1. We denote the Riemannian metric on M by $(-\ ,\ -)$ and so write the lagrangian density as

$$\mathcal{L} = \tfrac{1}{2}(\dot{X}, \dot{X})dt\ .$$

Here $\dot{X} = \frac{dX}{dt}$ is the tangent to the curve.

(a) We compute the variation by considering a path of X's, denoted X_u, $u \in (-\varepsilon, \varepsilon)$. The computation takes place on $(-\varepsilon, \varepsilon) \times \mathbb{R}$: There is a map $X\colon (-\varepsilon, \varepsilon) \times \mathbb{R} \to M$ and we use the pullback metric, the pullback tangent bundle, and the pullback of the Levi-Civita connection, which we denote ∇. We use the notation $\delta = \frac{\partial}{\partial u}$.

Since $[\partial_t, \delta] = 0$ and ∇ is torsion free, we have*

$$\nabla_t \delta X = \nabla_\delta \dot{X}\ . \tag{1.1}$$

So we make the computation

$$\begin{aligned} \delta\mathcal{L} &= \tfrac{1}{2}\delta(\dot{X}, \dot{X})dt \\ &= \left(\nabla_\delta \dot{X}, \dot{X}\right) dt \\ &= \left(\nabla_t \delta X, \dot{X}\right) dt \\ &= d\left(\delta X, \dot{X}\right) - \left(\delta X, \nabla_t \dot{X}\right) dt \end{aligned} \tag{1.2}$$

Now the first term in the final line is a total derivative, and the second is linear over the functions (as a function of δX). So the variational derivative is

$$\frac{\delta \mathcal{L}}{\delta X(t)} = -\nabla_t \dot{X}\, dt$$

* If x^i are local coordinates, and Γ^i_{jk} the corresponding Christoffel symbols on M, then the pullback by X makes the x^i functions of t, u and equation (1.1) follows, since

$$\frac{\partial^2 x^i}{\partial t \partial u} + \Gamma^i_{jk} \frac{\partial X}{\partial t} \frac{\partial X}{\partial u}$$

is symmetric in t, u. (In other words, Γ^i_{jk} is symmetric in j, k). A derivation without local coordinates is given at the end of the exercise in an appendix.

The space of critical points is then

$$W = \left\{ X\colon\ \nabla_t \dot{X} = 0 \right\} .$$

The image of any such X is a geodesic; it is parametrized at constant speed.

(b) We assume that M is complete. Then the standard theory of ordinary differential equations shows that there is a unique solution X to $\nabla_t \dot{X}$ given an initial position and velocity, i.e., a point of TM.

(c) To compute the symplectic structure we use the exact term in formula (1.2). Set

$$\gamma = \left(\delta X, \dot{X} \right)\Big|_{t_0} , \tag{1.3}$$

where t_0 is some fixed time. Then the symplectic structure is computed as the differential of γ in the space of paths:

$$\omega = \delta\gamma = -\left(\delta X, \nabla_\delta \dot{X} \right)\Big|_{t=t_0} = \left(\nabla_\delta \dot{X}, \delta X \right)\Big|_{t=t_0} \tag{1.4}$$

We elaborate: If ξ_1, ξ_2 are sections of $X^*(TM)$ — in other words, are tangent at X to the space of paths — then

$$\omega(\xi_1, \xi_2) = \left(\nabla_{\xi_1} \dot{X}(t_0), \xi_2(t_0) \right) - \left(\nabla_{\xi_2} \dot{X}(t_0), \xi_1(t_0) \right) .$$

Now the standard symplectic structure on T^*M is the differential of the 1-form

$$\theta_\alpha(\eta) = \alpha(\pi_* \eta), \qquad \alpha \in T^*M,\ \eta \in T_\alpha(T^*M), \tag{1.5}$$

where $\pi\colon T^*M \to M$ is projection. We identify $X \in W$ with the tangent vector $\left\langle X(t_0), \dot{X}(t_0) \right\rangle \in TM$. Then

$$\gamma(\eta) = \left(\pi_* \eta, \dot{X}(t_0) \right) , \qquad \eta \in T_{X(t_0)}(TM). \tag{1.6}$$

Under the map $TM \to T^*M$ given by the metric, it is easy to see that (1.5) pulls back to (1.6) at the point $\left\langle X(t_0), \dot{X}(t_0) \right\rangle \in TM$.

(d) The vector field v on W may be described by the equation

$$v_X = \dot{X} \in C^\infty(\mathbb{R}, X^*(TM)) \ .$$

To compute $\iota_v\omega$ we substitute $\delta X = \dot{X}$ into (1.4):

$$\begin{aligned} \iota_v\omega &= \Big(\nabla_t \dot{X}, \delta X\Big)\big|_{t_0} - \Big(\nabla_\delta \dot{X}, \dot{X}\Big)\big|_{t_0} \\ &= -\Big(\nabla_\delta \dot{X}, \dot{X}\Big)\big|_{t_0} \ , \end{aligned}$$

since X satisfies the eqution of motion $\nabla_t \dot{X} = 0$. It is easy to see that $\iota_v\omega = dH$ where H is the function

$$H(X) = -\tfrac{1}{2}\,|\dot{X}(t_0)|^2 \ .$$

(Of course, this is independent of t_0.)

Remark on Equation (1.1) (by P. Deligne).

An affine space A gives rise to a vector space V, the translations of A, and it is convenient (barycentric calculus) to view A and V as living in the same vector space B of dimension one more:

$$0 \longrightarrow V \longrightarrow B \xrightarrow{m} \mathbb{R} \longrightarrow 0, \qquad A = m^{-1}(1) \ .$$

If A is an affine space bundle over a manifold M (structural group the affine group $\mathrm{GL}(n) \ltimes \mathbb{R}^n$), a connection ∇ on A allows us to define $\nabla_X a$, for a a section of A, as a section of the associated vector bundle (vector bundle of translations of A), structural group $\mathrm{GL}(n)$, obtained by applying $\mathrm{GL}(n) \ltimes \mathbb{R}^n \to \mathrm{GL}(n)$.

Let T^{aff} be the tangent bundle T to M, viewed as a bundle of affine space. The vector bundle T is the vector bundle associated to T^{aff}. A connection ∇ on T defines as follows a connection ∇^{aff} on T^{aff}: ∇^{aff} induces ∇ on T and for z the zero section of T^{aff}, one has $\nabla^{\mathrm{aff}}_X z = -X$ [connexion sans glissement].

The torsion of ∇ is the translation part of the curvature of ∇^{aff}. View X as a map of $\mathbb{R}^2$ to the zero section of th tangent bundle of M. If ∇ is torsion free, vanishing of the translation part of the curvature gives

$$\nabla_t \nabla^{\mathrm{aff}}_u X - \nabla_u \nabla^{\mathrm{aff}}_t X = 0 \ , \qquad \text{i.e.}$$

$$-\nabla_t \partial_u X + \nabla_u \partial_t X = 0$$