TASI Lectures on Duality and Compactification

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Lecture III: Curved Backgrounds

As we saw in the previous lecture, certain natural backgrounds for string compactification which include D-branes (and yield singular supergravity solutions) break half of the supersymmetry of the original theory. The other natural way to study models with reduced supersymmetry is to introduced curved backgrounds.

The traditional way this has been done in string theory has been to decompose the 10-dimensional spacetime as a product $X^d \times M^{1,9-d}$ of a compact manifold X, and a flat spacetime M. To understand how much supersymmetry is preserved in such backgrounds, we must decompose the $(9+1)$-dimensional spinor rep. according to $\mathrm{Spin}(d) \times \mathrm{Spin}(1,9-d)$, and ask how many covariantly constant spinors will exist on X^d (with respect to the given metric on X^d) — these determine the unbroken supersymmetries.

A variant of this construction is given by the Freund-Rubin ansatz: we make a decomposition as a product $Y^{d-1} \times AdS^{1,10-d}$ together with a nontrivial field strength for one of the supergravity fields.

This time, the number of unbroken supersymmetries is determined by the number of <u>Killing spinors</u> on Y^{d-1}.

These constructions are closely related: if we decompose $AdS^{1,10-d}$ as a warped product of $\mathbb{R}^+$ and $M^{1,9-d}$, then we can rewrite $Y^{d-1} \times AdS^{1,10-d} = (Y^{d-1} \times \mathbb{R}^+) \times_{\text{warped}} M^{1,9-d}$ and the Killing spinors on Y^{d-1} go over to covariantly constant spinors on $Y^{d-1} \times \mathbb{R}^+$ (by a theorem of Bär). When Y^{d-1} is a sphere, we can regard this as a brane solution of the supergravity theory.

A more general ansatz which combines both of these ideas is a background of the form $X^d \times_{\text{warped}} M^{1,9-d}$ with a metric $ds^2 = ds_X^2 + \phi(x) ds_M^2$ with an appropriate (Ricci-flat) metric ds_X^2, a flat metric ds_M^2 on M, and a conformal factor $\phi(x)$ depending on $x \in X$. The space X should have finite volume, but might not be compact (due to the presence of branes).

We will mainly focus on the case where X is compact. In this case, the covariantly constant spinors are determined by the holonomy of the metric. (Similarly, the Killing spinors on Y in a Freund-Rubin ansatz will be determined by the Weyl-holonomy — but the holonomy is easier to determine.)

The classification of holonomy groups of Riemannian manifolds is given by the Berger-Simons theorem. Actually, it is important to bear in mind that there is a holonomy representation which is being classified, not just a group. If we start at a point $x \in X$ and follow a loop which begins and ends at x, parallel transport along that path will transport tangent vectors at x along to tangent vectors at intermediate points, finally reaching a tangent vector at x again. This gives a mapping from $T_{X,x}$ to itself, and the ~~gp~~ group generated by all such mappings is the holonomy group (with $T_{X,x}$ giving the holonomy representation space). Note that parallel transport can also be applied to differential forms and to spinors, so determining the covariantly constant ones is then a simple exercise in representation theory.

Here is the Berger-Simons classification of irreducible holonomy representations:

Cases with cov. spinor	
$\{1\}$ on $\mathbb{R}^n$	flat
$SU(n)$ on $\mathbb{R}^{2n}$ $(n \geq 3)$	Calabi-Yau
$Sp(n)$ on $\mathbb{R}^{4n}$	hyper-Kähler
G_2 on $\mathbb{R}^7$	G_2
$Spin(7)$ on $\mathbb{R}^8$	Spin 7

Cases without cov. spinor	
$SO(n)$ on $\mathbb{R}^n$	general Riemannian
$U(n)$ on $\mathbb{R}^{2n}$	Kähler
$Sp(1)\times Sp(n)/\mathbb{Z}_2$ on $\mathbb{R}^{4n}$	quaternionic Kähler
H	locally symmetric space G/H

To apply this, we also need to know that every compact manifold admits a finite (unramified) cover which can be decomposed as a product of a flat torus and a collection of manifolds with irreducible holonomy reps.

