A Survey of High-Dimensional Supergravity

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One of the first things you have to learn to peruse the physics literature are the acronyms SUSY, which means "supersymmetry," and SUGRA, which means "supergravity."

The details of supergravity are quite messy, and it is likely that factors of 2 and minus signs are incorrect in this lecture, but the correct signs can be looked up in standard texts like Polchinski's second volume on string theory. The aim of this lecture is to overview the general structure and features of supergravity, rather than worry about the particulars.

Outline of this lecture:

1. Why supergravity?
2. Why 11 dimensions?
3. Lagrangians, equations of motions (EoM), beyond SUGRA etc.

References:

W. Nahm, Nucl. Phys. B **135** (1978) 149
E. Cremmer, B. Julia, J. Scherk, Phys. Lett. B **76** (1978) 409.
J. Polchinski, "String Theory, vol. 2," Cambridge University Press, 1998.

1 Why SUGRA?

There are two basic motivations to study supergravity.

1.1

First, what kind of symmetries do we expect to be symmetries of nature? The symmetries of nature are not global symmetries, but are gauged. For example, electromagnetism, the strong interactions etc. If SUSY is an exact symmetry of nature, we expect it to be gauged at some energy scale. We therefore need to understand the implications of gauging SUSY.

From a global to a local symmetry: start with a SUSY algebra taking the following form in $D+1$ dimensions:

$$\{Q_\alpha, \overline{Q}_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu.$$

The P_μ are momenta, and the Q_α are spacetime spinors.[1] At the moment, we are dealing with a global rather than a local SUSY algebra. This algebra is an extension of the Poincaré algebra to a super-Poincaré algebra, obtained by adding the odd generators Q_α.

(A word on notation: $\overline{Q} = Q^\dagger \gamma^0$, where the γ's are generators of the Clifford algebra.)

In even dimensions, the spin representation is reducible, and can be decomposed into two irreps. of positive and negative chirality. In terms of the volume form γ_V for the $D+1$-dimensional Clifford algebra, a chiral supercharge Q_α satisfies,

$$\gamma_V Q_\alpha = \pm Q_\alpha.$$

In even dimensions, we therefore have the possibility of chiral supersymmetries. Indeed, type IIB string theory has its low-energy description type IIB SUGRA which is chiral. In special dimensions, like 10 and 11, we can also impose a reality condition on spinors, which allows us to reduce the number of supercharges by one-half.

There are many examples of supersymmetric theories: supersymmetric σ-models in 2 dimensions (with mirror symmetry as an application), Yang–Mills theories in 4 dimensions (with Seiberg–Witten theory as an application), and more mysterious theories of interacting anti-self-dual tensor multiplets in 6 dimensions (with luck, uncovering new mathematics might well be an application here!).

In all of these theories, SUSY acts in the following schematic way:

$$\delta\phi = \epsilon\psi$$

$$\delta\psi = \not{D}\phi\epsilon + \ldots$$

(Here ϵ is a SUSY parameter; namely, a Grassmann-valued spinor.) We generically denote a boson by ϕ, and a Grassmann-valued fermion by ψ. The omitted terms describe interactions particular to a given theory.

When we gauge a global symmetry, we replace a spacetime independent symmetry parameter ϵ by one which depends arbitrarily on the spacetime coordinates:

$$\epsilon \to \epsilon(x).$$

Gauging SUSY therefore requires us to consider supersymmetry parameters, ϵ, which depend on spacetime.

Gauging this symmetry, that is, studying local SUSY, necessarily leads to a SUGRA theory. This follows because SUSY closes on the generators of Poincaré. Gauging SUSY therefore implies that we have gauged Lorentz transformations. Naturally, we are led to a theory of gravity.

[1] As a caveat, I should point out that with extended SUSY, central charges can appear on the right hand side of the supersymmetry closure relation. These central charges commute with all the supercharges and typically measure some non-trivial background charge. In Yang-Mills theory, for example, this could be electric or magnetic charge.

1.2

A second motivation comes from string theory. We have a length scale in string theory, $\ell_s = \sqrt{\alpha'}$. (Work in $D+1$ dimensions again.)

