilde{X} & \to & X_J & \to & X. \end{array}$$

Let $D_j = \frac{1}{2}\eta^*(E_j)$ for $j \in J$; then

$$D_j^2 = \frac{1}{4}\eta^*(E_j)^2 = \frac{1}{4} \cdot 2(E_j)^2 = -1$$

so that D_j is an exceptional curve of the first kind. Blowing down all the D_j's for $j \in J$ with a map $\alpha : \widetilde{Y} \to Y$, it is easy to see that the induced rational map $Y \dashrightarrow X_J$ is in fact regular. So we get a bigger diagram:

$$\begin{array}{ccccc} \widetilde{Y} & \xrightarrow{\alpha} & Y & & \\ \downarrow{\scriptstyle \eta} & & \downarrow & & \\ \widetilde{X} & \to & X_J & \to & X. \end{array}$$

The inverse image of P_j in Y is the point $Q_j = \alpha(D_j)$, and the map $Y \to X_J$ is unramified away from the points P_j and Q_j ($j \in J$).

We need to compute some invariants of the surface Y. First, we have

$$K_{\widetilde{Y}} = \eta^*(K_{\widetilde{X}} + \frac{1}{2}\sum_{i \in J} E_i) = \sum_{i \in J} D_i$$

and

$$K_{\widetilde{Y}} = \alpha^*(K_Y) + \sum_{i \in J} D_i$$

which implies that $K_Y = 0$. Second, we can compute the topological Euler characteristic as follows (using the fact that for an unramified cover, the topological Euler characteristic multiplies by degree)

$$\begin{aligned} \chi_{\text{top}}(Y) &= \chi_{\text{top}}(Y - \bigcup_{i \in J} Q_i) + \#(J) \\ &= 2\chi_{\text{top}}(X_J - \bigcup_{i \in J} P_i) + \#(J) \\ &= 2\chi_{\text{top}}(\widetilde{X} - \bigcup_{i \in J} E_i) + \#(J) \\ &= 2(24 - 2 \cdot \#(J)) + \#(J) \\ &= 48 - 3 \cdot \#(J), \end{aligned}$$

since $\chi_{\text{top}}(\widetilde{X}) = 24$ (always the case for a smooth K3 surface).

Lemma (from classification of surfaces). *If $J \neq \emptyset$, then $\#(J) = 8$ or 16, with 16 points if and only if X is a Kummer surface and $X = X_J$.*

Proof: Suppose first that $\widetilde{Y}$ is Kähler (and so Y is Kähler). The only connected Kähler surfaces with $K_Y = 0$ are K3 surfaces (with $\chi_{\text{top}} = 24$) and complex tori (with $\chi_{\text{top}} = 0$). $\#(J) \neq 0$ implies that Y is connected, and the 2 cases correspond to $\#(J) = 8$ and $\#(J) = 16$ respectively. In the latter case, we must have $X = X_J$

because the exceptional curves of $X_J \to X$ lift to smooth rational curves on Y and there are no such curves on a complex torus.

To handle the non-Kähler case, there are several options. One can assume in the definition that X is Kähler in an appropriate sense; this has the disadvantage that the tools we are developing are used in the proof that every K3 surface is Kähler, so it is not a good idea to assume that fact here. Alternatively, there are arguments using deformation theory, or the analysis of the behavior of the signature under ramified covers, which can be used to eliminate the non-Kähler case. Both arguments are too far from our topics here, so we omit them. "Q.E.D."

We can construct an example of a cover with $\#(J) = 8$ in the following way. Let $T = \mathbb{C}^2/\Gamma$ be a complex torus and let $Y = T/\tilde{i}$ be its Kummer surface. (We retain the notation of Example Km from section 3.) Pick a point $t \in \frac{1}{2}\Gamma$ and let $\bar{\Gamma}$ be the group generated by Γ and t. Translation by t acts naturally on T, and the quotient by that action is another torus $\bar{T} = \mathbb{C}^2/\bar{\Gamma}$, which has a Kummer surface X.

$$\begin{array}{ccc} T & \to & \bar{T} \\ \downarrow & & \downarrow \\ Y & & X \end{array}$$

The translation by t descends to an automorphism of Y, because $z \mapsto -z \mapsto -z+t$ and $z \mapsto z+t \mapsto -z-t$ differ by an element $2t \in \Gamma$. Since X is the quotient of T by the group generated by $\tilde{i}$ and our translation, we get an induced map $Y \to X$.

Let us compute the fixed points of the action of the translation on Y. If the image of z is fixed, then either $z + t \equiv z \bmod \Gamma$ or $z + t \equiv -z \bmod \Gamma$. The first is clearly impossible, so we must have the second case: for some $\gamma \in \Gamma$, $z = \frac{1}{2}(-t+\gamma)$. Since $-z = \frac{1}{2}(-t + (2t - \gamma))$ has the same form, this set of 16 points on T descends to a set of 8 points on Y. The map $Y \to X$ is unramified away from those 8 points.

The set J on X over which $Y \to X$ ramifies is the image of $\{\frac{1}{2}(-t+\gamma) \bmod (\Gamma, \tilde{i})\}$ on X. Each such point satisfies $2(\frac{1}{2}(-t+\gamma)) = -t + \gamma \in \bar{\Gamma}$, so that the points all have order 2. There are 8 of them, and they form a subgroup which can naturally be identified with $\frac{1}{2}\Gamma/\Gamma + t \subset \frac{1}{2}\bar{\Gamma}/\bar{\Gamma}$. This is the set $J \subset \{$ points of order 2 on $\bar{T}\}$.

Conversely, if we had started with a hyperplane $J \subset \frac{1}{2}\bar{\Gamma}/\bar{\Gamma}$ we could construct the double cover as follows: $\bar{T} \to \bar{T}/J \to \bar{T}/\bar{T}_2 \cong \bar{T}$ where $\bar{T}_2$ is the subgroup of points of order 2. The map $\bar{T}/J \to \bar{T}$ is the quotient by the involution which is the non-trivial element in $\bar{T}_2/J \cong \mathbb{Z}/2\mathbb{Z}$ and repeating the construction above (with $\bar{T}/J$ in place of T) produces the cover $Y \to X$ branched on the points of J.

In order to efficiently use the fact that $\#(J)$ can only be 0, 8 or 16 we change our notation a bit and phrase our constructions in terms of *binary linear codes*.

Definition. Let I be a finite set, and let $\mathbb{F}_2^I$ denote the $\mathbb{F}_2$-vector space of maps from I to $\mathbb{F}_2$. A *binary linear code* is an $\mathbb{F}_2$-subspace $V \subset \mathbb{F}_2^I$.

In our situation of a K3 surface X with A_1 singularities at $P_1, \dots, P_k$, we let $I = \{1, \dots, k\}$ and use the isomorphism $L^\#/L \cong \mathbb{F}_2^I$ induced by sending $\frac{1}{2}e_i$ to the map φ_i such that $\varphi_i(j) = \delta_{ij}$ to produce the associated code $V \subset \mathbb{F}_2^I$ corresponding to $M/L \subset L^\#/L$. Explicitly, $\varphi \in V$ if and only if the set $J_\varphi = \{i \,|\, \varphi(i) = 1\}$ is one of our distinguished subsets: $\frac{1}{2}\sum_{i\in J_\varphi} E_i \in \operatorname{Pic}(\widetilde{X})$.

Another example of a binary linear code is the *universal binary linear code of dimension* α: for an $\mathbb{F}_2$-vector space W of dimension α, the space of linear maps $W^* = \operatorname{Hom}(W, \mathbb{F}_2)$ gives a code $W^* \subset \mathbb{F}_2^W$. A slight variant on this comes from noticing that $\varphi(0) = 0$ for any linear map, so that restricting linear maps to $W - \{0\}$ gives a code $W^* \subset \mathbb{F}_2^{W-\{0\}}$.

The code $W^* \subset \mathbb{F}_2^W$ is called "universal" because of the following construction. Let $V \subset \mathbb{F}_2^I$ be any code with $\dim V = \alpha$, let $W = \operatorname{Hom}(V, \mathbb{F}_2)$ and define a tautological map $\tau : I \to W$ by

$$\tau(i)(\varphi) = \varphi(i)$$

for $\varphi \in V$. Then $\tau^*(W^*) = V$; the only thing further which must be specified in order to describe $V \subset \mathbb{F}_2^I$ completely is the fibers of the map τ.

Theorem. *Let $V \subset \mathbb{F}_2^I$ be a binary linear code with $\alpha = \dim_{\mathbb{F}_2} V$ and $k = \#(I)$. Suppose that for every nonzero $\varphi \in V$, we have $\#\{i \,|\, \varphi(i) = 1\} = d$, a number independent of φ. Then $2^\alpha \,|\, 2d$, $k \geq \frac{d}{2^{\alpha-1}}(2^\alpha - 1)$, and for all nonzero $w \in \operatorname{Hom}(V, \mathbb{F}_2)$ we have $\#\{i \,|\, \tau(i) = w\} = \frac{d}{2^{\alpha-1}}$, where $\tau : I \to \operatorname{Hom}(V, \mathbb{F}_2)$ is the tautological map.*

Before giving the proof, we point out 2 applications to our situation.