If we are interested in compactifications of string theories (or M Theory) which preserve some supersymmetry, we should focus on the Calabi-Yau, hyper-Kähler, G_2 and Spin(7) cases. The last two "exceptional" cases are poorly understood, and will not be discussed further here.

The Calabi-Yau and hyper-Kähler manifolds can be given the following geometric characterizations (we assume the manifolds are compact):

Calabi-Yau manifolds (holonomy $SU(n)$, $n \geq 3$) have a non-vanishing holomorphic n-form Ω and a Kähler metric. There is a unique complex structure (up to complex conjugation) compatible with the metric. The Kähler metric can be described in terms of the Kähler form $\omega = \sum \frac{i}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$.
[Ω has a local description of the form $f(z)\, dz_1 \wedge \cdots \wedge dz_n$ with $f(z)$ holomorphic.]

hyper-Kähler manifolds (holonomy $Sp(n)$) have real dimension $4n$, with a distinguished 3-plane of 2-forms, and an S^2 of complex structures. If we choose one of the complex structures, it has a holomorphic 2-form of the form $\omega_1 + i\omega_2$ and a Kähler form $r\omega_3$ for some orthogonal basis $\omega_1, \omega_2, \omega_3$ of the three-plane and some positive constant r. The manifold also has holomorphic 4-form, 6-form, ..., $2n$-form given by taking powers of $\omega_1 + i\omega_2$. In particular, there is a form $\Omega = (\omega_1 + i\omega_2)^n$ of top degree. It is non-vanishing.

The metrics in all of these cases are Ricci-flat. Such metrics were studied by Calabi in the 1950's who showed that for a given complex structure and de Rham cohomology class of Kähler metrics, there is at most one Ricci-flat metric in the class. [If you are unfamiliar with de Rham cohomology, another way of stating this is: if ω is the Kähler form of a Ricci-flat metric, then there is no 1-form η on X such that $\omega + d\eta$ is also the Kähler form of a Ricci-flat metric.]

Calabi conjectured the existence of such metrics, and this was proved by Yau in the 1970's in the following form: given a compact complex manifold X ~~where~~ ~~of the~~ of complex dimension n which admits a non-vanishing holomorphic n-form Ω and given a Kähler form ω on X, there exists a Ricci-flat metric on X whose Kähler form is in the same de Rham class as ω, and for which Ω is covariantly constant.

The proof is a non-constructive existence proof. In particular, although we are certain that these metrics exist, it is very difficult to calculate any of their properties.

However, this theorem is very powerful as a tool for studying string backgrounds, since it reduces the search for solutions to the supergravity equations of motion to a search for complex Kähler manifolds which have a non-vanishing holomorphic n-form Ω.

In fact, the search can be restricted even further: it turns out that for every compact $SU(n)$ holonomy manifold ($n \geq 3$), the complex structure is algebraic (i.e., X comes from algebraic geometry); for hyper-Kähler manifolds, generically if you fix the Ricci-flat metric there will be choices out of the S^2 of complex structures for which X is algebraic.

So we can restrict our search to algebraic geometry, and employ a completely different set of tools to find and study such objects.

The "algebraic varieties" we now must study are complex submanifolds X of complex projective space $\mathbb{P}^N$. We describe $\mathbb{P}^N$ by means of "homogeneous coordinates" $[z_0, z_1, \ldots, z_N] \neq [0, 0, \ldots, 0]$

which do not label points uniquely but are subject to identifications $[z_0, z_1, \ldots, z_n] = [\lambda z_0, \lambda z_1, \ldots, \lambda z_n]$ for non-zero complex numbers λ. (We are using square brackets to emphasize that these are not ordinary coordinates.)

Given $X \subseteq \mathbb{P}^N$, each homogeneous coordinate z_i determines a codimension one subvariety

$$D_i = X \cap \{z_i = 0\}$$

on X. (We are assuming that $X \not\subseteq \{z_i = 0\}$; otherwise we would have treated X as a submanifold of $\{z_i = 0\} \cong \mathbb{P}^{N-1}$.) Such a codimension one subvariety is called a divisor on X. More generally, a combination $\sum m_i D_i$ with integer coefficients is also called a divisor.

