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Problem Set for Lecture 1. Simple functional integrals

PROBLEM 1. Feynman-Kac formula for harmonic oscillator

The Feynman Kac formula in the one-dimensional case ($d=0,\ \Sigma=[0,L]_{periodic}$) asserts for the 2-point function that, for $x_1 \leq x_2$,

$$\left(\int\limits_{\mathcal{C}_{per}([0,L])} \phi(x_1)\,\phi(x_2)\,d\mu_G(\phi) =\right) G(x_1,x_2) \;=\; \frac{\mathrm{tr}\ \mathrm{e}^{-x_1 H}\,\varphi\,\mathrm{e}^{(x_1-x_2)H}\,\varphi\,\mathrm{e}^{(x_2-L)H}}{\mathrm{tr}\ \mathrm{e}^{-LH}}\,. \tag{1}$$

In the normalizations from the lecture, $G=\frac{2\pi}{\beta}(-\frac{d^2}{dx^2}+m^2)^{-1}$ with $\frac{d^2}{dx^2}$ with periodic b.c. on $[0,L]$ and H is the Hamiltonian of a harmonic oscillator,

$$H \;=\; -\tfrac{\pi}{\beta}\,\tfrac{d^2}{d\varphi^2} \,+\, \tfrac{\beta m^2}{4\pi^2}\,\varphi^2 \,-\, \tfrac{m}{2} \;=\; m\,a^*a$$

where

$$a^* \;=\; -\sqrt{\tfrac{\pi}{\beta m}}\,\tfrac{d}{d\varphi} \,+\, \sqrt{\tfrac{\beta m}{4\pi}}\,\varphi\,, \qquad a \;=\; \sqrt{\tfrac{\pi}{\beta m}}\,\tfrac{d}{d\varphi} \,+\, \sqrt{\tfrac{\beta m}{4\pi}}\,\varphi\,, \qquad [a,a^*]=1\,,$$

are operators in $L^2(\mathbf{R},d\varphi)$. Recall that the spectrum of H is $m\{0,1,\dots\}$.

(a). Use the Fourier transform to write the left hand side. Show that its limit $G_\infty(x_1,x_2)$ when $L\to\infty$ exists.

(b). Prove that for $x_1,\dots,x_n>0$ and complex numbers $\lambda_1,\dots,\lambda_n$

$$\sum_{k,l=1}^{n} \bar\lambda_k\,\lambda_l\,G_\infty(-x_k,x_l) \;\geq\; 0\,.$$

(c). What is the limit of the right hand side of Eq. (1) when $L\to\infty$?

(d). Show that both sides of Eq. (1) coincide at $L=\infty$. Prove (b) using this result.

(e). Prove the relation (1) for finite L.

PROBLEM 2. Zeta-regularized determinants (for calculators)

(a). Using the identity $\lambda^{-s}=\Gamma(s)^{-1}\int_0^\infty t^{s-1}\mathrm{e}^{-\lambda t}\,dt$ show that $\zeta(0)=-\frac{1}{2}$ where ζ is the Riemann zeta function $\zeta(s)=\sum\limits_{n=1}^{\infty} n^{-s}$ (for $\mathrm{Re}s>1$, analytically continued elsewhere).

(b). For $\tau = \tau_1 + i\tau_2$ with τ_i real, $\tau_2 > 0$ show using the identity from (a) and the Poisson resummation that for $\mathrm{Re}\,s$ sufficiently large

$$\sum_{n=-\infty}^{\infty} |\tau+n|^{-2s} \;=\; \frac{\sqrt{\pi}}{\Gamma(s)} \left(\sum_{n\neq 0} \mathrm{e}^{2\pi i n \tau_1} \int_0^\infty t^{s-3/2}\, \mathrm{e}^{-\tau_2^2 t - \pi^2 n^2/t}\, dt \;+\; \tau_2^{-2s+1}\, \Gamma(s-1/2) \right) .$$

Note that the right hand side is analytic in s around $s=0$. Using (easy) relations $\Gamma(s)^{-1} = s + \mathcal{O}(s^2)$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\int_0^\infty t^{-3/2}\,\mathrm{e}^{-x(t+t^{-1})}\, dt \;=\; \sqrt{\pi}\, x^{-1/2}\,\mathrm{e}^{-2x}$ obtain:

$$\frac{d}{ds}\Big|_{s=0} \sum_n |\tau+n|^{-2s} \;=\; -\ln|1-q|^2 \;-\; 2\pi\tau_2$$

with the standard notation $q \equiv \mathrm{e}^{2\pi i\tau}$.