Corollary 1. *A K3 surface with 16 A_1-singularities is a Kummer surface.*

Corollary 2. *The code of a K3 surface with 15 A_1-singularities is isomorphic to $W^* \subset \mathbb{F}_2^{W-\{0\}}$ for an $\mathbb{F}_2$-vector space W of dimension 4.*

Proof of Corollaries. Suppose that the K3 surface X with k singularities of type A_1 is not a Kummer surface. Then for each non-empty distinguished subset J we have $\#(J) = 8$. This means that the theorem applies with $d = 8$, and we find: $\alpha \leq 4$ and $k \geq 2^{4-\alpha}(2^\alpha - 1)$.

On the other hand, we know that $k - 2\alpha \leq 22 - k$, i. e., that $\alpha \geq k - 11$. Since $\alpha \leq 4$ we conclude that $k \leq 15$. (Hence if there are 16 points, X *must* be a Kummer surface.) If $k = 15$, we have $\frac{d}{2^{\alpha-1}} = 1$ so that each $\tau^{-1}(w)$ has cardinality 1 for $w \neq 0$; since $W - \{0\}$ contains only 15 points, the tautological map induces an isomorphism between the code of X and the code $W^* \subset \mathbb{F}_2^{W-\{0\}}$. Q.E.D.

Proof of the Theorem. For each $w \in W = \operatorname{Hom}(V, \mathbb{F}_2)$, define

$$a_w = \#\{i \,|\, \tau(i) = w\}.$$

Now for each $\varphi \in V$ we can write

$$\{i \mid \varphi(i) = 1\} = \bigcup_{w \mid w(\varphi)=1} \{i \mid \tau(i) = w\}.$$

Thus, if $\varphi \neq 0$ we have

$$\sum_{w \mid w(\varphi)=1} a_w = d$$

which implies

$$\sum_{w \mid w(\varphi)=0} a_w = k - d.$$

while if $\varphi = 0$ then

$$\sum_{w \mid w(\varphi)=1} a_w = 0$$

and

$$\sum_{w \mid w(\varphi)=0} a_w = k.$$

We can combine these formulas as

$$(*) \qquad \sum_{w \in W} (-1)^{w(\varphi)} a_w = \begin{cases} k - 2d & \text{if} \quad \varphi \neq 0 \\ k & \text{if} \quad \varphi = 0. \end{cases}$$

Define a matrix $A = (A_{w\varphi})_{\substack{w \in W \\ \varphi \in V}}$ by $A_{w\varphi} = (-1)^{w(\varphi)}$. (This is a $2^\alpha \times 2^\alpha$ matrix). A is a *Hadamard matrix*, that is, an $N \times N$ matrix whose entries are all ± 1 such that $AA^T = \operatorname{diag}(N, \ldots, N)$. To see this, we compute

$$\begin{aligned} (AA^T)_{wu} &= \sum_{\varphi \in V} A_{w\varphi} A_{u\varphi} \\ &= \sum_{\varphi \in V} (-1)^{w(\varphi)} (-1)^{u(\varphi)} \\ &= \sum_{\varphi \in V} (-1)^{(w+u)(\varphi)} \\ &= 2^\alpha \delta_{wu} \end{aligned}$$

(since $w + u$ is identically 0 if and only if $w = u$; otherwise, $w + u$ takes the values 0 and 1 equally often.)

Now we compute using (*): on the one hand,

$$\sum_{\varphi \in V} \sum_{w \in W} (-1)^{w(\varphi)} a_w A_{u\varphi} = \sum_{w \in W} a_w (AA^T)_{wu} = 2^\alpha a_u$$

while on the other hand,

$$\begin{aligned}\sum_{\varphi\in V}\sum_{w\in W}(-1)^{w(\varphi)}a_w A_{u\varphi} &= k(-1)^{u(0)}+\sum_{\varphi\in V,\varphi\neq 0}(k-2d)(-1)^{u(\varphi)}\\ &= 2d(-1)^{u(0)}+(k-2d)\sum_{\varphi\in V}(-1)^{u(\varphi)}\\ &= \begin{cases}2d & \text{if } u\neq 0\\ 2d+2^\alpha(k-2d) & \text{if } u=0.\end{cases}\end{aligned}$$

Thus, $a_u=\frac{2d}{2^\alpha}\in\mathbb{Z}$ for $u\neq 0$ while $a_0=k-\frac{d}{2^{\alpha-1}}(2^\alpha-1)\geq 0$ for $u=0$, from which the theorem follows. Q.E.D.

We can derive some further structure in the case of Kummer surfaces on the code by using this theorem.

Proposition. *Let $V\subset\mathbb{F}_2^I$ be the code associated to a K3 surface with 16 A_1 singularities. Then $\dim V=5$, and the set I has a natural structure of an affine space of dimension 4 over $\mathbb{F}_2$ determined by: $J\subset I$ is an affine hyperplane if and only if $J=\{i\,|\,\varphi(i)=1\}$ for some $\varphi\in V$ with $\varphi\not\equiv 0$, $\varphi\not\equiv 1$.*

Proof. Pick a point $i_0\in I$, and define

$$V_0=\{\varphi\in V\,|\,\varphi(i_0)=0\}.$$

Then V_0 has codimension (at most) 1 in V, and $V_0\subset\mathbb{F}_2^{I-\{i_0\}}$ is a code satisfying the hypotheses of the theorem with $d=8$, $k=15$. It follows that $\dim V_0=4$ and so that $\dim V=5$ (since $\dim V\geq 5$).

Now if $W=\operatorname{Hom}(V_0,\mathbb{F}_2)$, then the code $V_0\subset\mathbb{F}_2^{I-\{i_0\}}$ is isomorphic to the code $W^*\subset\mathbb{F}_2^{I-\{i_0\}}$. If we extend the isomorphism $I-\{i_0\}\cong W-\{0\}$ (from corollary 2 above) to an isomorphism $I\cong W$, then for each $\varphi\in V_0$, $\varphi\not\equiv 0$ we have that $\{i\,|\,\varphi(i)=1\}$ is the complement of a linear hyperplane in W (and so is an affine hyperplane). Moreover, every element in $V-V_0$ can be written in the form $\varphi+\varphi_1$ with $\varphi\in V_0$, where $\varphi_1(i)=1$ for all i. For such elements, if $\varphi\not\equiv 0$ (i. e. $\varphi+\varphi_1\not\equiv 1$) we have that $\{i\,|\,(\varphi+\varphi_1)(i)=1\}$ *is* a linear hyperplane in W (and so also an affine hyperplane). Thus, $I\cong W$ has the desired structure. Q.E.D.

An *automorphism* of a code $V\in\mathbb{F}_2^I$ is an isomorphism $\sigma: I\to I$ such that $\sigma^*(V)=V$.

Proposition. *If $V\subset\mathbb{F}_2^I$ is the code associated to a K3 surface with 16 A_1 singularities, then $\operatorname{Aut}(V\subset\mathbb{F}_2^I)\cong AGL(4,\mathbb{F}_2)$, the affine general linear group (which is generated by the general linear group, and by translations). In particular, the affine $\mathbb{F}_2$-space structure on I is uniquely determined.*

Proof. We identify I with $W = \mathrm{Hom}(V_0, \mathbb{F}_2)$ as in the previous proposition. Consider $\sigma : W \to W$ defined by $\sigma(w) = w + w_0$ (for some fixed $w_0 \in W$). Then for $\varphi \in V$, $\sigma^*(\varphi)(w) = \varphi(w + w_0) = \varphi(w) + \varphi(w_0)$, so that

$$\sigma^*(\varphi) = \begin{cases} \varphi & \text{if } \varphi(w_0) = 0, \\ \varphi + \varphi_1 & \text{if } \varphi(w_0) = 1. \end{cases}$$

In particular, $\sigma^*(V) = V$.

So the translations lie in $\mathrm{Aut}(V \subset \mathbb{F}_2^I)$. Given an arbitrary $\sigma \in \mathrm{Aut}(V \subset \mathbb{F}_2^I)$, by composing with a translation we may assume $\sigma(i_0) = i_0$ (i. e. $\varphi(0) = 0$ under the identification $I \cong W$). But then σ preserves V_0, and so $\sigma \in \mathrm{Aut}(V_0 \subset \mathbb{F}_2^{I-\{i_0\}}) = \mathrm{Aut}(W^* \subset \mathbb{F}_2^{W-\{0\}})$. Now σ induces a linear automorphism $\sigma^* : V_0 \to V_0$, and thus a linear automorphism $(\sigma^*)^* : \mathrm{Hom}(V_0, \mathbb{F}_2) \to \mathrm{Hom}(V_0, \mathbb{F}_2)$; since this latter space is $\cong W$, we get that $\sigma = \sigma^{**}$ is a linear map on W. Conversely, any linear map $\sigma \in GL(W)$ preserves the subspace $W^* \subset \mathbb{F}_2^W$ and so acts on the code. Q.E.D.