If we consider two of these divisors, D_i and D_j, the ratio z_i / z_j makes sense as a function on $X - D_i - D_j$. (The individual homogeneous coordinates are not functions on X or on $\mathbb{P}^N$ due to the identifications, but the identifications cancel out in ratios.) This ratio z_i/z_j extends to a meromorphic function on X: its only singularities are poles.

Generally, for a meromorphic function f defined on X, we define its *divisor* to be

$$\mathrm{div}(f) = \{f=0\} - \{f=\infty\}$$

(where $\{f=0\}$ and $\{f=\infty\}$ are codimension one in X).
In the example at hand, we have

$$\mathrm{div}(z_i/z_j) = D_i - D_j.$$

This property is characteristic of divisors which occur as intersections with linear functions in $\mathbb{P}^N$ for the *same* embedding in projective space. (In general, a given algebraic variety will have many embeddings into projective spaces.)

To determine all of the ways to embed X into projective spaces, we can study *all* of the divisors on X.
Two divisors D and D' are said to be *linearly equivalent* if there is a meromorphic function f such that

$$\mathrm{div}(f) = D - D'.$$

The *linear system containing D* is the set

$$|D| = \{D' \text{ s.t. } D' \text{ is linearly equivalent to } D, \text{ and } D' = \sum n_i D_i \text{ with } n_i \geq 0 \text{ and } D_i \subseteq X \text{ codimension one}\}.$$

The last requirement in the definition comes from the observation that the divisors we encountered from $X \subseteq \mathbb{P}^N$ were actually ~~[illegible]~~ subsets of X

(with no negative coefficients allowed).

Given a linear system $|D|$, we chose a basis $D_0 = D, D_1, \ldots, D_n$ of the divisors in $|D|$, and let $f_1, f_2, \ldots, f_n$ be the meromorphic functions satisfying

$$\operatorname{div}(f_j) = D_j - D_0.$$

Then we can define a mapping $X \to \mathbb{P}^n$ by

$$(*) \qquad x \mapsto [1, f_1(x), f_2(x), \ldots, f_n(x)].$$

This is ill-defined along D_0, but by exploiting the equivalence in $\mathbb{P}^n$ we can rewrite this as

$$[1, f_1(x), f_2(x), \ldots, f_n(x)] = \left[\frac{1}{f_1(x)}, 1, \frac{f_2(x)}{f_1(x)}, \ldots, \frac{f_n(x)}{f_1(x)}\right]$$

which is ill-defined along D_1 instead of along D_0, and so on, for other divisors D_j.

Thus, if the divisors $D_0, D_1, \ldots, D_n$ have no points in common, our prescription $(*)$ can be extended to a well-defined mapping on all of X.

The linear system $|D|$ is said to be <u>very ample</u> if the associated mapping is actually an embedding into $\mathbb{P}^n$. Given a very ample linear system $|D|$, ~~or~~ ie, an embedding $X \subseteq \mathbb{P}^n$, we get a natural Kähler metric on X by restricting the Fubini–Study metric from $\mathbb{P}^n$. Explicitly, the Kähler form of this metric on $\mathbb{P}^n$ can be written $\omega = \partial\bar\partial \log \sum |z_i|^2$. Restricting to X, we get a form $\omega_{|D|} = \partial\bar\partial \log \sum |z_i|^2 \big|_X$ on X.

A key fact is that for Calabi-Yau manifolds, the Kähler classes $\omega_{|D|}$ coming from projective embeddings will generate all Kähler classes (using positive real linear combinations).

So this portion of our problem – determining the set of Kähler classes – can be solved using algebraic geometry. (The hyper-Kähler case is different and will be discussed in Lecture IV.)

The other portion of our problem – determining the set of complex structures – is also a problem in algebraic geometry. Once X has been embedded in $\mathbb{P}^n$, it can always be described by means of a finite set of homogeneous equations $f_1(z_0, \dots, z_n), \dots, f_k(z_0, \dots, z_n)$, with

$$X = \{[z_0, \dots, z_n] \in \mathbb{P}^n : f_j(z_0, \dots, z_n) = 0 \text{ for all } j\}.$$

In principle, the other complex structures are found by varying the coefficients in these defining equations. There are two difficulties with this in practice:

1) There may be some complex structures on this manifold which don't embed into the same projective space X

on X and meromorphic $(n-1)$-forms on D: given a meromorphic n-form $\dfrac{g(w_1,\dots,w_n)\,dw_1\wedge\cdots\wedge dw_n}{f(w_1,\dots,w_n)}$ with a simple pole on D (using local coordinates $w_1,\dots,w_n$ on X), its Poincaré residue is

$$\left.\frac{g(w)\,dw_1\wedge\cdots\wedge dw_{n-1}}{\partial f/\partial w_n}\right|_D$$

(well-defined if $\partial f/\partial w_n \neq 0$) with similar, equivalent, formulas when $\partial f/\partial w_j \neq 0$. (If D is a sub<u>manifold</u>, then at every point one of the $\partial f/\partial w_j$'s must be $\neq 0$.)

It is common to express the properties of meromorphic n-forms in terms of <u>divisors</u>; if $\alpha(w_1,\dots,w_n)\,dw_1\wedge\cdots\wedge dw_n$ is a meromorphic n-form, we define the <u>canonical divisor of X</u> to be

$$K_X = \mathrm{div}(\alpha) = \{\alpha=0\} - \{\alpha=\infty\}.$$

Thus, in our Poincaré residue formula, we see

$$K_X = \mathrm{div}(g) - \mathrm{div}(f) = \mathrm{div}(g) - D.$$

while $\quad K_D = \mathrm{div}(g)|_D \quad$ (since $\partial f/\partial w_j \neq 0$).

Thus, $\quad K_D = (K_X + D)|_D$. This is known as the <u>adjunction formula</u>.

2) The number of equations needed to describe X is larger than $\dim \mathbb{P}^n - \dim X$, and the equations don't meet transversally; thus, if we vary the coefficients arbitrarily we will find a common intersection which is smaller than X. [So we must vary the coefficients judiciously, and it is hard to see explicitly how to do this.]

We will encounter both of these phenomena in our discussion of K3 surfaces below. Over the years, algebraic geometers have developed some rather sophisticated machinery to address these issues. Very little of this machinery has been applied to cases of interest in physics (to date!).

So we have seen that the complex structures can be studied by varying coefficients, and the Kähler classes can be studied by locating all (very ample) divisors. The issue we have not yet addressed is: how can we recognize whether or not there exists a non-vanishing holomorphic n-form?

A very useful tool in studying this issue is the "adjunction formula". Given a complex submanifold $D \subseteq X$ defined by a single equation $f=0$ (locally), there is a "Poincaré residue formula" relating meromorphic n-forms

The interpretation of $D|_D$ is this: find a divisor D' which is linearly equivalent to D, and treat $D'|_D$ as a divisor on D. (All facts about these divisors are being considered up to linear equivalence only.) This is the divisorial version of the ~~notation~~ "normal bundle" of D.

The requirement in Yau's Theorem is $K_X = 0$: this means that there exists a ~~holo~~ mero n-form whose divisor is trivial, i.e., it has neither zeros nor poles.

<u>Key example</u> $K_X = 0$ and $D \subseteq X$ is a codimension one submanifold. The adjunction formula tells us that $K_D = D|_D$. Note that $D|_D$ becomes quite concrete if we embed X in $\mathbb{P}^n$ using the linear system $|D|$: then $D'|_D$ represents the intersection of D with some $z_i = 0$. In other words, D is embedded by the canonical linear system $|K_D|$.

The case of $\dim X = 2$ (a K3 surface) is instructive. Then D is a Riemann surface, which must have some genus g. The degree of the canonical divisor is well-known: $\deg(K_D) = 2g-2$. Also, the canonical linear system $|K_D|$ embeds D into $\mathbb{P}^{g-1}$.

The interpretation of these facts in terms of X is that X should embed in $\mathbb{P}^g$, and its degree (the number of points of intersection $X \cap \{z_i = 0\} \cap \{z_j = 0\}$) should be $2g-2$.

Remarkably, surfaces $X \subseteq \mathbb{P}^g$ of this type exist for every $g \geq 2$, and in every case, the number of independent deformations of complex structure is 19. These are the algebraic K3 surfaces.

Let us consider these surfaces for low values of g.

