Lecture 5

String algebras

In this lecture and the next I shall discuss two-dimensional field theories for which the state space $\mathcal{H}_{S^1}$ associated to a circle is a cochain complex rather than just a vector space. The main reason for being interested in these comes from string theory, and we have already met examples in the last lecture. Now, however, I shall make a fresh start, setting out from classical algebraic topology.

5.1 Homotopy type and cochain complexes

The most characteristic tool of algebraic topology is cohomology: one associates to a space X a *cochain complex*

$$C^\cdot = (\cdots \xrightarrow{d} C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \cdots),$$

i.e. a sequence $\{C^k\}_{k\in\mathbb{Z}}$ of abelian groups and homomorphisms d such that $d \circ d = 0$, and defines the cohomology group

$$H^k = \{\ker d : C^k \to C^{k+1}\}/\{\text{image } d : C^{k-1} \to C^k\}.$$

A cochain complex is a model of the homotopy type of a space. Cochain complexes form an additive category in which there is a notion of homotopy between morphisms: a morphism

$$f : C^\cdot \to \widetilde{C}^\cdot$$

is a sequence of homomorphisms $f^k : C^k \to \widetilde{C}^k$ such that $d \circ f^k = f^{k+1} \circ d$, and a homotopy h between morphisms $f, g : C^\cdot \to \widetilde{C}^\cdot$ is a sequence $h^k : C^k \to \widetilde{C}^{k-1}$ such that

$$\{d, h\} = d \circ h + h \circ d = g - f.$$

There are many ways to assign a cochain complex to a space — singular, Cech, Alexander-Spanner, simplicial, cellular, ... — but for reasonable spaces they are all *canonically* homotopy equivalent. Thus we can speak of a *functor* from spaces to the category of cochain complexes and homotopy classes of morphisms. I shall refer to this additive category as the "homotopy category of cochain complexes". It is a fairly good first approximation to the category of topological spaces and homotopy classes of maps. But really it is an approximation to the "stable" homotopy category.

The stable homotopy category is another fundamental discovery of algebraic topology. A *stable map* from X to Y, where X and Y are compact spaces, is a homotopy class of base-point-preserving maps $f : S^nX \to S^nY$ for some n, where S^nX is the n-fold suspension of X, i.e. the one-point compactification of $X \times \mathbb{R}^n$ (with ∞ as base-point). The stable maps form an abelian group $\{X;Y\}$, and composition

$$\{X;Y\} \times \{Y;Z\} \to \{X;Z\}$$

is bilinear. Thus compact spaces and stable maps form an additive category, which is, in a precise sense, the minimal additive category that can be made from the homotopy category of spaces. I shall say more about it in the next section. For the moment I shall just remark that rationally it is the same as the homotopy category $\mathcal{C}$ of rational cochain complexes, i.e.

$$\{X;Y\} \otimes \mathbb{Q} \;\cong\; \mathrm{Hom}_{\mathcal{C}}\,(C^{\cdot}(Y;\mathbb{Q})\;;\;C^{\cdot}(X;\mathbb{Q})).$$

Passing from homotopy to stable homotopy obliterates the distinction between spaces which become the same when "thickened" by suspension. The essential loss is the notion of intersection of cycles: two intersecting cycles will always miss each other when deformed slightly in a larger ambient dimension. Intersection of cycles corresponds to multiplication of cohomology classes : the algebraic model of an actual space, rather than a stable space, is not a cochain complex but a cochain *algebra*. Like the cochains themselves, the multiplication $C^{\cdot} \otimes C^{\cdot} \to C^{\cdot}$ is really defined only up to homotopy. It induces an associative anticommutative product on the cohomology. If we use cochains with rational or real coefficients then the cochain multiplication itself can be chosen to give us an anticommutative differential graded algebra : the de Rham complex of a smooth manifold is such a choice.

Among compact spaces, those that are orientable manifolds — i.e. that are locally identical with Euclidean space $\mathbb{R}^n$ — form an important special class. Surprisingly enough, this local geometric property corresponds to a global algebraic property of the cohomology ring, namely Poincaré duality. If X is a compact oriented manifold there is an operation of integration

$$\int_X : H^n(X) \to \mathbb{Q},$$

and the bilinear form

$$H^p(X) \times H^{n-p}(X) \to \mathbb{Q}$$

given by $(\alpha, \beta) \mapsto \int_X \alpha \cdot \beta$ makes h^p and H^{n-p} into dual vector spaces. (For simplicity I am using cohomology with rational coefficients here.) In other words, the cohomology ring $H^*(X)$ is a Frobenius algebra (in the graded sense).

Even more surprising is that the converse is true too, at least for simply connected space. If the cohomology ring of a compact space X obeys Poincaré duality then X is rationally homotopy equivalent to a manifold. I cannot think of any other significant example of an equivalence like this between a local topological property and a global one. In any event, it seems a very basic fact that compact manifolds correspond to cochain complexes which are Frobenius algebras up to homotopy.

There is an extensive theory of "algebraic structures up to homotopy", more than adequate for studying the topology of manifolds. Still, it is interesting that topological field theory gives us a new variant of a Frobenius algebra up to homotopy. The field theory commutativity property seems considerably weaker than that of the cochains of an actual manifold, and an optimist will hope that this reflects a fundamental aspect of the way in which string theory generalizes the idea of a manifold.

A topological field theory with cochain values can be defined in a variety of ways, just as there are many ways of defining cochains. Any definition will assign a cochain complex C_S to each closed 1-manifold S, and a cochain map

$$U_\Sigma : C_{S_0} \to C_{S_1}$$

to each cobordism $\Sigma : S_0 \rightsquigarrow S_1$. If we have a smoothly varying family $\{\Sigma_t\}$ of cobordisms from S_0 to S_1 then we expect the maps $U_{\Sigma_t}\}$ to be homotopic. In fact, we want a map from the "space" of cobordisms $S_0 \rightsquigarrow S_1$ to the "space" represented by the cochain complex Hom $(C_{s_0}; C_{S_1})$. One way to make sense of this is to work with categories rather than spaces, i.e. to treat the cobordisms from S_0 to S_1 as a category, and to ask for a functor from it to the category of cochain homotopies. Another way is to rigidify the category of surfaces by equipping the surfaces with Riemannian metrics, i.e. we let the cochain map U_Σ depend on a choice of a metric on Σ. Then there is an infinite dimensional manifold $\mathcal{M}(S_0; S_1)$ of isomorphism classes of cobordisms, and the natural way to "map" $\mathcal{M}(S_0; S_1)$ to Hom $(C_{S_0}; C_{S_1})$ is to give a cochain on $\mathcal{M}(S_0; S_1)$ with values in Hom $(C_{S_0}; C_{S_1})$, say

$$\widehat{U}_\Sigma \in C^{\cdot}(\mathcal{M}(S_0; S_1)\,;\ \mathrm{Hom}\,(C_{S_0}; C_{S_1})).$$

This second approach is more in the spirit of convential quantum field theory, especially in the de Rham version where the C_S are complexes of real vector spaces and $\widehat{U}_\Sigma$ is a differential form on $\mathcal{M}(S_0; S_1)$. These structures are usually called "string backgrounds", but I shall use the name *string algebra* for the version I shall define in §5.3.