The description of the affine space structure we have given above is an abstract one, based solely on the code. Using however the fact that the K3 surface in question is actually a Kummer surface, we can give an alternate description of this structure.

Lemma. *let $T = \mathbb{C}^2/\Gamma$, let $T_2 = \frac{1}{2}\Gamma/\Gamma$, and let X be the Kummer surface of T. If we fix an origin on T, then T_2 has a natural $\mathbb{F}_2$-vector space structure; forgetting the choice of origin leads to an affine $\mathbb{F}_2$-space structure. Under the map $T_2 \cong \mathrm{Sing}(X)$ induced by the quotient map $\xi : T \to X$, this coincides with the structure determined by the code.*

Proof. What needs to be checked is that the subsets $J \subset I$ such that $\frac{1}{2}\sum_{i \in J} E_i \in \mathrm{Pic}(\widetilde{X})$ and $|J| = 8$ exactly correspond to the affine hyperplanes of T_2. But we already checked that the affine hyperplanes are among the subsets J (in our construction of an example of the cover branched on 8 points); since the number of such subsets is 30 (= the number of hyperplanes), these must be all of the subsets. Q.E.D.

As the final step in our analysis of Kummer surfaces, we consider again the basic diagram

$$\begin{array}{ccc} \widetilde{T} & \xrightarrow{\rho} & T \\ \big\downarrow{\scriptstyle\eta} & & \\ \widetilde{X} & \to & X \end{array}$$

which relates the surfaces, and consider the map

$$r = \eta_* \rho^* : H^2(T, \mathbb{Z}) \to H^2(\widetilde{X}, \mathbb{Z}).$$

If x and y are cohomology classes in $H^2(T,\mathbb{Z})$, then by the projection formula, $\eta^*(r(x))\cdot\eta^*(r(y)) = 2r(x)\cdot r(y)$. Thus,

$$\begin{aligned} r(x)\cdot r(y) &= \frac{1}{2}\eta^*(r(x))\cdot\eta^*(r(y)) \\ &= \frac{1}{2}\eta^*\eta_*(\rho^*(x))\cdot\eta^*\eta_*(\rho^*(y)) \\ &= \frac{1}{2}(2\rho^*(x))\cdot(2\rho^*(y)) \\ &= 2\rho^*(x)\cdot\rho^*(y) \\ &= 2x\cdot y. \end{aligned}$$

In other words, the map r induces an inclusion $H^2(T,\mathbb{Z})\hookrightarrow H^2(\widetilde{X},\mathbb{Z})$ which multiplies the intersection form by 2. Note that $\operatorname{Im} r\subset M^\perp$, and $\operatorname{rank} M^\perp = 22-16 = 6 = \operatorname{rank}\operatorname{Im} r$.

If $x_1,\dots,x_6$ is a basis for $H^2(T,\mathbb{Z})$, since $H^2(T,\mathbb{Z})$ is unimodular (i. e., is isomorphic to $H^2(T,\mathbb{Z})^\#$) it is easy to see that $(\operatorname{Im} r)^\#$ is generated by $\frac{1}{2}r(x_1),\dots,\frac{1}{2}r(x_6)$ and so that

$$(\operatorname{Im} r)^\#/(\operatorname{Im} r)\cong(\mathbb{Z}/2\mathbb{Z})^6.$$

Lemma. $M^\perp = (\operatorname{Im} r)$. *(That is, in our previous notation, $N=\operatorname{Im} r$.) In particular,* $\operatorname{Im} r$ *is saturated.*

Proof. Since N is the saturation of $\operatorname{Im} r$, we have

$$2^6 = |(\operatorname{Im} r)^\#/\operatorname{Im} r| = |N^\#/N|[N\colon\operatorname{Im} r]^2.$$

On the other hand, $N^\#/N\cong M^\#/M\cong(\mathbb{Z}/2\mathbb{Z})^{k-2\alpha}$ and for a Kummer surface $k=16$, $\alpha=5$. Thus, $|N^\#/N| = 2^6$ so that $[N\colon\operatorname{Im} r]=1$, i. e., $N=\operatorname{Im} r$. Q.E.D.

As a consequence of this lemma, r induces an isomorphism (which we also denote by r):

$$r: H^2(T,\mathbb{F}_2)\xrightarrow{\cong} N^\#/N.$$

There are several other groups isomorphic to these—we introduce names for the isomorphisms. First, let $T_2=\frac{1}{2}\Gamma/\Gamma$ be the set of points of order 2 on T (which we identify with $I\cong\operatorname{Sing}(X)$ in the natural way). There is then a isomorphism

$$s:\operatorname{Hom}(\Lambda^2T_2,\mathbb{F}_2)\xrightarrow{\cong} H^2(T,\mathbb{F}_2).$$

Second, property 2 of discriminant-groups gives us an isomorphism

$$q: N^\#/N\xrightarrow{\cong} M^\#/M$$

(since M and N are saturated). Finally, the identification $L^\#/L\cong\mathbb{F}_2^{T_2}$ which sends M/L to the code $V\subset\mathbb{F}_2^{T_2}$ induces an inclusion

$$p: M^\#/M\to\mathbb{F}_2^{T_2}/V.$$

The image of p is U/V, where U is the subspace of $\mathbb{F}_2^{T_2}$ corresponding to $M^\#/L \subset L^\#/L$.

We let $t = pqrs : \mathrm{Hom}(\Lambda^2 T_2, \mathbb{F}_2) \to \mathbb{F}_2^{T_2}/V$ be the composite map.

Proposition. *Let $\varphi, \psi \in \mathrm{Hom}(T_2, \mathbb{F}_2)$ be nonzero linear functions, and let $\chi_{\varphi\psi} \in \mathbb{F}_2^{T_2}$ be the (nonlinear) function defined by*

$$\chi_{\varphi\psi}(i) = \begin{cases} 1 & \text{if } \varphi(i) = \psi(i) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

($\chi_{\varphi\psi}$ is the characteristic function of the intersection of hyperplanes $\varphi = \psi = 0$.) Then

$$t(\varphi \wedge \psi) \equiv \chi_{\varphi\psi} \mod V.$$

Proof. By definition of p,

$$p\left(-\frac{1}{2} \sum_{i \,|\, \varphi(i)=\chi(i)=0} e_i\right) = \chi_{\varphi\psi} \mod V.$$

Thus, the definition of $q : N^\#/N \xrightarrow{\cong} M^\#/M$ shows that it suffices to prove that

$$rs(\varphi \wedge \psi) + -\frac{1}{2} \sum_{i \,|\, \varphi(i)=\chi(i)=0} e_i \in H^2(\widetilde{X}, \mathbb{Z})$$

i. e., is an integral class; in other words, that

$$2rs(\varphi \wedge \psi) - \sum_{i \,|\, \varphi(i)=\chi(i)=0} e_i$$

is divisible by 2 in $H^2(\widetilde{X}, \mathbb{Z})$. (Note that $rs(\varphi \wedge \psi) \in H^2(\widetilde{X}, \mathbb{Q})$!)

Now $\varphi, \psi \in \mathrm{Hom}(T_2, \mathbb{F}_2)$ are induced by some homomorphisms $\tilde{\varphi}, \tilde{\psi} \in \mathrm{Hom}(\Gamma, \mathbb{Z})$ (using $T_2 = \frac{1}{2}\Gamma/\Gamma \cong \Gamma/2\Gamma$), and the cohomology classes in $H^1(T, \mathbb{Z})$ are Poincaré dual to the (real) hypersurfaces $\tilde{\varphi} = 0$ and $\tilde{\psi} = 0$ respectively. Thus, the class of $\tilde{\varphi} \wedge \tilde{\psi}$ is dual to $\{\tilde{\varphi} = \tilde{\psi} = 0\}$; let x denote this class. Note that $rs(\varphi \wedge \psi) = \frac{1}{2}r(x)$. We have that x passes through the points $\{i \,|\, \varphi(i) = \psi(i) = 0\} \subset T_2$ (and only those points of T_2), so that the proper transform of x on $\widetilde{T}$ is the integral class

$$\rho^*(x) - \sum_{i \,|\, \varphi(i)=\chi(i)=0} D_i \in H^2(\tilde{T}, \mathbb{Z}).$$

But this proper transform is invariant under the involution on $\widetilde{T}$, so applying η_* yields a cohomology class which is divisible by 2 in $H^2(\widetilde{X}, \mathbb{Z})$. That is, 2 divides (in

$H^2(\widetilde{X}, \mathbb{Z})$) the class

$$\begin{aligned} \eta_*(\rho^*(x) - \sum_{i \mid \varphi(i)=\chi(i)=0} D_i) &= \eta_*\rho^*(x) - \sum_{i \mid \varphi(i)=\chi(i)=0} e_i \\ &= r(x) - \sum_{i \mid \varphi(i)=\chi(i)=0} e_i \\ &= 2rs(\varphi \wedge \psi) - \sum_{i \mid \varphi(i)=\chi(i)=0} e_i, \end{aligned}$$

as required. Q.E.D.