(c). By taking $\tau \to 0$ in the last formula show that $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$.

(d). Prove that for the periodic b.c. operator $\frac{d^2}{dx^2}$ on $[0,L]$ the zeta-regularized determinant

$$\mathrm{det}'\!\left(-\tfrac{\beta}{2\pi}\tfrac{d^2}{dx^2}\right) \;=\; 2\pi L^2 .$$

(c). Show that the spectrum of the Laplacian Δ_τ on the torus $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ in the metric $|dz|^2$ is given by $\lambda_{m,n} = -(\frac{2\pi}{\tau_2})^2\,|\tau m+n|^2$ for $n,m \in \mathbf{Z}$.

(d). Proceeding as in (b) decompose

$$\sum_{(m,n)\neq(0,0)} |\tau m+n|^{-2s} \;=\; \sum_{m\neq 0,\; n} |\tau m+n|^{-2s} \;+\; 2\zeta(2s)$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)} \left(\sum_{m,n\neq 0} \mathrm{e}^{2\pi i m n \tau_1} \int_0^\infty t^{s-3/2}\, \mathrm{e}^{-m^2\tau_2^2 t - \pi^2 n^2/t}\, dt \;+\; \sum_{m\neq 0} m^{-2s+1}\, \tau_2^{-2s+1}\, \Gamma(s-1/2) \right) \;+\; 2\zeta(2s)$$

and show that (after analytic continuation)

$$\zeta_{-\Delta'_\tau}(s) \;\equiv\; \sum_{(m,n)\neq(0,0)} (-\lambda_{m,n})^{-s} \;=\; -1 \;-\; 2s\,\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 \;-\; 2s\,\ln\tau_2 \;+\; \tfrac{1}{3}\pi\,\tau_2\, s\,.$$

Infer that

$$\zeta_{-\Delta'_\tau}(0) \;=\; -1\,, \qquad \zeta'_{-\Delta'_\tau}(0) = -2\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 \;-\; 2\ln\tau_2 \;+\; \tfrac{1}{3}\pi\,\tau_2\,,$$

and that

$$\mathrm{det}'(-\Delta_\tau) \;=\; \tau_2^2\,|\eta(\tau)|^4$$

where the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$.

PROBLEM 3. Rationality of the free field with values in S^1

The toroidal partition function of the free field compactified on the circle of radius squared r^2 is

$$Z_{r^2}(\tau) = \sum_{n,m\in\mathbf{Z}} \mathrm{e}^{\pi i\tau\,(mr+n/r)^2/2\,-\,\pi i\bar\tau\,(mr-n/r)^2/2}\,|\eta(\tau)|^{-2}. \tag{2}$$

Show that for rational r^2 it is a finite sesqui-linear combination of functions analytic in τ.

PROBLEM 4. Toroidal compactification to a complex torus

Let (Σ,γ) be a Riemann surface. Let $R = R_1 + iR_2$ and $T_1 + iT_2$ be two numbers in the upper half plane. Consider on the torus $(\mathbf{R}/2\pi\mathbf{Z})^2$ the complex structure given by the coordinate $\psi = \phi^1 + T\phi^2$, the metric $g = \frac{R_2}{T_2}|d\psi|^2$ and the 2-form $\omega = i\frac{R_1}{T_2}d\psi\wedge d\bar\psi$. Show that the partition functions $Z_{R,T}$ of the free field on Σ with values in $(\mathbf{R}/2\pi\mathbf{Z})^2$ with action $S(\phi) = \frac{1}{4\pi}(\|d\phi\|^2_{L^2} + i\int_\Sigma \phi^*\omega)$ satisfy the identities:

$$Z_{R,T} = Z_{R+1,T} = Z_{R,T+1} = Z_{R,-T^{-1}} = Z_{-R^{-1},-T^{-1}}.$$

additionally to the mirror symmetry $Z_{R,T} = Z_{T,R}$ shown in the lecture.

PROBLEM 9. Conformal anomaly

Using the short time expansion of the heat kernel of the Laplacian $-\Delta$ on a Riemann surface (Σ,γ)

$$\mathrm{e}^{t\Delta}(x,x) = \frac{1}{4\pi t} + \frac{1}{12\pi}\,r(x) + \mathcal{O}(t)$$

where r stands for the scalar curvature, prove that under the infinitesimal local rescaling of the metric $\gamma\mapsto \mathrm{e}^{\sigma}\gamma$

$$\frac{\delta}{\delta\sigma(x)}\bigg|_{\sigma=0} \ln\left(\frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\right) = -\frac{1}{12\pi}\,r(x). \tag{3}$$

