Witten's Problems, Set Two — N°. 4 (solution written by P. Deligne)

(a) Let P be the subbundle of the tangent bundle T of W generated by D, and P_2 be generated by D and $D^2 := \frac{1}{2}[D, D]$. We have

$$B_- \oplus P_2 \xrightarrow{\sim} T$$

and by the Frobenius pairing, $P_2/P \simeq P \otimes P$.

The filtration $T \supset P_2 \supset P$ of T, with successive quotients B_-, P_2/P and P, gives a canonical isomorphism

$$\begin{aligned} \mathrm{Ber}(T) &\simeq \mathrm{Ber}(B_-) \otimes \mathrm{Ber}(P_2/P) \otimes \mathrm{Ber}(P) \\ &\simeq B_- \otimes P_2/P \otimes P^\vee . \end{aligned}$$

As $P_2/P \simeq P^{\otimes 2}$, this gives

$$\mathrm{Ber}(T)^\vee = B_-^\vee$$

and the lagrangian density

$$\left\langle dX\big|_{B^-}, dX\big|_{P^\vee} \right\rangle$$

is canonically a section of the bundle of volume elements.

(b) Not all W are locally isomorphic to the flat model given. Indeed, P_2 and B_- are integrable distributions, giving locally a decomposition of W as

$$W^{1,1} \times W^1 .$$

The odd distribution P is in the tangent space of the fibers of the projection pr_2 to W^1. All fibers are identified with $W^{1,1}$ by the projection to $W^{1,1}$ and the given flat model corresponds to the case where the induced distributions on the fibers $\mathrm{pr}_2^{-1}(t) \xrightarrow{\sim} W^{1,1}$ are all equal.

Let us compute the Euler-Lagrange equations for the extremals for the "flat" W given. Because $D^2 = \partial_+$, the relevant trivialization of $\mathrm{Ber}(T)$ is the one corresponding to the basis $(\partial_-, \partial_+, D)$ or, what amounts to the same, corresponding to the basis $(\partial_u, \partial_v, \partial_\theta)$ (triangular change of basis). The lagrangian density is

$$\langle \partial_- X, DX \rangle \,.(\text{volume form } [u, v, \theta] \text{ given by the coordinates}).$$

If we let X depend on an additional even parameter a, the variation $\frac{d}{da}$ of the lagrangian density is given as follows at $a = 0$: let X be what X is at $a = 0$, and δX be $\delta_a X$: a section of $X^* T_M$. Defining derivatives of vectors using the Levi-Civita connection, using that this connection preserves $\langle\ ,\ \rangle$ and that it is torsion free (cf. exercise 1 of set 1), we get at $a = 0$ that

$$\begin{aligned}
\tfrac{d}{da}\mathfrak{L} &= (\langle \partial_a \partial_- X, DX \rangle + \langle \partial_- X, \partial_a DX \rangle) && [u, v, \theta] \\
&= (\langle \partial_- \partial_a X, DX \rangle + \langle \partial_- X, D\partial_a X \rangle) && [u, v, \theta] \\
&= (\langle \partial_- \delta X, DX \rangle + \langle \partial_- X, D\delta X \rangle) && [u, v, \theta] \\
&= (\partial_- \langle \delta X, DX \rangle + D \langle \partial_- X, \delta X \rangle) && [u, v, \theta] \\
&\qquad - (\langle \delta X, \partial_- DX \rangle - \langle D\partial_- X, \delta X \rangle) && [u, v, \theta]
\end{aligned}$$

where the first term is

$$d(\langle \delta X, DX \rangle\, \partial_- \perp [u, v, \theta] + \langle \partial_- X, \delta X \rangle\, D \perp [u, v, \theta])\,,$$

because the Lie derivative $\mathfrak{L}_D([u, v, \theta])$ vanishes, and the second is

$$\delta \mathfrak{L} = -2\, \langle \delta X, \partial_- DX \rangle \quad [u, v, \theta].$$

This gives the equation of motion

(1) $$\partial_- DX = 0.$$

If one works over a purely even parameter space B, a map $X\colon W \to M$ factors as

$$W \to W_{\mathrm{red}} \xrightarrow{x} M,$$

the first map being $(u, v, \theta) \to (u, v)$. Indeed, the even functions on $W \times B$ are just the functions on $W_{\mathrm{red}} \times B$, as $W \times B$ is of dimension $(*, 1)$. The equation (1) amounts to

$(1)_{\mathrm{red}}$ $$\partial_- \partial_+ x = 0$$

for x: the reduced space $Y_{\rm red}$ of the space Y of extremals is the space of "harmonic" maps from $W_{\rm red}$ (viewed as a quotient of W) to M. Beware that "harmonicity" is of an hyperbolic character, as we are in a Minkowsky space setting.