As a concluding remark about the combinatorics of Kummer surfaces, consider the case of a *projective* Kummer surface, say a quartic Kummer surface in $\mathbb{P}^3$. There is then a nef line bundle with a class $\lambda \in M^\perp$, $\lambda^2 = 4$. Let $\widetilde{M} = M \oplus \mathbb{Z}(\lambda)$; then $\widetilde{M}^\#/\widetilde{M} \cong (\mathbb{Z}/2\mathbb{Z})^6 \times \mathbb{Z}/4\mathbb{Z}$ while $\widetilde{M}^\perp$ has rank 5! There must thus be an additional class in $(\widetilde{M} \otimes \mathbb{Q}) \cap \Lambda$.

In fact, this class has the form $\frac{1}{2}(\lambda - \sum_{i=1}^6 e_i)$ where $e_1, \ldots, e_6$ have been chosen so that if e_6 is the origin of a vector space structure on T_2 then $e_1, \ldots, e_4$ form a basis and e_5“ = ”$e_1 + \cdots + e_4$ in this basis. There are 16 such classes, and each leads to a hyperplane in $\mathbb{P}^3$ passing through 6 of the nodes of the quartic, which is everywhere tangent to the quartic along a plane conic. Moreover, each of the 16 singular points lies in 6 of these planes, and each of the 16 planes contains 6 of the singular points. This famous “16-6 configuration” can be viewed in the photograph from Hudson's book.

9. The Torelli theorem for Kummer surfaces

A *Hodge structure of weight* n is a free $\mathbb{Z}$-module of finite rank $H_{\mathbb{Z}}$ together with a direct sum decomposition $H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p=0}^n H^{p,n-p}$ such that $\overline{H^{p,n-p}} = H^{n-p,p}$. The n^{th} cohomology group of an algebraic variety (or a Kähler manifold) has a natural Hodge structure of weight n, and the *Torelli problem* asks whether the Hodge structure determines the isomorphism type of the variety. Generally in order for the Torelli problem to have a positive answer, some extra structure (like a polarization of the Hodge structure) must be added.

The easiest case of a Torelli theorem is that of complex tori (weight 1 Hodge structures). If $T = \mathbb{C}^n/\Gamma$ is a complex torus, then we can identify $H_1(T, \mathbb{Z}) \cong \operatorname{Hom}(H^1(T, \mathbb{Z}), \mathbb{Z})$ and recover the Albanese variety as $(H^{0,1})^*/\operatorname{Hom}(H^1_{\mathbb{Z}}, \mathbb{Z})$. Since a torus is isomorphic to its Albanese, the Hodge structure determines the isomorphism type.

Since a Kummer surface is built out of a torus, it is reasonable to expect a similar phenomenon for Kummer surfaces. However, H^1 of a Kummer surface is trivial; we must use H^2, which contains $H^2(T, \mathbb{Z}) = \Lambda^2 H^1(T, \mathbb{Z})$ as a subgroup. The difficulties in the Torelli problem for Kummer surfaces come from the necessity of passing from an isomorphism $\Lambda^2 H^1(T) \to \Lambda^2 H^1(T')$ to an isomorphism $H^1(T) \to H^1(T')$.

In order to solve the Torelli problem, we must consider some extra structure on the cohomology; we describe this extra structure for an arbitrary smooth K3 surface X. First, notice that $H^{2,0}(X)$ is necessarily 1-dimensional: if we let ω be a nowhere-vanishing holomorphic 2-form and α be any other holomorphic 2-form, then $\frac{\alpha}{\omega}$ defines a global holomorphic function and is therefore constant. It follows that $\dim_{\mathbb{C}} H^{2,0}(X) = \dim_{\mathbb{C}} H^{0,2}(X) = 1$, and thus that $\dim_{\mathbb{C}} H^{1,1}(X) = 20$. (Recall that $b_2 = 22$).

Next, there are certain compatibilities between the intersection form and the Hodge structure which are guaranteed by the Hodge index theorem: we have $H^{p,q}(X) \perp H^{p',q'}(X)$ *unless* $p+p' = q+q' = 2$; moreover, the intersection form is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$, and has signature (1,19) on $H^{1,1}(X) \cap H^2(X, \mathbb{R})$. (This last statement is a more general version of the Hodge index theorem.) It follows that it has signature (3,19) overall.

Given an isomorphism $\Phi : X' \to X$ between 2 smooth Kähler K3 surfaces, the induced isomorphism $\Phi^* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ has several properties:

(1) Φ^* preserves the intersection form, i. e., $\Phi^*(x) \cdot \Phi^*(y) = x \cdot y$.
(2) Φ^* preserves the Hodge structure, i. e., $\Phi^*(H^{p,q}(X)) = H^{p,q}(X')$.
(3) Φ^* preserves the effective classes, i. e., $\Phi^*(\mathcal{E}(X)) = \mathcal{E}(X')$, where

$$\mathcal{E}(X) = \{\text{cohomology classes in } H^2(X, \mathbb{Z}) \text{ of effective divisors}\},$$

(4) for some Kähler class κ on X and some Kähler class κ' on X', $\Phi^*(\kappa) \cdot \kappa' > 0$.

(Here, a *Kähler class* is the cohomology class in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$ of a Kähler metric on X. Property (4) could have been stated: for every κ and every κ', but it is in fact enough to check it for 1 as we will see in the next section.)

Definition. Let X, X' be smooth Kähler surfaces. An *effective Hodge isometry* is an isomorphism $\varphi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ such that

(1) $\varphi(x) \cdot \varphi(y) = x \cdot y$ if $x, y \in H^2(X, \mathbb{Z})$
(2) $\varphi(H^{p,q}(X)) = H^{p,q}(X')$
(3) $\varphi(\mathcal{E}(X)) = \mathcal{E}(X')$
(4) for some Kähler classes κ on X and κ' on X', $\varphi(\kappa) \cdot \kappa' > 0$.

Remark. If X is the minimal desingularization of a Kummer surface, then X is Kähler. In fact, if $r : H^2(T, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the natural map from the cohomology of the torus, and κ is any Kähler class on T then $r(\kappa) + \varepsilon \sum_{i=1}^{16} e_i$ is a Kähler class on X for sufficiently small $\varepsilon > 0$.

Theorem (The Torelli theorem for Kummer surfaces). *Let X be the minimal desingularization of a Kummer surface, and let X' be a smooth Kähler K3 surface.*

If $\varphi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is an effective Hodge isometry, then there is an isomorphism $\Phi : X' \to X$ such that $\Phi^* = \varphi$.

Proof. There are several steps.

Step 1. We first check that X' is also the minimal desingularization of a Kummer surface. Let $e_1, \dots, e_{16}$ be the classes of the 16 disjoint smooth rational curves on X coming from the resolution of the Kummer surface's singularities. Since φ preserves effective classes, each $\varphi(e_i)$ is effective; we must check that it is irreducible and then it will follow that X' is also the desingularization of a Kummer surface. If we write $\varphi(e_i) = \sum f_{ij}$ where each f_{ij} is the class of an irreducible component of $\varphi(e_i)$ (with repetitions allowed), then $\varphi^{-1}(f_{ij})$ must be an effective curve on X and $e_i = \sum \varphi^{-1}(f_{ij})$. But the curve E_i does not move; hence there can be only one f_{ij} so that $\varphi(e_i)$ is irreducible.

Step 2. Now we have tori T and T' whose Kummer surfaces are X and X'. We use the notation of section 8. The map φ induces an isomorphism between the singular sets $\operatorname{Sing}(X) \cong T_2$ and $\operatorname{Sing}(X') \cong T_2'$ and an isomorphism $L^\#/L \stackrel{\cong}{\to} L'^\#/L'$ which sends M/L to M'/L' and $M^\#/L$ to $M'^\#/L'$. Thus, we get an isomorphism of codes $V \subset \mathbb{F}_2^{T_2} \cong V' \subset \mathbb{F}_2^{T_2'}$. The natural structures of affine spaces on T_2 and T_2' must be preserved by this isomorphism; hence, if we fix origins $i_0 \in T_2$ and $i_0' \in T_2'$ for T and T' such that $\varphi(e_{i_0}) = e_{i_0'}$ we have an induced isomorphism $\varphi_1 = H^1(T,\mathbb{F}_2) \to H^1(T',\mathbb{F}_2)$ (as $\mathbb{F}_2$-vector spaces).

Now we also have, by considering $N = M^\perp$ mapping to $\varphi(N) = N' = M'^\perp$, a natural isomorphism $\varphi_2 : H^2(T,\mathbb{Z}) \to H^2(T',\mathbb{Z})$. The compatibility condition between $N^\#/N$ and $M^\#/M$ then guarantees (since $M^\#/M$ is mapped to $M'^\#/M'$) that $\varphi_2 \equiv \varphi_1 \wedge \varphi_1 \mod 2$. (In fact, we only checked this for *reducible* elements in $H^2(T,\mathbb{F}_2) = \Lambda^2 \operatorname{Hom}(T_2,\mathbb{F}_2)$, but the reducible elements generate $\Lambda^2 \operatorname{Hom}(T_2,\mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space.)

Step 3. Consists of the following.

