Witten's Problems, Set Two — N°. 1 (solution written by P. Deligne)

We will write $\mathbb{R}^{d+1}$ for $\mathbb{R}^n$, coordinates $t = x_0, x_1, \ldots, x_d$.

(a) In the Minkowski signature, $*$ is defined as follows:

1. Extend the given bilinear form on V to $\overset{p}{\wedge} V$, by

$$(v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p) = \det(v_i, w_j).$$

Same for V^*, identified with V.

2. An orientation has to be chosen. Together with the metric, it defines a form ω of maximal degree: in standard coordinates, $dt \wedge dx_1 \wedge \ldots \wedge dx_d$.

3. The operation $*$ is defined by $\alpha \wedge * \beta = (\alpha, \beta).\omega$.

The lagrangian density $\mathcal{L}$ is

$$-\tfrac{1}{4e^2}\,(F,F).(\text{standard density } dv)$$

In this way of writing, no orientation is needed. However, in the computations which follow, we choose an orientation to identify densities and forms of maximal degree. The variation of the lagrangian density by $A \mapsto A + \delta A$ is

$$\begin{aligned} -\tfrac{1}{2e^2}\,(d\delta A, dA) dv &= -\tfrac{1}{2e^2}\, d\delta A \wedge * dA \\ &= d\left(-\tfrac{1}{2e^2}\,\delta A \wedge * dA\right) - \tfrac{1}{2e^2}\,\delta A \wedge d * dA; \end{aligned}$$

the variational derivative $\delta\mathcal{L}$ is $-1/2e^2\ \delta A \wedge d * dA$, giving as Euler-Lagrange equation for the extremals (a.k.a. equation of motion)

$$d * dA = 0,$$

i.e., for d^* the adjoint of d, $d^* dA = 0$.

(b) The general rule for defining ω_0 is as follows:

1. Pick a space like hypersurface Γ (e.g. $t = 0$).

2. Let a be the d-form whose exterior derivative is used to justify the integration by part leading to Euler-Lagrange equations. Here,

$$a = -\tfrac{1}{2e^2}\,\delta A \wedge * dA.$$

it is a 1-form on the space of fields, with values in d-forms, and $\delta A \mapsto a(\delta A)$ is linear over C^∞-functions.

3. Define a 1-form α_Γ on the space of fields by integrating a on Γ.

4. Restrict α_Γ to the space of extremals, and define

$$\omega_0 = d\alpha_\Gamma \ .$$

For this to make sense, one needs some control at spatial infinity. it is not unreasonable to consider only A which vanish at spatial infinity: we will see in (c) that if at $t = 0$, A and $\dot{A}$ vanish for $\left(\sum x_i^2\right)^{1/2} > R$, and if A obeys the equations of motion $d * dA = 0$, then some gauge transform of A vanishes for $\left(\sum x_i^2\right)^{1/2} > R + |t|$.

On the space of extremals vanishing at spatial infinity, the 1-form α_Γ makes sense. Its exterior derivative is

$$\omega_0(\delta_1 A, \delta_2 A) = \int_\Gamma -\tfrac{1}{2e^2}\,(\delta_2 A \wedge * d\delta_1 A - \delta_1 A \wedge * d\delta_2 A).$$

The space of extremals is an affine space, and ω_0 is translation invariant. The general lagrangian formalism ensures that this 2-form is independent of Γ.

(c) Gauge transformations $A \mapsto A + d\phi$ preserve the lagrangian density, hence the space of extremals. Of course, if one imposes to A to vanish at spatial infinity, one has to impose to ϕ to be locally constant at infinity (hence constant at spatial infinity if $d + 1 \neq 2$). Suppose $d + 1 \neq 2$: one can impose to ϕ to be zero at spatial infinity, by subtracting a constant. The formula for ω_0 makes it clear that if $A \mapsto A + \delta_1 A$ is an infinitesimal gauge transformation with support disjoint from Γ, then $\omega_0(\delta_1 A, \delta_2 A) = 0$. As Γ can be chosen as it pleases, it follows that ω_0 comes from the quotient by gauge transformations: decompose an infinitesimal gauge transformation δA_1 in gauge transformations with support in $t < 1$,

$-1 < t < 1$ and $-1 < t$, and for each evaluate $\omega_0(\delta_1 A, \delta_2 A)$ using a Γ for which it obviously vanishes.

Direct proof: if B is a tangent vector to the space of extremals: $d * dB = 0$ and B vanishing at spatial infinity, then

$$\int_\Gamma (d\phi) \wedge * dB + B \wedge * d(d\phi) = \int_\Gamma d(\phi . * dB) = 0 \ .$$

To investigate the quotient of the space of extremals by the group of gauge transformation, we now fix a decomposition of space-time as space $\times$ time. Coordinates $x_0 = t$, x_i. Using $\phi = \int_0^t A_0(x, \tau) d\tau$ (vanishing at spatial infinity if A does), one gauge transforms A into a new A with $A_0 = 0$. The equation $d * dA$ becomes

$$\partial_0 \left(\sum \partial_i A_i \right) = 0 \tag{1}$$

$$\Box A = -d \sum \partial_i A_i \tag{2}$$

if one gives A at $t = 0$, (1) gives $f := \sum \partial_i A_i$ at all times: it is independent of t. If $\dot{A}$ is given at $t = 0$, one can then solve $\Box A = -df$ with the given Cauchy data $(A, \dot{A})$ at $t = 0$ (hyperbolicity of $\Box$). The support of the resulting A is contained in what can be reached by a time-like vector from the support of $(A, \dot{A})$ at $t = 0$.

