(1) Let M be a fixed Riemannian manifold, and let $X : \mathbf{R} \to M$ be a map from the t-line $\mathbf{R}$ (endoowed with the metric $(dt)^2$) to M.

Let

$$L = \frac{1}{2}\int dt g_{IJ}\frac{dX^I}{dt}\frac{dX^j}{dt}. \tag{1}$$

(a) Describe the space W of critical points of L. (You should find that it is closely related to the space of geodesics on M.)

(b) Show that given any choice of a point $t_0 \in \mathbf{R}$, W has a natural identification with T^*M. (Use the metric on M to identify TM and T^*M.)

(c) Compute the symplectic structure ω on W. Show that for any choice of t_0, W becomes identified (under the identification in (b)) with T^*M with its natural symplectic structure.

(c) The vector field d/dt on $\mathbf{R}$ induces a vector field v on W. The "Hamiltonian" is a function H on W such that

$$dH = i_v\omega. \tag{2}$$

This is usually expressed by saying that "H generates time translations via Poisson brackets." Compute H.

(2) Consider $\mathbf{R}^{1,1}$ with even and odd coordinates t and θ and an odd distribution $\mathcal{A}$ generated by the vector field

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}. \tag{3}$$

Thus, $D^2 = -\partial/\partial t$.

As in the last exercise, let M be a fixed Riemannian manifold. Let X be a map from $\mathbf{R}^{1,1}$ to M. Consider the Lagrangian

$$L = \frac{1}{2}\int_{\mathbf{R}^{1,1}} g_{IJ}\frac{\partial X^I}{\partial t}DX^j. \tag{4}$$

(a) Make sense of the definition of L by interpreting the integrand as a section of the Berezinian of the tangent bundle of $\mathbf{R}^{1,1}$.

(b) Setting $X^I = x^I + \theta\psi^I$, write L explicitly as an ordinary integral, over the t-line $\mathbf{R}^{1,0}$, of a function of x^I and ψ^I.

(c) Describe the space Y of critical points of L. You should get a description of the following kind. Describe the reduced space Y_{red} as an ordinary manifold, and the normal bundle to Y_{red} in Y as ΠV, where V is a vector bundle over Y_{red}. What are the "initial data" at a given point $t = t_0$ in $\mathbf{R}^{1,1}$ needed to determine a critical point? (This is the analog of problem 1(b) above.)

(d) Compute the symplectic structure on Y.

(e) The vector fields

$$\frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial t} \tag{5}$$

induce vector fields on Y. Find the functions Q and H that generate these vector fields by Poisson brackets. Explain why $\{Q, Q\} = 2H$.