Proposition. *Let H and H' be two free $\mathbb{Z}$-modules of rank 4 on which an orientation has been chosen. (The orientations determine isomorphisms $\Lambda^4 H \cong \mathbb{Z}$ and $\Lambda^4 H' \cong \mathbb{Z}$ and thereby determine symmetric bilinear forms on $\Lambda^2 H$ resp. $\Lambda^2 H'$ by*

$$\Lambda^2 H \times \Lambda^2 H \to \Lambda^4 H \cong \mathbb{Z}$$

and similarly for H'.) Let $\psi : \Lambda^2 H \to \Lambda^2 H'$ be an isomorphism preserving this bilinear form. The following are equivalent:

(i) *There exists an isomorphism $\lambda : H \to H'$ such that $\psi = \pm\lambda \wedge \lambda$*
(ii) *There exists an isomorphism $\lambda : G \otimes \mathbb{F}_2 \to H' \otimes \mathbb{F}_2$ such that $\psi \equiv \lambda \wedge \lambda$ mod 2.*

A proof of this proposition can be found on pp. 103–105 of the seminar notes "Geometrie des surfaces K3 … " edited by Beauville et al., or on p. 138 of the book by Barth-Peters-Van de Ven.

To finish the proof, by steps 2 and 3 there is an isomorphism $\lambda : H^1(T,\mathbb{Z}) \to H^1(T',\mathbb{Z})$ such that $\lambda\wedge\lambda = \pm\varphi_2 : H^2(T,\mathbb{Z}) \to H^2(T',\mathbb{Z})$. If $\lambda\wedge\lambda = -\varphi_2$, let κ and κ' be Kähler classes on T and T'. Since λ is induced by an isomorphism $T' \overset{\cong}{\to} T$ we have $(\lambda\wedge\lambda)(\kappa)\cdot\kappa' > 0$. On the other hand,

$$\begin{aligned}(\lambda\wedge\lambda)(\kappa)\cdot\kappa' &= -\varphi_2(\kappa)\cdot\kappa' \\ &= -\varphi(\kappa+\varepsilon\sum e_i)\cdot(\kappa'+\varepsilon'\sum e_i) \\ &< 0.\end{aligned}$$

Since $\kappa+\varepsilon\sum e_i$, $\kappa'+\varepsilon\sum e_i'$ are Kähler classes on X and X'; this is a contradiction. Thus, $\lambda\wedge\lambda = \varphi_2$.

There is an isomorphism $\Lambda : T' \to T$ inducing λ; by composing with a translation we may assume $\Lambda(i_0') = i_0$. Since the $\mathbb{F}_2$-space structures on T_2, T_2' are preserved, we see that the induced isomorphism $\Phi : X' \to X$ between the Kummer surfaces satisfies $\Phi^* = \varphi$. Q.E.D.

10. Nef and ample bundles, Kähler classes, and the Weyl group

We consider in this section only smooth K3 surfaces X. Recall that every (nonzero) effective irreducible divisor D on X satisfies $D^2 \geq -2$, and if $D^2 \geq 0$ then D moves in a nontrivial linear system (i. e. $h^0(\mathcal{O}_X(D)) \geq 2$); in particular, $D^2 \geq 0$ implies that D is nef. Now every effective divisor can be written as a nonnegative linear combination of irreducible ones, so the effective divisors form a *cone* generated by the smooth rational curves (i. e. effective irreducibles with $D^2 = -2$) and the nef divisors.

Conversely, if $D^2 \geq -2$ then either D or $-D$ is effective. But even when $D^2 \geq 0$ it may fail to be nef, and even if $D^2 = -2$ it may fail to be irreducible. So the characterization of nef divisors and of smooth rational curves requires further work.

If X has an ample line bundle L, this bundle can be used to distinguish the effective divisors. For $L\cdot D > 0$ whenever D is effective, and thus the sign of $L\cdot D$ determines whether D or $-D$ is effective (when $D^2 \geq -2$). More generally, if X has a Kähler metric with cohomology class κ [something which certainly holds in the case of an ample bundle, whose class becomes κ] then again $\kappa\cdot D > 0$ for all effective divisors. Note that $\kappa^2 > 0$ (and $L^2 > 0$ in the ample line bundle case) so that this is also a necessary condition for a class to be Kähler. A first attempt at identifying the set of Kähler class is then to consider the set

$$(*)\quad \{x \in H^{1,1}(X)\cap H^2(X,\mathbb{R}) \,|\, x^2 > 0, x\cdot d > 0 \text{ for all cohomology classes } d \text{ of effective divisors}\}.$$

The Kähler classes certainly lie in this set; but as we will see below, for non-algebraic K3 surfaces one additional piece of information must be added.

Dually, we can hope to use the set (*) (or some refinement of it) to identify the nef divisors and the smooth rational curves: a nonzero divisor D with $D^2 \geq -2$ will be effective (we hope!) exactly when $D \cdot x > 0$ for some (and hence for every) x in the set (*).

For *algebraic* K3 surfaces, all of this works without further modification: this is guaranteed by the Nakai-Moishezon criterion for ampleness. In fact, that criterion (combined with the Lefschetz (1,1)-theorem) says exactly that

$$\begin{aligned}\{\text{classes of ample divisors on } X\} = \{&x \in H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \mid \\ &x^2 > 0, x \cdot d > 0 \text{ for all cohomology classes } d \text{ of effective divisors}\}.\end{aligned}$$

(The difference between the right hand side of this equation and (*) is that this time we have required the class to be *integral*, not just real.) The set of classes of nef divisors will simply be the closure of the set of classes of ample divisors.

To introduce the refinement I mentioned in the non-algebraic case, we need some notation: let $H^{1,1}_{\mathbb{R}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. The intersection form has signature (1,19) when restricted to this space, and this implies that the set

$$C(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \mid x^2 > 0\}$$

has 2 connected components. The picture is this:

$x \cdot x > 0$ (first component)

$x \cdot x = 0$

$x \cdot x < 0$ (second component)

($x \cdot x = 0$ is sometimes called the *light cone* in analogy with the theory of special relativity, where forms with this signature [or rather, the opposite signature to this] appear.) The two components are interchanged by the map $x \mapsto -x$.

There is a convenient characterization of the components of $C(X)$: x and y (both in $C(X)$) belong to the same component if and only if $x \cdot y > 0$. To see this, consider the line segment $tx + (1-t)y$ for $0 \leq t \leq 1$; since

$$(tx + (1-t)y)^2 = t^2x^2 + 2t(1-t)x \cdot y + (1-t)y^2 > 0$$

this never leaves the component. Moreover, if $x \cdot y < 0$ then x and $-y$ are in the same component so x and y cannot be.

Similarly if $x \in C(X)$, $y \in \overline{C(X)}$ (the closure), then y belongs to the closure of the component containing x if and only if $x \cdot y \geq 0$.

Now fix a Kähler class κ on X; any other Kähler class must satisfy $\kappa \cdot x > 0$, and so belongs to the same component of $C(X)$ as does κ. (This is because any convex combination $t\alpha_\kappa + (1-t)\alpha_x$ of Kähler forms α_κ and α_x with $0 \le t \le 1$ is again a Kähler form.) Furthermore, any class d of an effective divisor satisfies $\kappa \cdot d > 0$; for such classes d with $d^2 \ge 0$ we see that d belongs to the closure of the component of $C(X)$ containing κ: It automatically follows that $x \cdot d > 0$. So consider the set

$$(**) \quad \{x \in H^{1,1}_{\mathbb{R}}(x) \mid x^2 > 0, x \cdot \kappa > 0 \text{ and for all classes of irreducible effective divisors } d \text{ with } d^2 = -2 \text{ we have } x \cdot d > 0\}$$

This is a subset of the previous one; the only "new" condition that has been added is $x \cdot \kappa > 0$, and all previous conditions continue to hold.

To describe this set more efficiently, we define

$$\Delta(X) = \{\delta \in H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \mid \delta^2 = -2\}$$

$$\Delta^+(X) = \{\delta \in \Delta(X) \mid \delta \text{ is the class of an effective divisor}\}.$$

(So for each $\delta \in \Delta(X)$, either $\delta \in \Delta^+(X)$ or $-\delta \in \Delta^+(X)$.) We can now re-describe the set (**) as:

$$V^+(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \mid x^2 > 0, x \cdot \kappa > 0 \text{ and } x \cdot \delta > 0 \text{ for all } \delta \in \Delta^+(X)\}.$$

This set is independent of the choice of Kähler metric κ; it only depends on the component of $C(X)$ in which κ lies. The remainder of the section is devoted to studying properties of this set.