Show that this infinitesimal relation is equivalent to the global one

$$\frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\bigg|_{\mathrm{e}^{\sigma}\gamma} = \mathrm{e}^{-\frac{1}{48\pi}\,(\|d\sigma\|^2_{L^2} + 4\int_\Sigma \sigma\, r\, dv)}\ \frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\bigg|_{\gamma}.$$

PROBLEM 10. Massless Fermions on Riemann surface

Let (Σ,γ) be a Riemann surface. Spin structure on Σ may be identified with the square root L of the canonical bundle $K = T^{*1,0}(\Sigma)$. A Dirac spinor $\Psi = (\psi,\tilde\psi)$ is an element of $\Gamma(L\oplus\bar L)$ where $\bar L$ is the bundle complex conjugate to L. The conjugate spinor is $\bar\Psi = (\tilde\chi,\chi)\in\Gamma(\bar L\oplus L)$

and in the euclidean Dirac theory it should be treated as an independent field ($\chi = \psi$, $\tilde{\chi} = \tilde{\psi}$ for Majorana fermions). Denote by $\bar{\partial}_L$ the $\bar{\partial}$ operator of L and by $\partial_{\bar{L}}$ its complex conjugate which may be naturally identified with $\bar{\partial}_L^*$. The action is a function on the odd vector space $\Pi(\Gamma(L \oplus \bar{L}) \oplus \Gamma(\bar{L} \oplus L))$:

$$S(\Psi, \bar{\Psi}) \;=\; \tfrac{1}{\pi} \int_\Sigma (\chi \bar{\partial}_{\bar{L}} \psi \;+\; \tilde{\chi} \partial_L \tilde{\psi})$$

(note that the integrand is naturally a 2-form). Partition functions of the Dirac fermions are given by the formal Berezin integral

$$Z_L \;=\; \int \mathrm{e}^{-S(\Psi, \bar{\Psi})}\, D\bar{\Psi}\, D\Psi \;=\; \det(\partial_{\bar{L}}) \det(\bar{\partial}_L) \;=\; \det(\bar{\partial}_L^* \bar{\partial}_L)\,.$$

The last determinant may be zeta-regularized giving a precise sense to the partition function Z_L of the Dirac field on Σ.

On the elliptic curve $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ with τ in the upper half-plane, the canonical bundle K may be trivialized by the section dz and spin structures correspond to the choice of periodic or anti-periodic boundary conditions under $z \to z+1$ and $z \to z+\tau$:

$$L \;=\; pp,\;\; pa,\;\; ap,\;\; aa\,.$$

(a). Show that the eigenvalues of $\bar{\partial}_L^* \bar{\partial}_L$ are

$$\lambda_{m,n} \;=\; (\tfrac{\pi}{\tau_2})^2 |\tau m + n|^2$$

with

$$\begin{array}{llll} m \in \mathbf{Z}\,, & n \in \mathbf{Z} & \text{for} & L = pp\,, \\ m \in \mathbf{Z}\,, & n \in \mathbf{Z} + \frac{1}{2} & \text{for} & L = pa\,, \\ m \in \mathbf{Z} + \frac{1}{2}\,, & n \in \mathbf{Z} & \text{for} & L = ap\,, \\ m \in \mathbf{Z} + \frac{1}{2}\,, & n \in \mathbf{Z} + \frac{1}{2} & \text{for} & L = aa\,. \end{array}$$

(b). Infer that

$$Z_{pp}(\tau) \;=\; 0\,.$$

(c). Show that

$$\zeta_{\bar{\partial}_{pa}^* \bar{\partial}_{pa}}(s) \;=\; 2^{2s} \left(\zeta_{-\Delta'_{2\tau}}(s) \;-\; \zeta_{-\Delta'_\tau}(s) \right).$$

Infer from Eq. (2) that

$$Z_{pa}(\tau) \;=\; 4 |q^{1/24} \prod_{n=1}^{\infty} (1 + q^n)|^4\,.$$

In the Hilbert space picture

$$Z_{pa}(\tau) \;=\; \mathrm{tr}_{\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R}\, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24}\,. \tag{4}$$

The "Ramond sector" Hilbert space is $\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R$ with

$$\mathcal{H}_R = \mathbf{C}^2 \otimes \left(\wedge (\bigoplus_{n=1}^{\infty} \mathbf{C}) \right)^{\otimes 2} \tag{5}$$

and $\tilde{\mathcal{H}}_R$ is another copy of $\mathcal{H}_R$. L_0 acts in the first copy. It has eigenvalue $\frac{1}{8}$ on $\mathbf{C}^2$ (the "Ramond ground states") and the occupied n^{th} mode in the fermionic Fock space adds n to it. The periodic partition function is

$$Z_{pp}(\tau) = \mathrm{tr}_{\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R} (-1)^{F+\tilde{F}} \, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24} \equiv \mathrm{str}_{\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R} \, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24} \tag{6}$$

where $(1,0),(0,1) \in \mathbf{C}^2$ correspond to the eigenvalues $+1,-1$ of $(-1)^F$ and each occupied fermionic Fock space mode adds 1 to F. $Z_{pp}(\tau)$ vanishes since modes with odd and even Fermi numbers are paired.