$g=2$ Riemann surfaces of genus 2 are hyperelliptic, and map 2:1 onto $\mathbb{P}^1$. So X will map 2:1 onto $\mathbb{P}^2$. The map on D must have 6 branch points, so the map $X \to \mathbb{P}^2$ must be branched over a curve of degree 6. We can describe X as

$$y^2 = z_0^6 + z_1^6 + z_2^6 + \cdots$$

(the degree six equation can be arbitrary), and regard this as an equation in a weighted projective space, $\mathbb{P}^{1,1,1,3}$ in which $[z_0, z_1, z_2, y] = [\lambda z_0, \lambda z_1, \lambda z_2, \lambda^3 y]$. (The superscripts in the notation denote the power of λ, the so-called weights of the homogeneous variables.)

$\underline{g=3}$ The general Riemann surface of genus 3 embeds as a degree 4 curve in $\mathbb{P}^2$; X should be a surface of degree 4 in $\mathbb{P}^3$, for example

$$z_0^4+z_1^4+z_2^4+z_3^4=0.$$

$\underline{g=4}$ This time, $D\subseteq\mathbb{P}^3$ is the intersection of surfaces of degrees 2 and 3, and $X \subseteq \mathbb{P}^4$ will be the intersection of hypersurfaces of degrees 2 and 3. degree $=2\cdot 3=6$

$\underline{g=5}$ $D\subseteq\mathbb{P}^4$ and $X\subseteq\mathbb{P}^5$ can be described as intersection of 3 hypersurfaces of degree 2. degree $=2\cdot 2\cdot 2=8$.

$\underline{g\geq 6}$ $D\subseteq\mathbb{P}^{g-1}$ and $X\subseteq\mathbb{P}^g$ require more defining equations than their codimension. This makes the moduli problem tricky — coefficients must be varied judiciously.

One interesting feature to note about this set of examples: the complex dimension of the space of Riemann surfaces of genus g is $3g-3$, whereas the dimension of those which lie on a K3 surface is at most $19+g$ (19 parameters for X and g parameters for the choice of D when $X\subseteq\mathbb{P}^g$). Thus, when $g>11$, not every curve lies on a K3 surface.

Another interesting feature, to be discussed further in lecture IV, is that the set of all complex structures on a K3 surface has complex dimension 20, and form a single family containing all of the algebraic K3 surfaces of every g. This is an example of the phenomenon mentioned above in which not all deformations of complex structure may happen in the given projective space.

However, when the holonomy is $SU(n)$, $n \geq 3$, there always exist embeddings $X \subseteq \mathbb{P}^N$ for which all nearby complex structures can be obtained within the same $\mathbb{P}^N$. (Warning: if somebody hands you $X \subseteq \mathbb{P}^N$, it might not have this property: some projective embeddings are "deficient" in this sense.)

The theory of K3 surfaces is understood in great detail, and will be explored further in the next lecture. We know much less about Calabi-Yau or hyper-Kähler manifolds of higher dimension. There are two strategies which might be followed:

1) try to directly generalize constructions like the $g \leq 5$ cases
2) try to study in general the possible divisors D and whether they occur on Calabi-Yau or hyper-Kähler manifolds.

The first strategy has led to an extensive study of Calabi-Yau "complete intersections" in projective spaces, and more generally in weighted projective spaces or toric varieties (a further generalization of weighted projective space). Tens of thousands of examples have been produced in this way. And yet, as the above story about K3 surfaces illustrates, such constructions may have only barely scratched the surface.

It is instructive to see why the set of complete intersection Calabi-Yau manifolds (of fixed dimension) in projective space is finite. Suppose $X^d \subset \mathbb{P}^n$ has been defined as the intersection

$$X = Y_1 \cap Y_2 \cap \cdots \cap Y_{n-d}$$

of $n-d$ hypersurfaces. Each Y_j is linearly equivalent to $m_j H$, where $H = \{z_0 = 0\}$, and m_j is the degree of the homogeneous polynomial defining Y_j. We use the fact that $K_{\mathbb{P}^n} = -(n+1)H$ (which can be seen from the globally well-defined meromorphic n-form

$$\frac{z_0\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n + z_1\, dz_2 \wedge \cdots \wedge dz_n \wedge dz_0 + \cdots}{z_0 z_1 \cdots z_{n-1} z_n}$$