For $\delta \in \Delta(X)$, define the *reflection in* δ to be the mapping

$$s_\delta : x \mapsto x + (x \cdot \delta)\delta.$$

This acts on $H^2(X, \mathbb{Z})$, preserving the Hodge decomposition and the intersection form, since $(x + (x \cdot \delta)\delta)^2 = x^2$. (The Hodge decomposition is preserved since $\delta \in H^{1,1}$ and $H^{2,0} \oplus H^{0,2} \subset \delta^\perp$.) It therefore acts on $H^{1,1}_{\mathbb{R}}(x)$ as well. We define the *Weyl group of* X to be the group $W(X)$ generated by $\{s_\delta \mid \delta \in \Delta\}$; this can be regarded as a subgroup of $\mathrm{Aut}(H^2(X, \mathbb{Z}))$ or $\mathrm{Aut}(H^{1,1}_{\mathbb{R}}(X)) \cong O(1, 19)$, where the automorphisms in question are those preserving the intersection form. Note that the action of $W(X)$ (even of all of $O(1, 19)$) on $H^{1,1}_{\mathbb{R}}(X)$ preserves the subset $C(X)$.

Lemma 1. *$W(X)$ is a discrete group which acts properly discontinuously on $C(X)$.*

Proof. There is a $W(X)$-equivariant isomorphism

$$C(X) \cong C^1(X) \times \mathbb{R}^+$$

where $C^1(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \mid x^2 = 1\}$ and $\mathbb{R}^+$ denotes the positive reals. It suffices to show that the action on $C^1(X)$ is properly discontinuous. Now $\mathrm{Aut}(H^{1,1}_{\mathbb{R}}(X)) \cong O(1, 19)$ acts transitively on $C^1(X)$, and the stabilizer of a point is a compact group isomorphic to $O(19)$.

$W(X)$, being a subgroup of $\mathrm{Aut}(H^2(X,\mathbb{Z}))$, is discrete in $\mathrm{Aut}(H^2(X,\mathbb{R}))$ and hence also in the subgroup $\mathrm{Aut}(H^{1,1}_{\mathbb{R}}(X))$ in which it lies. Thus, the action of $W(X)$ on $O(1,19)$ is properly discontinuous, which implies that the induced action on $C^1(X) \cong O(1,19)/O(19)$ is also properly discontinuous. Q.E.D.

Lemma 2. *If a discrete group W acts properly discontinuously on a space C and if S is a subset of W then*

$$F = \bigcup_{s\in S} \{x \mid s(x) = x\}$$

is closed in C.

Proof. For $y \in X - F$ let $W_y = \{w \in W \mid w(y) = y\}$ be the stabilizer; we have $W_y \cap S = \emptyset$. Since the action is properly discontinuous, there is a neighborhood U of y such that $wU \cap U = \emptyset$ for all $w \in W - W_y$; in particular, for all $w \in S$. But then no point of U is fixed by any element of S, so $U \subset X - F$. Q.E.D.

Corollary. $\bigcup_{\delta\in\Delta(x)} \delta^{\perp}$ *is closed in $C(X)$.*

(Because $\delta^{\perp}$ is the fixed locus of the reflection S_δ.)

The hyperplanes $\delta^{\perp}$ are called the *walls* in $C(X)$, and the connected components of $C(X) - \bigcap_{\delta\in\Delta(x)} \delta^{\perp}$ are called the *chambers* of $C(X)$. Chambers are open subsets of $C(X)$ (and of $H^{1,1}_{\mathbb{R}}(X)$.).

Theorem (it's in Bourbaki ...). *The group $W(X) \times \{\pm 1\}$ acts transitively on the set of chambers of $C(X)$.*

The use of this theorem in the study of K3 surfaces is this: a chamber V is determined by the set $\Delta_V = \{\delta \in \Delta(x) \mid \delta \cdot x > 0 \text{ for } x \in V\}$ which says which side of each wall V lies on. ($V^+(X)$ is one of the chambers). If we happen to have an isomorphism of Hodge structures which does *not* preserve effective classes and Kähler classes (i. e. does not preserve V^+), then this group $W(X) \times \{\pm 1\}$ can be used to alter the isomorphism so that these things *are* preserved.

The theorem can also be used to derive conclusions about the *irreducible* classes in $\Delta^+(X)$: These are the ones whose walls actually meet the closure of $V^+(X)$. Unfortunately, we do not have sufficient time to explore this topic.

Proof of the Theorem. Since ± 1 interchanges the two connected components of $C(X)$ while $W(X)$ preserves them (because: $x \cdot s_\delta(x) = x^2 + (x\cdot\delta)(x\cdot\delta) > 0$), it suffices to check the transitivity of the action of $W(X)$ on the chambers in one of the components of $C(X)$.

Let $x, y \in C(x)$ such that $x \cdot y > 0$ and $x \cdot \delta \neq 0$, $y \cdot \delta \neq 0$ for all $\delta \in \Delta(X)$. We must show that for some $w \in W(X)$, $w(x)$ and y lie in the same connected component of $C(x) - \bigcup_{\delta\in\Delta(x)} \delta^{\perp}$. Let $\ell = x^2$.

For each $a \in \mathbb{R}$, the set

$$\{z \in C(x) \,|\, 0 \leq y \cdot z \leq a, z^2 = \ell\}$$

is compact. (Pictorially,

$z^2 = \ell, y \cdot z > 0$

$y \cdot z = a$

$y \cdot z = 0$

Since the action of $W(X)$ on $C(X)$ is properly discontinuous, it follows that

$$\{w \in W(x) \,|\, y \cdot w(x) \leq a\}$$

is a finite set. Note that $0 \leq y \cdot w(x)$ and $w(x)^2 = \ell$. Thus, the function $z \mapsto y \cdot z$ on the orbit Wx of x attains its minimum at a point $z_0 = w_0 x$. But then for all $\delta \in \Delta$ we have

$$y \cdot w_\delta(w_0 x) \geq y \cdot w_0 x$$

i. e.

$$y \cdot (z_0 + (\delta \cdot z_0)\delta) \geq y \cdot z_0$$

so that

$$(\delta \cdot z_0)(y \cdot \delta) \geq 0.$$

Thus, z_0 and y are on the same side of every wall ($(\delta \cdot z_0) > 0$ if and only if $(y \cdot \delta) > 0$) and so belong to the same connected component of $C(X) - \bigcup_{\delta \in \Delta(x)} \delta^\perp$. Q.E.D.

10.1. Addendum to section 10. Two things should be noticed about the application of this result to K3 surfaces: (1) the set

$$V^+(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \,|\, x^2 > 0, x \cdot \kappa > 0, x \cdot \delta > 0 \text{ for all } \delta \in \Delta^+(X)\}$$

is a chamber (independent of the choice of Kähler class κ), and (2) a map $\varphi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfies $\varphi(V^+(X)) = V^+(X')$ if and only if it satisfies both (a) $\varphi(\mathcal{E}(X)) = \mathcal{E}(X')$ and (b) $\varphi(\kappa) \cdot \kappa' > 0$ for some Kähler classes κ and κ'. In particular, the definition of "effective Hodge isometry" can be reformulated as: preserves Hodge structures and intersection forms, and maps $V^+(X)$ isomorphically to $V^+(X')$.

11. The period mapping for K3 surfaces

Recall two key properties of the intersection form on the second cohomology group of a K3 surface.

Property 1. If X is a K3 surface, the intersection form on $H^2(X,\mathbb{Z})$ is unimodular, by Poincaré duality. (We first encountered this property in section 8.)

Property 2. If X is a K3 surface, the intersection form on $H^2(X,\mathbb{Z})$ has signature (3,19), by the extended version of the Hodge index theorem. (We first encountered this property in section 9.)

There are two additional key facts about K3 surfaces which we have not yet mentioned.

Topological Fact. Let X be a (smooth) compact complex surface. Suppose that $H^1(X,\mathbb{Z}/2\mathbb{Z})=0$. Then for all $\gamma\in H^2(X,\mathbb{Z})$, $\gamma\cdot\gamma - c_1(X)\cdot\gamma\equiv 0 \mod 2$. (This is clear for algebraic classes, i. e. those coming from divisors D: $D\cdot D+K_X\cdot D = 2g(D)-2$ must be *even*. The proof in general uses the Wu formula and Stiefel-Whitney classes; a good reference is the book of Milnor and Stasheff.) In particular, for a smooth K3 surface X we have $\gamma\cdot\gamma\equiv 0 \mod 2$ for all $\gamma\in H^2(X,\mathbb{Z})$.

Number-Theoretic Fact. There is a *unique* (up to isomorphism) free $\mathbb{Z}$-module Λ with a unimodular symmetric bilinear form $\Lambda\times\Lambda\to\mathbb{Z}$ of signature (3,19) such that $\gamma\cdot\gamma\equiv 0 \mod 2$ for all $\gamma\in\Lambda$. This form Λ is isomorphic to $(-E_8)^{\oplus 2}\oplus U^{\oplus 3}$ where E_8 is the unimodular even positive definite form of rank 8, and U is the hyperbolic plane. We call Λ the *K3 lattice*. (A good reference for this is Serre's *Cours d'Arithmétique*.) In particular, for every smooth K3 surface there exists an isomorphism $\alpha: H^2(X,\mathbb{Z})\to\Lambda$ preserving the bilinear forms; a choice of such an isomorphism is called a *marking* of X.

By using a marking $\alpha: H^2(X,\mathbb{Z})\to\Lambda$, we get a Hodge structure on Λ; we want to describe the set of all Hodge structures of the appropriate type.