(d). Show that

$$\zeta_{\bar{\partial}^*_{ap}\bar{\partial}_{ap}}(s) = 2^{4s}\zeta_{-\Delta'_{\tau/2}}(s) - 2^{2s}\zeta_{-\Delta'_{\tau}}(s) \qquad \text{and} \qquad Z_{ap}(\tau) = |q^{-1/48} \prod_{n=0}^{\infty}(1 - q^{n+1/2})|^4 .$$

The Hilbert space interpretation is

$$Z_{ap}(\tau) = \mathrm{str}_{\mathcal{H}_{NS} \otimes \tilde{\mathcal{H}}_{NS}} \, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24} \tag{7}$$

where the "Neveu-Schwarz sector" Hilbert space is

$$\mathcal{H}_{NS} = \left(\wedge (\bigoplus_{n=0}^{\infty} \mathbf{C}) \right)^{\otimes 2} . \tag{8}$$

The "Neveu-Schwarz ground state" has eigenvalue zero of L_0 and the n^{th} occupied zero mode contributes $(n+\frac{1}{2})$ to it. The fermion number of the NS-ground state vanishes and each occupied fermionic mode adds 1 to it.

(e). Show that

$$\zeta_{\bar{\partial}^*_{aa}\bar{\partial}_{aa}}(s) = 2^{2s}\zeta_{\bar{\partial}^*_{pa}\bar{\partial}_{pa}}(s)\Big|_{\tau/2} - \zeta_{\bar{\partial}^*_{pa}\bar{\partial}_{pa}}(s)\Big|_{\tau} \qquad \text{and} \qquad Z_{aa}(\tau) = |q^{-1/48} \prod_{n=0}^{\infty}(1 + q^{n+1/2})|^4$$

and that

$$Z_{aa}(\tau) = \mathrm{tr}_{\mathcal{H}_{NS} \otimes \tilde{\mathcal{H}}_{NS}} \, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24} . \tag{9}$$

(f). Prove the modular properties:

$$\begin{array}{lll} Z_{pa}(\tau+1) = Z_{pa}(\tau), & Z_{ap}(\tau+1) = Z_{aa}(\tau), & Z_{aa}(\tau+1) = Z_{ap}(\tau), \\ Z_{pa}(-1/\tau) = Z_{ap}(\tau), & Z_{ap}(-1/\tau) = Z_{pa}(\tau), & Z_{aa}(-1/\tau) = Z_{aa}(\tau). \end{array}$$

PROBLEM 10. <u>Bosonization</u>

The spin structure is called even (odd) if the dimension of the kernel of $\bar{\partial}_L$ is even (odd). Denote by $\sigma(L)$ the parity of L. The bosonization formula asserts that

$$\sum_L (-1)^{\sigma(L)} Z_L \;=\; C^{h-1} Z_{1/2} \tag{10}$$

where on the right hand side we have the partition function of the bosonic free field with values in the circle of radius squared $\frac{1}{2}$, C is a constant and h the genus of the Riemann surface Σ. These equalities extend to correlations. For example, the fermionic fields $(\psi\tilde{\psi})(x)$ correspond to bosonic fields $:\mathrm{e}^{i\phi(x)}:$ and $(\tilde{\chi}\chi)(x)$ to $:\mathrm{e}^{-i\phi(x)}:$. What are their conformal weights? Prove identity (10) for $\Sigma = \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ using the expression (2) for $Z_{1/2}(\tau)$ and the classical product expressions for the theta functions

$$\vartheta(z|\tau) \equiv \sum_{n\in\mathbf{Z}} \mathrm{e}^{\pi i\tau n^2 + 2\pi i n z} \;=\; \prod_{n=1}^{\infty} (1-q^n)\,(1+\mathrm{e}^{2\pi i z}\, q^{n-1/2})\,(1+\mathrm{e}^{-2\pi i z}\, q^{n-1/2}).$$

What is the Hilbert space interpretation of the left hand side of Eq. (10) on the elliptic curve?