Over a general parameter space B, a map X: $W \to M$ (i.e. a map from $W \times B$ to M) can be described by

(a) its restriction x to the subvariety $\theta = 0$ of W (i.e. its restriction to $W_{\rm red} \times B$ (not: $(W \times B)_{\rm red}$);

(b) its derivative $\psi = \partial_\theta X$, restricted to $\theta = 0$. It is an odd section of $x^* T_M$.

If we use local coordinates on M, this means writing

$$X^i = x^i + \theta \psi^i,$$

with x^i and ψ^i depending only on u and v.

Let $\overline{\mathcal{L}}$ be obtained from the lagrangian density by integrating along the fibers of $(u, v, \theta) \to (u, v)$. It is

$$\overline{\mathcal{L}} = (- \langle \partial_- x, \partial_+ x \rangle + \langle \nabla_- \psi, \psi \rangle)\, du\, dv\ .$$

We can view $\overline{\mathcal{L}}$ as a lagrangian density for the space of fields (x, ψ) on the (u, v)-space $W_{\rm red}$: x a map to M, ψ an odd section of $x^* T_M$. For general reasons, the lagrangian density $\overline{\mathcal{L}}$ give rise to the same space of extremals as $\mathcal{L}$ does, and to the same 2-form on the space of extremals.

As in Set 1, N° 2, the Euler Lagrange equations for (x, ψ) are of the form

(1) $\nabla_- \partial_+ x =$ quadratic in ψ (involving the curvature),

(2) $\nabla_- \psi = 0$.

If the parameter space B is purely even, the odd section ψ of $x^* T_M$ (on $W_{\rm red} \times B$) is zero, and one finds again that $Y_{\rm red}$ is a space of "harmonic" maps.

Over a general B, if I is the ideal defining $B_{\rm red}$ and if one works modulo I^2, (1) again reduces to $\nabla_- \partial_+ x = 0$. This gives the normal bundle of $Y_{\rm red}$ in Y: at x in $Y_{\rm red}$, it is ΠV, for V the space of sections s of $x^* TM$ such that $\nabla_- s = 0$.

Because of the quadratic terms in (1), I don't know any natural way to map Y to Y_{red}, to view Y as a bundle over Y_{red}. Taking $(x, \psi) \mapsto x = (x, 0)$ will not do.

However, the quadratic terms in (1) don't spoil the kind of Cauchy data for which (1) (2) is a well posed system of equations. With the notations of (d): $u = t + x$, $v = t - x$, a Cauchy data consist in x, $\dot{x}$ and ψ at $t = 0$. In terms of $X\colon W \to M$, this is X and its derivatives $\dot{X}$ and $\partial_\theta X$ at $t = \theta = 0$. The projection $(x, \dot{x}, \psi) \mapsto (x, \dot{x})$: Cauchy data for $Y_{\text{red}} \to$ Cauchy data for Y, gives a projection $Y \to Y_{\text{red}}$. It is, however, not "natural" in the sense that it is not invariant by time evolution: it depends on having chosen the line $t = 0$.

Suppose the parameter space B is of dimension $(*, b)$. Let I be the ideal defining B_{red} in B. To solve (1) (2) for a given Cauchy data, one can proceed as follows:

(1) Solve (1) (2) over B_{red}. After pull back to B_{red}, we have $\psi = 0$ by parity reason and one has to solve for an "harmonic" map: $\nabla_- \partial_+ x_0 = 0$.

(2) One can view x_0 as being over B/I^2: modulo I^2, the even part of $C^\infty(W \times B)$ is $C^\infty(W \times B_{\text{red}})$. One solves (1) (2) modulo I^2, using x_0 given by (1) and solving (2). This amounts to solving an equation $\nabla_- \psi = 0$ on $W_{\text{red}} \times B_{\text{red}}$, for ψ a section of $x_0^* T_M \otimes \mathrm{pr}_2^*(I/I^2)$. We now have (x_1, ψ_1) modulo I^2.

(3) If we already have (x_r, ψ_r) which is a solution $\mod I^{r+1}$, i.e. a r-jet of solution along $B_{\text{red}} \subset B$, we extend it in any way to (x^0, ψ^0) modulo I^{r+2}. Then correct (x^0, ψ^0) into a solution (x_{r+1}, ψ_{r+1}), by a correction trivial modulo I^{r+1}. At each step, this amounts to solving on $W_{\text{red}} \times B_{\text{red}}$ an equation

$$\nabla_- \partial_+ x = \text{second member}, \qquad \text{or}$$
$$\nabla_- \psi = \text{second member},$$

with x or ψ in $x_0^* T_M \otimes \mathrm{pr}_2^* I^{r+1}/I^{r+2}$.

This process terminates when $r = b$.

Assuming M complete, and some control at spatial infinity, we can now identify the space Y of extremals with the space of Cauchy data.