More generally, suppose we have a free $\mathbb{Z}$-module L with bilinear form of signature $(2h^{2,0}+1, h^{1,1}-1)$, and we want to consider Hodge decompositions $L\otimes\mathbb{C}=H^{2,0}\oplus H^{1,1}\oplus H^{0,2}$ such that $H^{p,q}\perp H^{p',q'}$ unless $p+p'=q+q'=2$, and such that the form is positive definite on $(H^{2,0}\oplus H^{0,2})\cap(L\otimes\mathbb{R})$, and has signature $(1,h^{1,1}-1)$ on $H^{1,1}\cap(L\otimes\mathbb{R})$. Such a Hodge structure is completely specified by $H^{2,0}$, since $H^{0,2}=\overline{H^{2,0}}$ and $H^{1,1}=(H^{2,0}\oplus H^{0,2})^{\perp}$. In fact, the natural parameter space for all such Hodge structures is the space

$$\Omega_L=\{\mu\in \mathrm{Gr}(h^{2,0},L\otimes\mathbb{C})\,|\,\mu \text{ is totally isotropic for the bilinear form,}$$
$$\text{and for each } x\in\mu,\ x\neq 0 \text{ we have } x\cdot\bar{x}>0\}$$

($H^{2,0}=\mu$, $H^{0,2}=\bar{\mu}$, $H^{1,1}=(\mu\oplus\bar{\mu})^{\perp}$ is the Hodge structure).

Now it is not difficult to see that the tangent space to $\operatorname{Gr}(h^{2,0}, L\otimes\mathbb{C})$ at μ is given by

$$T_\mu \operatorname{Gr}(h^{2,0}, L\otimes\mathbb{C}) \cong \operatorname{Hom}(\mu, (L\otimes\mathbb{C})/\mu)$$

in a natural way. Slightly more difficult is the isomorphism

$$T_\mu\Omega_L \cong \operatorname{Hom}(\mu, \mu^\perp/\mu).$$

In fact, since $\mu^\perp = \mu \oplus H^{1,1}$ we have

$$T_\mu\Omega_L \cong \operatorname{Hom}(H^{2,0}, H^{1,1}).$$

Returning to the K3 lattice Λ, we let $\Omega = \Omega_\Lambda$; in this case, the Grassmannian is a projective space and we can write

$$\Omega = \{[\omega] \in \mathbb{P}(\Lambda\otimes\mathbb{C}) \,|\, \omega\cdot\omega = 0, \omega\cdot\bar\omega > 0\}$$

with associated Hodge structure

$$H^{2,0} = \mathbb{C}(\omega), H^{0,2} = \mathbb{C}(\bar\omega), H^{1,1} = <\omega,\bar\omega>^\perp.$$

Thus we see that Ω is an open subset in a quadric in $\mathbb{P}^{21}$.

The next topic to be discussed is deformation theory; for lack of time, I refer you to the chapter by Gauduchon "Théorème de Torelli locale pour les surfaces K3" in the seminar notes "Géometrie des surfaces K3 ... " edited by Beauville et al.

For a smooth K3 surface X, the sheaf of holomorphic vector fields Θ_X is naturally isomorphic to the sheaf of holomorphic 1-forms Ω^1_X via contraction with the nowhere vanishing holomorphic 2-form ω. Thus, $H^2(\Theta_X) = H^2(\Omega^1_X) = 0$; it follows that there is a smooth local universal deformation of X: this is a map $\pi : \mathcal{X} \to S$ with smooth fibers such that $\pi^{-1}(0) = X$ and the Kodaira-Spencer map is an isomorphism. (The Koraira-Spencer map sends T_0S to $H^1(\Theta)$.)

We choose a marking $\alpha : H^2(X,\mathbb{Z}) \to \Lambda$, and extend this to a continuous family of markings $\alpha_s : H^2(\pi^{-1}(s),\mathbb{Z}) \to \Lambda$ by using the differentiable triviality of the family $\mathcal{X} \to S$. (That is, there is a C^∞ isomorphism $\mathcal{X} \cong_{C^\infty} X\times S$.) We then get the *period mapping* of the family $S \to \Omega$ which sends s to the Hodge structure on Λ given by $\alpha(H^2(\pi^{-1}(S),\mathbb{Z})$.

Now we have

$$\begin{tikzcd} T_0S \arrow[rr, "\text{differential}", "\text{of period map}"'] \arrow[d, "\text{Kodaira-Spencer}"', "\cong"] & & T_\mu\Omega \arrow[d] \\ H^1(\Theta_X) & & \operatorname{Hom}(H^{2,0}(X), H^{1,1}(X)). \end{tikzcd}$$

Theorem (The local Torelli theorem for K3 surfaces). *The differential of the period map $S \to \Omega$ is an isomorphism.*

The proof is based on the more general fact that in the diagram above, the natural map $H^1(\Theta_X) \to \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X))$, or $H^1(\Theta_X) \to \mathrm{Hom}(H^0(\Omega^2_X), H^1(\Omega^1_X))$ is in fact given by the mapping

$$H^1(\Theta_X) \otimes H^0(\Omega^2_X) \to H^1(\Omega^1_X)$$

induced by contraction of a vector field with a 2-form to produce a 1-form. In the case of a K3 surface, as has already been pointed out, this map is an isomorphism. Q.E.D.

As a consequence of the local Torelli theorem, the set of Hodge structures corresponding to K3 surfaces is an open set in the 20-dimensional complex manifold Ω. We can discover a lot about the moduli of K3 surfaces by examining the period space Ω.

Where are the algebraic K3 surfaces in our picture? If we choose a class $\lambda \in \Lambda$ with $\lambda^2 > 0$ (corresponding to an ample divisor under a marking of X, say), then we can look at the subset

$$\Omega_\lambda = \{[\omega] \in \Omega \,|\, \omega \cdot \lambda = 0\}.$$

This has codimension 1 in Ω, and parametrizes all K3 surfaces for which a marking α exists such that $\alpha^{-1}(\lambda)$ is the class of a *divisor*. As a consequence, we see that the set of algebraic K3 surfaces is a union of codimension 1 subvarieties in the set of *all* K3 surfaces. In particular, every algebraic K3 surface has arbitrarily close deformations which are non-algebraic.

More generally, given a Hodge structure on Λ, we may consider $\Lambda \cap H^{1,1}$; if $\alpha : H^2(X,\mathbb{Z}) \to \Lambda$ is a marking of X preserving the Hodge structures, then $\alpha^{-1}(\Lambda \cap H^{1,1})$ is the Neron-Severi group of X (by the Lefschetz (1,1)-theorem). We can try to preassign the structure of the Neron-Severi group by finding submodules $M \subset \Lambda$ and looking at Hodge structures for which M lies in $H^{1,1}$, as follows:

Given $M \subset \Lambda$ such that the signature of M is $(1, r-1)$ or $(0, r)$, define

$$\Omega_M = \{[\omega] \in \Omega \,|\, \omega \cdot \mu = 0 \text{ for all } \mu \in M\}.$$

It turns out that Ω_M is non-empty and has dimension equal to $22 - r = 22 - \mathrm{rank}(M)$.

As an example of this, let $e_1, \ldots, e_{16} \in \Lambda$ be classes coming from some Kummer surface (so $e_i \cdot e_j = -2\delta_{ij}$), and let M be the saturation of the lattice L which they generate. ($M/L \subset L^\#/L$ is necessarily isomorphic to the Kummer code.) The resulting space Ω_M has dimension 4, and parametrizes Kummer surfaces. However, for any $\gamma \in O(\Lambda)$ (the integral automorphism group of the unimodular form Λ), if we change the marking on the Kummer surfaces by using γ we discover that $\Omega_{\gamma(M)}$ *also* parametrizes Kummer surfaces.

Key Fact. $\bigcup_{\gamma \in O(\Lambda)} \Omega_{\gamma(M)}$ is dense in Ω.

This "key fact" is an essential step in the proof of the global Torelli theorem for K3 surfaces. Using it, one proceeds like this: given an effective Hodge isometry $\varphi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$, pick sequences of Kummer surfaces X_i, X'_i whose periods tend to those of X, X' and show that isomorphisms φ_i can be chosen "converging" to φ. There exist isomorphisms $\Phi_i : X'_i \to X_i$ with $\Phi_i^* = \varphi_i$, one needs to know that these converge in some appropriate sense.

The technical work needed to implement this method is somewhat difficult, and we will not go into it here.

12. The structure of the period mapping

For a smooth K3 surface X, we have identified a particular chamber of $C(X)$

$$V^+(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \,|\, x^2 > 0, x \cdot \lambda > 0 \text{ and for all } \delta \in \Delta^+(X), x \cdot \delta > 0\}$$

which we call the *Kähler chambers*. (Here, κ is any Kähler class on X.) Now if X is projective, $V^+(X) \cap H^2(X, \mathbb{Z})$ is exactly the set of ample divisors on X. the set of nef and big divisors can be identified with $\overline{V^+(X)} \cap C(X) \cap H^2(X, \mathbb{Z})$.