At long wavelengths, both $(\partial_\mu \phi)^2$ and $\psi \not{D} \psi$ are equally important in the description of the low-energy physics. There is a long-wavelength or momentum expansion of the spacetime effective action. Terms in the effective action are ordered according to

$$n = N_\partial + \frac{1}{2} N_f.$$

Here N_∂ is the number of derivatives appearing in a given term, while N_f is the number of fermions. In this expansion, we have

$$S_{\text{eff}} = S_2 + S_4 + \dots$$

where S_n are terms with value n. The lowest order terms in S_2 comprise the supergravity action, S_4 contains terms such as $\int \sqrt{g} R^4$ and $\int \lambda^{16}$ where R^4 is some combination of curvatures containing 8 derivatives and λ is some fermion.

The bosonic variation in a supergravity theory always takes the form

$$\delta\phi = \epsilon\psi$$

where ϕ is the boson and ψ is some fermion. By a field redefinition, this variation can be taken as the definition of the fermion. (There can be problems in trying to do this when there are many flavors of fermion. The freedom to perform field redefinitions may not suffice to put all boson variations into this simple form.) The fermionic variation takes the schematic form

$$\delta\psi = \epsilon\partial\phi + M\epsilon$$

where M has terms with $n = 1$ while ϵ has $n = -1/2$. The action S_2 takes the schematic form

$$S_2 \sim \int (\partial\phi)^2 + (\psi \not{D} \psi) + (\psi\psi\partial\phi) + (\psi)^4.$$

Note that as long as M has only terms with $n = 1$, the SUSY algebra can close on the finite set of interactions appearing in S_2. However, as soon as we add higher order terms, this is no longer the case. The SUSY algebra can no longer close on a finite set of terms. We shall return to this point at the end of this discussion.

Global supersymmetry satisfies the algebra,

$$\{Q, \overline{Q}\} = \gamma^\mu P_\mu,$$

typically only on-shell. This is even the case for free theories with extended supersymmetry. Generally,

$$\{Q, \overline{Q}\} = \gamma^\mu P_\mu + \partial_\Lambda(\dots) + \dots$$

withe ∂_Λ a gauge transformation with a possibly field-dependent parameter.

As a simple example where this happens, let us consider abelian Yang–Mills theory with $F = dA$:

$$\delta A_\mu = \epsilon\gamma_\mu\psi$$
$$\delta\psi = \gamma^{\mu\nu}F_{\mu\nu}\epsilon.$$

As a further word on notation, note that

$$\delta A_\mu = [\epsilon Q, A_\mu].$$

Again, I should warn the reader that factors of i and 2 are omitted in these expressions. Evaluating the commutator of two supersymmetry transformations gives,

$$[\delta,\delta']A_\mu = \epsilon'\gamma^\nu\epsilon\partial_\nu A_\mu - \epsilon'\gamma^\nu\epsilon\partial_\mu A_\nu.$$

The first term is the desired action of Poincaré on A_μ. The second term, $\epsilon'\gamma^\nu\epsilon\partial_\mu A_\nu$, is a field-dependent gauge transformation $\delta_\Lambda A_\mu$, where the symmetry acts in the following way:

$$\delta_\Lambda A_\mu = \Lambda_{,\mu}.$$

The parameter Λ appearing in closure of the algebra is then $\epsilon'\gamma^\nu\epsilon A_\nu$.

Now at first, this decomposition into the action of Poincaré on A_μ and a field-dependent gauge transformation looks quite unnatural. It seems more natural to say that the two combine to give the field strength, $F_{\mu\nu}$. The natural geometric formulation is to say that Poincaré is realized by the covariant derivative D_ν on the connection D_μ. We should then reformulate closure of the algebra in the following way:

$$[\delta,\delta']D_\mu = \epsilon'\gamma^\nu\epsilon[D_\nu, D_\mu].$$

I find that both views are useful. In this case where we have a clear idea of the relevant geometry – namely, bundles and connections – the latter approach is natural. In other situations where we do not understand the geometry – models with tensor fields, for example – the first way of looking at the closure condition let's us get a handle on the natural gauge-invariant combinations.

2 Why 11 dimensions?

(i) Massive particles

We now want to study representations of the SUSY algebra. These representations encode the particle content of a supermultiplet. In the case of a massive particle, we can always move to a very special frame where the momentum takes the form

$$P_\mu = (-m, 0, \ldots, 0)$$

(the mass is m). Define $\widetilde{Q} = Q/\sqrt{m}$. Then

$$\{\widetilde{Q}_\alpha, \widetilde{Q}^\dagger_\beta\} = \delta_{\alpha\beta}.$$

Here $\alpha = 1, \dots, 2^{[\frac{D+1}{2}]}$. Representations of this algebra are easy to construct. Take a vacuum $|0\rangle$ with $\widetilde{Q}_\alpha|0\rangle = 0$, and get states $|0\rangle$, $\widetilde{Q}^\dagger_\alpha|0\rangle$, $\widetilde{Q}^\dagger_\alpha\widetilde{Q}^\dagger_\beta|0\rangle$, etc. for a total of $2^{2^{[\frac{D+1}{2}]}}$ states. This representation decomposes into reducible representations of $\mathrm{Spin}(D,1)$.