We have

$$\begin{aligned} &\sum \partial_i A_i = f \quad \text{at } t = 0,\ f \text{ independent of } t \\ &\Box A_i = -\partial_i f \end{aligned}$$

It follows that

$$\Box \left(\sum \partial_i A_i - f \right) = \sum \partial_i \Box A_i - \Box f = \sum -\partial_i \partial_i f - \Box f = Df - Df = 0$$

and if $\frac{d}{dt} \left(\sum \partial_i A_i - f \right) = 0$ at $t = 0$, i.e. if (1) holds at $t = 0$, then $\sum \partial_i A_i - f = 0$ always, i.e. (1) holds.

It follows that we have a bijection

$$A \longmapsto (A, \dot{A}) \quad \text{at} \quad t = 0$$

from the space of extremals A (vanishing at spatial infinity), with $A_0 = 0$, to the space of Cauchy data $(A, \dot{A})$ at $t = 0$, vanishing at infinity with $A_0 = 0$, $\dot{A}_0 = 0$, $\sum \partial_i \dot{A}_i = 0$. The last condition is what keeps making sense, on the Cauchy data, of the equation of motion $d * dA = 0$.

Transported to the affine space of Cauchy data, the 2-form ω_0 becomes the translation invariant 2-form

$$(3) \qquad (A, \dot{A}), (B, \dot{B}) \longmapsto \int (A, \dot{B}) - (\dot{A}, B)$$

(up to a constant). It is degenerate, as we still have to divide by the time independent gauge transformations ϕ: we have vanishing for $(A, \dot{A}) = (d\phi, 0)$. Conversely, if $(A, \dot{A})$ is in the kernel, we have $\int (\dot{A}, B) = 0$ for any compactly supported B, hence $\dot{A} = 0$, and $\int (A, \dot{B}) = 0$ if $d^* \dot{B} = 0$. One can take $\dot{B} = d^* C$ and integration by part gives $\int (dA, C) = 0$, hence $dA = 0$ and $A = d\phi$ for some ϕ.

This can be presented as follows: one the space of all $(A, \dot{A})$ at $t = 0$ ($d^* \dot{A} = 0$ not required), we have the symplectic structure (3). We took the symplectic quotient $L^\perp / L$ by the isotropic linear subspace L of the $(d\phi, 0)$.

Geometrically, the hyperbolicity modulo gauge of the equation of motion $d * dA = 0$ has the following meaning.

One thinks of the 1-form A as being a connection ∇ on the trivial $\mathbb{R}$-torsor (= principal $\mathbb{R}$-bundle) T. For electromagnetism, it might be better to use rather $\mathbb{R}/\mathbb{Z} = U^1$, but if one wants a change of trivializations to be given by a function ϕ with values in $\mathbb{R}$, not $\mathbb{R}/\mathbb{Z}$, one needs $\mathbb{R}$. The gauge transform $A + d\phi$ of A is the 1-form describing the same connection ∇, in the trivialization ϕ of the torsor T.

Call "space" the hypersurface $t = 0$ and let U be a region of space. Suppose given a 1-jet P_1 of $\mathbb{R}$-torsor along U. If s_1 is a trivialization of P_1, any other trivialization can be written as $s_1 + \phi_1$, for ϕ_1 a 1-jet of function along U: $\phi_1 = (\phi, \dot{\phi})$ on U. Suppose a connection ∇_1 is given on P_1. Given a trivialization of P_1, it is the data of A_0, A_i, $\dot{A_i}$ ($i \neq 0$) in U (hence at $t = 0$). Suppose lastly that what makes sense of the equation of motion holds. In coordinates:

$$\sum_{i \neq 0} \partial_i^2 A_0 - \partial_i \dot{A}_i = 0$$

This is the “Cauchy data”.

Let V be the region of space time which, in terms of causality

for $t > 0$: is influence only by U;

for $t < 0$: influences only U.

Otherwise said: v is in V if all time like trajectories through v hit $t = 0$ in U. For U the ball $\left(\sum x_i^2\right)^{1/2} < R$, of space, i.e. at $t = 0$, V is the region $\left(\sum x_i^2\right)^{1/2} < R - |t|$.

Hyperbolicity gives that the Cauchy data (P_1, ∇_1) extends uniquely to a torsor with connection obeying the equation of motion on V: an extension exists, any two extensions are isomorphic by an isomorphism whose first jet along U is the identity, and such an isomorphism is unique.

Example. If U is the region $\sum x_i^2 > R$, then V is the region $\sum x_i^2 > R + |t|$. If (P, ∇) obeys the equation of motion and has a 1-jet along U isomorphic ot the trivial flat torsor, it follows that (P, ∇) is isomorphic to the trivial flat torsor on V.

In the preceding stories, one can replace “space” $t = 0$ by a space like hypersurface. Proofs are similar. One begins by mapping V to U by sending $v = (t, x)$ to the point of U with the same x-coordinates, and by using the connection along the time-lines to write the torsors with connection on V as pull back, as torsors, of a torsor on U (“gauge transformation making $A_0 = 0$”).

In this geometric language, the restriction to A which vanish at spatial infinity, means considering only (P, ∇) such that P admits a trivialization which is horizontal at spatial infinity: flatness at spatial infinity, and, for $d = 2$, trivial monodromy at spatial infinity and vanishing of the integral of the curvature on a space like hypersurface, hence on all of them.