Now if λ is the class of a nef and big divisor, λ corresponds to a line bundle $\mathcal{O}_X(L)$. The linear system $|3L|$ factors through the contraction $\pi : X \to \bar{X}$ of all curves C_i such that $C_i^2 = -2$, $C_i \cdot L = 0$, and embeds the resulting surface $\bar{X}$. Moreover, $\mathcal{O}_X(L) = \pi^*(\mathcal{O}_{\bar{X}}(\bar{L}))$ for an ample linear system $|\bar{L}|$ on $\bar{X}$. The ample line bundles on $\bar{X}$ exactly correspond to classes $\lambda \in \overline{V^+(X)} \cap C(X) \cap H^2(X, \mathbb{Z})$ such that $\lambda^\perp \cap \Delta(X)$ exactly corresponds to the (-2)-classes supported on the exceptional set $\text{exc}(\pi)$.

To generalize this to the Kähler clase, we should introduce the notion of a *generalized Kähler metric* on a singular surface. In fact, we have no time to do this—please believe that such a notion exists, that it determines a set of Kähler classes on every K3 surface, and that the integral Kähler classes always correspond exactly to the ample line bundles.

Suppose now that X is an arbitrary K3 surface (with rational double points allowed), fix a Kähler class κ on X, and let $\pi : \widetilde{X} \to X$ be the minimal desingularization. We make several definitions in parallel to the smooth case.

For $R = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$, let $H^2_R(X)$ denote the orthogonal complement of the components of the exceptional set $\text{exc}(\pi)$ in $H^2(\widetilde{X}, R)$. $H^2_{\mathbb{Z}}(X)$ inherits a Hodge structure from $H^2(\widetilde{X}, \mathbb{Z})$, since the components of $\text{exc}(\pi)$ give classes in $H^{1,1}_{\mathbb{R}}$. $H^{1,1}_{\mathbb{R}}(X)$ denotes $H^2_{\mathbb{R}}(X) \cap H^{1,1}(X)$. We define

$$\begin{aligned}
\Delta(X) &= \Delta(\widetilde{X}) \\
\Delta^+(X) &= \Delta^+(\widetilde{X}) \\
R(x) &= \text{the } \textit{root system of } X = \{\delta \in \Delta(\widetilde{X}) \,|\, \pm\delta \text{ is supported on } \text{exc}(\pi)\} \\
V^+(x) &= \{x \in H^{1,1}_{\mathbb{R}}(X) \,|\, X^2 > 0, x \cdot \kappa > 0 \text{ and } x \cdot \delta > 0 \\
&\qquad \text{for all } \;\delta \in \Delta^+(X) - (\Delta^+(X) \cap R(X)\}
\end{aligned}$$

(That is, $x \cdot \delta \geq 0$ for $\delta \in \Delta^+(X)$ with equality if and only if $\delta \in R(X)$.) Note that when X is projective, $V^+(X) \cap H^2_{\mathbb{Z}}(X)$ is the set of ample classes.

Definition.

(1) A map $\varphi : H^2_{\mathbb{Z}}(X) \to H^2_{\mathbb{Z}}(X')$ is an *effective Hodge isometry* if and only if φ preserves the intersection forms and Hodge structures, and $\varphi(V^+(X)) = V^+(X')$.
(2) A map $\varphi : H^2_{\mathbb{Z}}(X) \to H^2_{\mathbb{Z}}(X')$ is *liftable* if and only if there exists an isomorphism $\psi : H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{X}', \mathbb{Z})$ preserving intersection forms such that $\psi|_{H^2_{\mathbb{Z}}(X)} = \varphi$.

We can now state the 3 main theorems about the structure of the period mapping for K3 surfaces. For the proofs, we refer to the book of Barth-Peters-Van de Ven, the seminar notes "Géométrie des surfaces K3 … " edited by Beauville et al., and the references contained there.

Global Torelli Theorem. *If $\varphi : H^2_{\mathbb{Z}}(X) \to H^2_{\mathbb{Z}}(X')$ is a liftable effective Hodge isometry between K3 surfaces X and X', there exists an isomorphism $\Phi : X' \to X$ such that $\Phi^* = \varphi$.*

Surjectivity Theorem. *Given a submodule $R \subset \Lambda$ generated by classes e_i with $e_i^2 = -2$ on which the form is negative definite, a Hodge structure $\{H^{p,q}\}$ on Λ of K3 type, and a connected component V^+ of $(R^\perp \cap C(H^{1,1})) - \bigcup_{\delta\in\Delta - R} \delta^\perp$ (where $C(H^{1,1}) = \{x \in H^{1,1} \mid x^2 > 0\}$ and $\Delta = \{\delta \in H^{1,1} \cap \Lambda \mid \delta^2 = -2\}$) there exists a K3 surface X and a marking $\alpha : H^2(\widetilde{X}, \mathbb{Z}) \to \Lambda$ preserving Hodge structures and intersection forms, such that $\alpha(R(X)) = R$ and $\alpha(V^+(X)) = V^+$.*

Existence of Kähler Metrics Theorem. *Every K3 surface is Kähler. Moreover, the set of Kähler classes is exactly $V^+(X)$.*

I want to describe how these theorems can be used to construct K3 surfaces whose Neron-Severi groups have specified properties. Typically, we are given a free $\mathbb{Z}$-module M with an intersection matrix, and perhaps some elements $\mu_1, \ldots, \mu_k \in M$ and we would like to find a K3 surface X such that $\mathrm{NS}(X) = M$, and the μ_i correspond to effective, or irreducible, or ample, or very ample divisors. If such is the case, we have $M \subset \Lambda$ embedded in a saturated way; and the discriminant-groups give a condition which M must satisfy. Namely, let $N = M^\perp$ so that $M^\#/M \cong N^\#/N$; we must have

$$(\text{minimum number of generators of } M^\#/M) \leq \mathrm{rank}(N).$$

The converse is a theorem of Nikulin.

Theorem (Nikulin). *Let M be a free $\mathbb{Z}$-module with a nondegenerate symmetric bilinear form of signature $(1, r-1)$ or $(0, r)$. If*

$$(\textit{minimum number of generators of } M^\#/M) \leq 22 - r$$

then there is an embedding $M \subset \Lambda$ with saturated image.

In any specific application, there are then several further steps to carry out. We illustrate all of this with an example: a free $\mathbb{Z}$-module M generated by λ_1, λ_2 with intersection matrix $\binom{4\ 8}{8\ 4}$.

Step 1. M can be generated with at most 2 generators, well less than 20. (In general, $\operatorname{rank}(M) \leq 11$ implies that M can be embedded, since $r \leq 22 - r$.)

Step 2. We construct a Hodge structure on Λ such that $\Lambda \cap H^{1,1} = M$, as follows. Choose $\omega \in \Lambda \otimes \mathbb{C}$ such that $\omega \cdot \omega = 0$, $\omega \cdot \bar{\omega} > 0$ and the smallest $\mathbb{Q}$-vector subspace of $\Lambda \otimes \mathbb{Q}$ containing ω is $M^\perp$. (Essentially, this means choosing the coefficients in a basis of $M^\perp$ to be algebraically independent transcendentals, except for the relation imposed by $\omega \cdot \omega = 0$.) Then $H^{1,1} \cap (\Lambda \otimes \mathbb{Q}) = M^{\perp\perp} \otimes \mathbb{Q} = M \otimes \mathbb{Q}$, so (since $M \subset \Lambda$ is saturated) $H^{1,1} \cap \Lambda = M$.

Step 3. Note that for $a_1\lambda_1 + a_2\lambda_2 \in M$ we have $(a_1\lambda_1 + a_2\lambda_2)^2 \equiv 0 \mod 4$, so that there are no (-2)-classes in M (i. e. $\Delta = \emptyset$). Let V^+ be the chamber containing λ_1—this is simply the connected component of $C(H^{1,1})$ containing λ_1.

Step 4. By the surjectivity theorem, there is a smooth K3 surface X and an isomorphism $\alpha : H^2(X, \mathbb{Z}) \to \Lambda$ preserving Hodge structures and intersection forms, such that $\alpha(V^+(X)) = V^+$. We have $\mathrm{NS}(X) = \alpha^{-1}(M) \cong M$, and $\alpha^{-1}(\lambda_1)$ corresponds to an ample line bundle $\mathcal{O}_X(L_1)$. Since $\lambda_1 \cdot \lambda_2 > 0$, $\alpha^{-1}(\lambda_2)$ also corresponds to an ample line bundle L_2.

(In more general examples, where M contains (-2)-classes, the analysis of Δ, V^+, the ampleness question, and whether or not X is smooth is much more complicated.)

Step 5. Are L_1 and L_2 very ample? Since $L_i^2 \neq 2$ or 8, to answer this we must know whether there can be a class $d \in M$ such that $d^2 = 0$, $d \cdot \lambda_i = 1$ or 2. We showed in section 1 that this is not the case for this particular M.

It follows that $|L_1|$ and $|L_2|$ are very ample, so there is a smooth $C \in |L_2|$. If we embed X by $\varphi_{|L_1|}$, we get a smooth quartic surface X with a smooth curve C of degree 8 and genus 6. Thus, this set of lectures ends exactly where it began.

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