Because this gives a multiplet of massive particles, there are no real physical constraints here. The spins can become arbitrarily high without a physical inconsistency appearing in the low-energy theory.

(ii) Massless particles

The situation is now different. We can take a convenient frame,

$$P_\mu = (-|p|, p, 0, \dots, 0).$$

Again, we rescale our generators, $\widetilde{Q} = Q/\sqrt{|p|}$, to give the algebra:

$$\{\widetilde{Q}_\alpha, \widetilde{Q}^\dagger_\beta\} = -\{\gamma^0(\gamma^0 - \gamma^1)\}_{\alpha\beta}.$$

To see how the representation of the algebra changes, let us pick a convenient basis for the first two elements of the Clifford algebra. Let us choose

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to give

$$\{\widetilde{Q}_\alpha, \widetilde{Q}^\dagger_\beta\} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}_{\alpha\beta}.$$

We see that half the supersymmetry generators are represented trivially so the size of our representation is reduced.

Now restrict to $D = 11$. We can impose a Majorana condition and take $Q^\dagger = Q$. We then have a $2^{[11/2]} = 32$-dimensional Majorana representation that gives rise to $2^8 = 256$ states. This representation, as in all these cases, must have an equal number of bosons and fermions. Each application of Q_α to a state changes a boson to a fermion, and vice-versa.

In this case, we get 128 bosons and 128 fermions. We need to determine how this representation transforms under $\mathrm{Spin}(10,1)$. This will give us the physical particle content of the supermultiplet. With a little algebra, one can check that the 128 bosons and 128 fermions decompose in the following way:

Bosons: a metric $g_{\mu\nu}$ with 44 states, and a 3-form $C_{\mu\nu\lambda}$ with 84 states.

Fermions: $\psi_{\mu\alpha}$ (with a constraint $\gamma^\mu\psi_\mu = 0$).

The fermion is a gravitino, or Rarita-Schwinger field, and it takes values in the bundle $(TM \otimes S - S)$. Here S denotes the spin-bundle. This field is often called a spin 3/2 particle (a vanilla spinor is typically called a spin 1/2 particle). There are no fermions beyond the gravitino in 11 dimensions.

For $D > 11$, if you impose SUSY, and follow the same construction, you will find higher spin fields. Massless fields with spin greater than 2 cannot (in a way that we currently understand) consistently be coupled to gravity.

There is a strong belief that such theories do not make sense. In this respect, 11-dimensional supergravity is quite unique. It is the minimal and maximal supergravity possible in 11 dimensions, and we believe that it gives a good description of M theory[2] on large smooth spaces.

3 Lagrangians, equations of motions, etc.

We start with 11-dimensional supergravity, with Lagrangian

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}(R - \frac{1}{2}G\wedge \star G) - \frac{1}{6}\int C\wedge G\wedge G + \text{fermions}$$

(here, $\frac{1}{2\kappa_{11}^2} = \frac{1}{(\ell_\rho^{11})^9}$, and $G = dC$).

Introduce e_μ^a a "veilbein" (or in this case, "elfbein"), where a is a local Lorentz index and μ is a spacetime index. These satisfy

$$e_\mu^a e_b^\mu = \delta_b^a$$

$$e_\mu^a e_b^\nu = \delta_\mu^\nu.$$

The supersymmetry transformations are roughly,

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu$$

$$\delta C_{\mu\nu\rho} = -\frac{\sqrt{2}}{8}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}$$

$$\delta\psi_\mu = D_\mu(\hat{\omega})\epsilon + \frac{\sqrt{2}}{288}(\gamma_\mu^{abcd} - 8\delta_\mu^a\gamma^{bcd})\epsilon\widehat{G}_{abcd},$$

where the followng 'supercovariant' combinations appear:

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} + \frac{1}{8}\overline{\psi}^\nu\gamma_{\nu\mu ab\lambda}\psi^\lambda,$$

$$\widehat{G}_{abcd} = G_{abcd} - 3\overline{\psi}_{[a}\gamma_{bc}\psi_{d]}.$$

Here ω is the spin connection. These particular combinations are nice because under a SUSY variation, no derivatives of the parameter ϵ appear. By including the correct signs and factors of 2, one can check that these transformations give a closed SUGRA algebra on-shell.