with poles along all $n+1$ coordinate hyperplanes) and apply the adjunction formula repeatedly:

$$(\cdots((K_{\mathbb{P}^n}+Y_1)|_{Y_1}+Y_2)|_{Y_2}+\cdots+Y_{n-d})|_{Y_{n-d}} = K_X$$

$$\| $$

$$(-(n+1)H+m_1H+m_2H+\cdots+m_{n-d}H)|_X.$$

So the condition is: $\sum_{j=1}^{n-d} m_j = n+1$. Moreover, we can choose $m_j \geq 2$ for each j since otherwise X would sit in a linear subspace (a smaller $\mathbb{P}^n$). So the condition is

$$\sum_{j=1}^{n-d}(m_j-1) = d+1 \quad , \quad m_j - 1 \geq 1$$

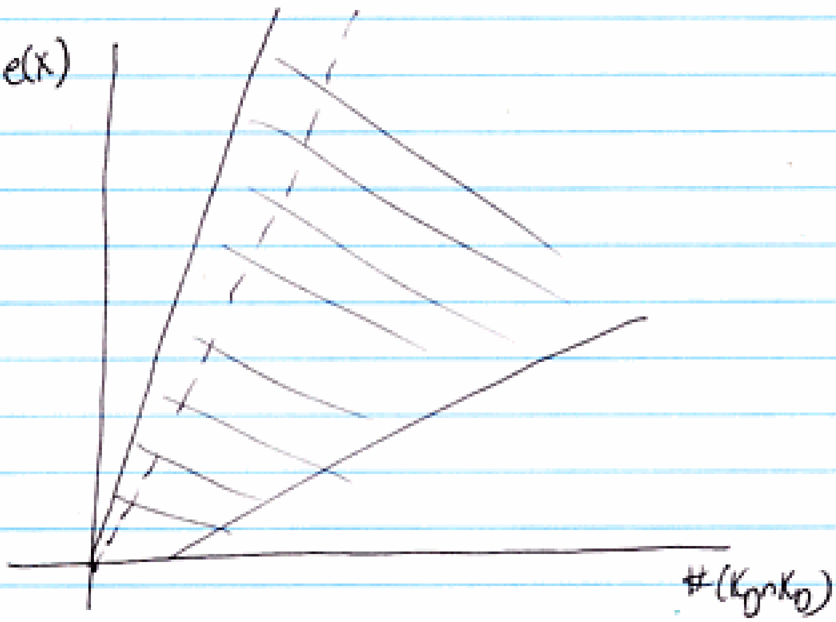
and with a fixed d there are clearly only a finite number of solutions. The K3 examples are reproduced by $3=3$, $3=2+1$, $3=1+1+1$ giving degrees 4, $(3,2)$, and $(2,2,2)$.

The second strategy would seem to be a more general one: first, we study all surfaces D for which $|K_D|$ gives an embedding (or at least a reasonable map), and then we try to decide which ones can be on Calabi-Yau 3-folds. As the remark about curves on K3's indicated, the second part will be highly non-trivial. However, even the first part is quite hard: for curves, we had a simple invariant (the genus), and rather complete knowledge about the set of curves of genus g.

For surfaces, there are several invariants, including the Euler number $e(X) = \chi_{top}(X)$ and the degree of the canonical divisor $\#(K_D \cap K_D)$. There are constraints such as

$$12 \mid e(X) + \#(K_D \cap K_D)$$

and also inequalities which restrict the invariants to the shaded region:



Above the dotted line, there are fewer surfaces than below it.

For each point on the graph, there are at most a finite number of families, but it is not known how many, nor what are their dimensions, etc.

It has often been speculated that the number of families of Calabi-Yau 3-folds might be finite. Certainly, the vast array of possibilities for D, together with the phenomenon of algebraic K3 surfaces for every $g \geq 2$ casts some doubt on this. (However, as we shall see, the K3 surfaces are in fact unified into a single family of hyper-Kähler manifolds.) Of course, many ~~each~~ Calabi-Yau 3-folds ~~all~~ have a wide variety of divisors D on them, so there will be much duplication. At the moment, it's hard to tell whether the expectation that the number of families is finite is reasonable or not.