[2] M theory is the mysterious non-perturbative completion of string theory. We expect it to look 11-dimensional at long wavelengths. There is considerable evidence (including the uniqueness of 11-dimensional supergravity) that it will be unique, should it exist. There are even dynamical predictions following from the existence of M theory (and not required by other string dualities), some of which have been verified. The consistency and beauty of the entire string duality picture leaves no doubt in my mind that M theory exists - although we can't yet say much about it.

3.1 Some comments on $D = 10$

There is an easy way to get a maximally supersymmetric (32 real supercharges) $D = 10$ SUGRA. We just dimensionally reduce the 11-dimensional theory.

On reduction, $\psi_{\mu\alpha}$ reduces to 2 Majorana–Weyl gravitinos of opposite chirality, together with 2 Majorana–Weyl spinors λ_α (also of opposite chirality). The three-form C_3 reduces to a three-form in $D = 10$ C_3 together with a two-form B_2. The metric g reduces to a metric g, a connection C_1, and a scalar ϕ. This is the field content of type IIA SUGRA which is the low-energy description of type IIA string theory.

There is really only one novelty in this non-chiral theory which comes about in the following way: we define $H_3 = dB_2$ and take the Hodge dual $\star H_3 = H_7$. This gives us a dual 6-form potential $H_7 = dA_6$. We view the potential as coupling to NS 5-branes in precisely the same way that B_2 couples to fundamental strings.

Following the same procedure, we see that $\star dC_3 = dC_5$ defines a 5-form potential and $\star dC_1 = dC_7$ defines a seven-form potential. The 5-form couples to D4-branes while the 7-form couples to D6-branes. The novelty is the possibility of adding a coupling to a nine-form potential C_9 with some modification of the action. This potential couples to D8-branes. Note that C_9 has a 10-form field strength and so has trivial dynamics. However, it does carry energy density which leads to new physics. This modified theory is massive type IIA SUGRA. It does not descend from 11 dimensions. Within string theory, however, this possibility is realized.

There is a second $D = 10$ theory which does not descend by simple reduction from 11 dimensions. It is the chiral type IIB SUGRA. In this case, there are again two gravitinos but they have the same chirality (which also determines the chirality of the SUSY charges). Again, there are two dilatinos with chirality opposite to that of the gravitinos. The field content is more interesting: there is a complex scalar τ specifying the string couping, 2 two-forms B_2, C_2 and a four-form C_4. The four-form is subtle because it's field strength is self-dual (actually a combination of dC_4, $B_2 \wedge dC_2$ and $dB_2 \wedge C_2$ is self-dual.) This theory is quite fascinating because we believe it has an exact $SL(2, Z)$ duality acting on the coupling τ and the 2 two-forms. Further exploration (unfortunately) will take us beyond our alloted time today. However, more details can be found in Polchinski's text together with references.

3.2 Beyond SUGRA

By way of closing comments, I want to mention one of the outstanding issues connected with SUSY. This issue is not special to SUGRA but appears in essentially all supersymmetric models. As we have discussed, the leading terms in a long wavelength expansion with $n = 2$ form a closed SUSY algebra.

However, there are typically corrections to these terms. These kinds of terms are required for SUGRA since by themselves, SUGRA theories do not define quantum field theories. For example, there are known $\int \sqrt{g} R^4$ corrections to 11-dimensional SUGRA, which correspond to M-theoretic corrections to supergravity. Without miraculous cancellations, it is clear that an infinite set of corrections are needed for a closed SUGRA or SUSY algebra. How do we determine all these corrections? Can we repackage the problem in a way that separates the terms required by symmetry from terms with honest new dynamics? This is fascinating issue that may require developing some deeper variant of superspace. If we can answer this question, it would be an incredible step forward, taking us far beyond our current understanding of M theory and string theory.

I should point out that the answer cannot be simple. We already know that SUSY in type IIB SUGRA is strong enough to give a differential equation for a modular form with a particular eigenvalue.[3] Any framework strong enough to give a set of equations of motion closed under SUSY must at least reproduce that result.

[3]For an explanation, see M. B. Green and S. Sethi, hepth/9808061.